Internat. J. Math. & Math. Sci.
Vol. 1 (1978) 433-438

ON SEPARABLE EXTENSIONS OF GROUP RINGS AND QUATERNION RINGS

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(Received November 29, 1977)

ABSTRACT. The purposes of the present paper are (1) to give a necessary and sufficient condition for the uniqueness of the separable idempotent for a separable group ring extension RG (R may be a non-commutative ring), and (2) to give a full description of the set of separable idempotents for a quaternion ring extension RQ over a ring R, where Q are the usual quaternions i,j,k and multiplication and addition are defined as quaternion algebras over a field. We shall show that RG has a unique separable idempotent if and only if G is abelian, that there are more than one separable idempotents for a separable quaternion ring RQ, and that RQ is separable if and only if 2 is invertible in R.

KEY WORDS AND PHRASES. Group Rings, Idempotents in Rings, Separable Algebras

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. Major: 16A26 Minor: 16A16, 16A32

1. INTRODUCTION.

M. Auslander and O. Goldman ([1] and [2]) studied separable algebras over a commutative ring. Subsequently, the investigation of separable algebras (in particular, Brauer groups and Azumaya algebras) has attracted a lot of researchers, and rich results have been obtained (see References). K. Hirata and K. Sugano ([5]) generalized the concept of separable algebras to separable ring extensions; that is, let S be a subring of a ring T with the same identity. Then T is called a separable ring extension of S if there exists an element $\sum a_i \otimes b_i$ in $T \otimes_S T$ such that $x(\sum a_i \otimes b_i) = (\sum a_i \otimes b_i)x$ for each x in T and $\sum a_i b_i = 1$. Such an element $\sum a_i \otimes b_i$ is called a separable idempotent for T. We note that a separable idempotent takes an important role in many theorems (for example, see [6], Section 5,6, and 7). It is easy to verify that $(1/n)(\sum g_i \otimes g_i^{-1})$ and $\sum e_{i1} \otimes e_{1i}$ ([4], Examples II and III, P. 41) are separable idempotents for a group algebra RG and a matrix ring $M_m(R)$ respectively, where $G = \{g_1, \ldots, g_n\}$ with n invertible in R and e_{ij} are matrix units. We also note that the separable idempotent for a commutative separable algebra is unique ([6], Section 1, P. 722).

2. PRELIMINARIES.

Throughout, G is a group of order n, R is a ring with an identity 1. The group ring $RG = \{\sum r_i g_i \ / \ r_i$ in R and g_i in $G\}$, