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On Semiprime Rings with Generalized Derivations

Mohd Rais Khan and Mohammad Mueenul Hasnain*
Department of Mathematics, Jamia Millia Islamia(A Central University), New Delhi, 110-025, India
e-mail : mohdrais_khan@yahoo.co.in *and* mhamu786@gmail.com

Abstract. In this paper, we investigate the commutativity of a semiprime ring R admitting a generalized derivation F with associated derivation D satisfying any one of the properties: (i) $F(x) \circ D(y) = [x, y]$, (ii) $D(x) \circ F(y) = F[x, y]$, (iii) $D(x) \circ F(y) = xy$, (iv) $F(x \circ y) = [F(x), y] + [D(y), x]$, and (v) $F[x, y] = F(x) \circ y - D(y) \circ x$ for all x, y in some appropriate subsets of R.

1. Introduction

The commutativity of prime rings with derivation was initiated by Posner in [13]. Thereafter, several authors have proved commutativity theorems for prime or semiprime rings admitting automorphisms or derivations which are centralizing or commuting on some appropriate subsets of R (see [1-7,9,10,12,14] where further references can be found).

Throughout this paper, R will represent an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ and $(x \circ y)$ stand for the commutator $xy - yx$ and the anti-commutator $xy + yx$ respectively. A ring R is called a 2-torsion free if whenever $2x = 0$ with $x \in R$, then $x = 0$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = 0 \Rightarrow a = 0$ or $b = 0$, and it is called semiprime if for any $a \in R$, $aRa = 0 \Rightarrow a = 0$.

We shall make extensive use of the following basic identities without any specific mention; For all $x, y, z \in R$,

(i) $[xy, z] = x[y, z] + [x, z]y$ & $[x, yz] = y[x, z] + [x, y]z$,

(ii) $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$,

(iii) $(xy) \circ z = (x \circ z)y - x[y, z] = x(y \circ z) + [x, z]y$.

An additive mapping $D : R \longrightarrow R$ is called a derivation of R, if $D(xy) = xD(y) + D(x)y$ holds for all $x, y \in R$. In particular, for a fixed $a \in R$, the mapping

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* Corresponding Author.

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$I_a : R \to R$ given by $I_a(x) = [x, a]$ is a derivation called an inner derivation. Let I be a nonempty subset of R. A mapping $g : R \longrightarrow R$ is called centralizing on I if $[g(x), x] \in Z(R)$ for all $x \in R$. In particular, it is called commuting on I if $[g(x), x] = 0$, for all $x \in R$. An additive map $F : R \to R$ is called a generalized inner derivation if $F(x) = ax + xb$ for fixed $a, b \in R$. For such mapping F, it is easy to see that

$$F(xy) = F(x)y + x[y, b] = F(x)y + xI_b(y) \text{ for all } x, y \in R.$$

This observation leads to the following definition, given in [9] and [11]; an additive mapping $F : R \to R$ is called generalized derivation with associated derivation D if

$$F(xy) = F(x)y + xD(y) = D(x)y + xF(y) \text{ for all } x, y \in R.$$

Familiar examples of generalized derivations are derivations and generalized inner derivations and latter includes left multiplier i.e. an additive mapping $F : R \to R$ satisfying $F(xy) = F(x)y$ for all $x, y \in R$. Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x) = cx + D(x)$, where c is fixed element of R and D a derivation of R, is a generalized derivation; and if R has 1, all generalized derivations have this form. The purpose of this paper is to prove some results which are of independent interest and related to generalized derivations on semiprime rings.

2. Main Results

We begin our discussion with the following known results which are essentially proved in [7] and [8] respectively.

Lemma A. *If R is a semiprime ring , I a nonzero ideal of R. Let D be a derivation on R, which is nonzero on I, centralizing on I, then R has a nonzero central ideal.*

Lemma B. *If R is a 2-torsion free semiprime ring, I a nonzero ideal of R and $a, b \in R$. Then the following conditions are equivalent:*

(i) $axb = 0$, *for all* $x \in R$,

(ii) $bxa = 0$, *for all* $x \in R$,

(iii) $axb + bxa = 0$, *for all* $x \in R$.

Theorem 2.1. *Let R be a 2 torsion free semiprime ring, I be a nonzero ideal of R and R admits a generalized derivation F with associated nonzero derivation D. Then R contains a nonzero central ideal if it satisfies any one of the following properties:*

(i) $F(x) \circ D(y) = [x, y]$ *for all* $x, y \in I$ (ii) $D(x) \circ F(y) = F[x, y]$, *for all* $x, y \in I$.

Proof. **(i)** Given that

$$F(x)D(y) + D(y)F(x) = xy - yx, \quad for\ all\ x, y \in I. \tag{2.1}$$

Replace x by xy in (2.1), we have

$$F(x)yD(y) + xD(y)D(y) + D(y)F(x)y + D(y)xD(y) = xyy - yxy, \quad for\ all\ x, y \in I.$$

In the light of (2.1), above expression yields to

$$F(x)[y, D(y)] + (x \circ D(y))D(y) = 0, \quad for\ all\ x, y \in I. \tag{2.2}$$

Again replacing x by yx in (2.2) to obtain

$$yF(x)[y, D(y)] + D(y)x[y, D(y)] + y(x \circ D(y))D(y) + [y, D(y)]xD(y) = 0, \quad for\ all\ x, y \in I.$$

By using (2.2) & an application of Lemma B in above equation to get

$$[y, D(y)]xD(y) = 0, \quad for\ all\ x, y \in I. \tag{2.3}$$

i.e. $I[y, D(y)]RI[y, D(y)] = 0$, for all $x, y \in I$, then by semiprimeness of R, we find that $I[y, D(y)] = 0$, for all $x, y \in I$. This yields that $[y, r][y, D(y)] = 0$, for all $x, y \in I$, $r \in R$. Now, replacing r by $D(y)r$ to obtain

$$[y, D(y)]R[y, D(y)] = 0, \ for\ all\ x, y \in I.$$

Again by the semiprimeness of R we find that $[y, D(y)] = 0$, for all $y \in I$. Hence, by Lemma A we get required result.
(ii) By hypothesis, we have

$$F[x, y] = D(x)F(y) + F(y)D(x), \quad for\ all\ \ x, y \in I. \tag{2.4}$$

Taking yx instead of y in (2.4) to get

$$\begin{aligned} &F[x, y]x + [x, y]D(x) \\ =&D(x)F(y)x + D(x)yD(x) + F(y)xD(x) + yD(x)D(x), \quad for\ all\ \ x, y \in I. \end{aligned}$$

In view of (2.4), the above expression yields that

$$[x, y]D(x) = F(y)[x, D(x)] + (D(x) \circ y)D(x), \quad for\ all\ \ x, y \in I.$$

Replacing x by xy in the last relation and using it's application to find that

$$[x, D(x)]yD(x) + D(x)y[x, D(x)] = 0, \quad for\ all\ x, y \in I.$$

In the light of Lemma B, we have

$$[x, D(x)]yD(x) = 0, \quad \textit{for all } x, y \in I.$$

Using similar approach as used in the proof of theorem 2.1 (i), we find that $[x, D(x)] = 0$, for all $x \in I$. Thus by Lemma A, we get required result. □

Theorem 2.2. *Let R be a 2 torsion free semiprime ring, I be a nonzero ideal of R and R admits a generalized derivation F with associated nonzero derivation D such that $D(x) \circ F(y) = xy$, for all $x, y \in I$* or *$D(x) \circ F(y) + xy = 0$, for all $x, y \in I$. Then R contains a nonzero central ideal.*

Proof. By the given hypothesis, we have

$$D(x)F(yx) + F(yx)D(x) = xyx, \quad \textit{for all } x, y \in I.$$

i.e.

$$D(x)F(y)x + D(x)yD(x) + F(y)xD(x) + yD(x)D(x) = xyx, \quad \textit{for all } x, y \in I.$$

Using hypothesis in above relation to find

$$F(y)[x, D(x)] + (y \circ D(x))D(x) = 0, \quad \textit{for all } x, y \in I.$$

Taking xy instead of y in last relation and using it, we obtain

$$D(x)y[x, D(x)] + [x, D(x)]yD(x) = 0, \quad \textit{for all } x, y \in I.$$

In light of Lemma B, we have

$$D(x)y[x, D(x)] = 0, \quad \textit{for all } x, y \in I.$$

Using the same process as used in the proof of Theorem 2.1 (i), we get required result. On other hand, if $D(x) \circ F(y) + xy = 0$, for all $x, y \in I$, then using same techniques as above with necessary variations, we get required result. □

Theorem 2.3. *Let R be a 2 torsion free semiprime ring, I be a nonzero ideal of R. If R admits a generalized derivation F with associated nonzero derivation D such that $F(x \circ y) = [F(x), y] + [D(y), x]$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. For all $x, y \in I$, we have

$$F(x \circ y) = [F(x), y] + [D(y), x], \quad \textit{for all } x, y \in I. \tag{2.5}$$

Replacing y by yx in (2.5) and using (2.5), we find that

$$(x \circ y)D(x) = y[F(x), x] + y[D(x), x] + [y, x]D(x), \quad \textit{for all } x, y \in I.$$

Again taking yz instead of y in above relation and using it's application to get

$$2[x,y]zD(x) = 0, \quad \textit{for all} \quad x,y,z \in I.$$

Since R is 2-torsion free, we get

$$[x,y]zD(x) = 0, \quad \textit{for all} \quad x,y,z \in I.$$

Replacing y by $D(x)y$ in above relation to find

$$[x,D(x)]yzD(x) = 0, \quad \textit{for all} \quad x,y,z \in I. \tag{2.6}$$

Replacing z by zx in (2.6), we have

$$[x,D(x)]yzxD(x) = 0, \quad \textit{for all} \quad x,y,z \in I. \tag{2.7}$$

Right multiplication of (2.6) by x gives

$$[x,D(x)]yzD(x)x = 0, \quad \textit{for all} \quad x,y,z \in I. \tag{2.8}$$

Combining (2.7) & (2.8), we obtain

$$[x,D(x)]yz[x,D(x)] = 0, \quad \textit{for all} \quad x,y,z \in I, \ r \in R.$$

Take zr instead of z in last expression to obtain

$$[x,D(x)]yzR[x,D(x)]yz = 0, \quad \textit{for all} \quad x,y,z \in I. \tag{2.9}$$

In the light of semiprimeness of R, we have $[x,D(x)]yz = 0$, for all $x,y,z \in I$, this implies that $[x,D(x)]yR[x,D(x)]y = 0$, for all $x,y \in I$. Again by semiprimeness of R, we find $[x,D(x)]y = 0$, for all $x,y \in I$ and hence using similar method as in proof of theorem 2.1 (i), we find that $[x,D(x)] = 0$, for all $x \in I$. Now by Lemma A, we obtain the required result. □

Theorem 2.4. *Let R be a 2-torsion free semiprime ring, I be a nonzero ideal of R. If R admits a generalized derivation F with associated nonzero derivation D such that $F[x,y] = F(x) \circ y - D(y) \circ x$, for all $x,y \in I$, then R contains a nonzero central ideal.*

Proof. For all $x,y \in I$, we have

$$F[x,y] = F(x) \circ y - D(y) \circ x, \quad \textit{for all} \quad x,y \in I. \tag{2.10}$$

Taking yx instead of y in (2.10) and using it to obtain

$$2[x,y]D(x) = -y[F(x),x] - y(D(x) \circ x), \quad \textit{for all} \quad x,y \in I.$$

Again replace y by yz in last relation to get

$$2[x,y]zD(x) = 0, \ \textit{for all} \ x,y,z \in I.$$

Since R is 2 torsion free we get

$$[x,y]zD(x)=0, \quad for\ all \quad x,y,z\in I.$$

Putting $y=D(x)y$ in above expression to obtain $[x,D(x)]yzD(x)=0$, for all $x,y,z\in I$. Now by using the same technique as in Theorem 2.3, we get the required result. □

Following example shows that the restrictions imposed on the hypothesis of the various results are not superfluous.

Example 2.6 Let $R=\left\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a,b\in S\right\}$, where S is any ring and $I=\left\{\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} : b\in S\right\}$ be an ideal of R. Define $F:R\to R$ by $F\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ a+b & 0 \end{pmatrix}$. Then it easy to see that R is not semiprime and F is a generalized derivation with associated derivation D such that $D\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$. It is straight forward to see that F satisfies the properties: (i) $F(x)\circ D(y)=[x,y]$, (ii) $D(x)\circ F(y)=F[x,y]$, (iii) $D(x)\circ F(y)=xy$, (iv) $F(x\circ y)=[F(x),y]+[D(y),x]$, and (v) $F[x,y]=F(x)\circ y-D(y)\circ x$ for all $x,y\in I$. However, I is not central.

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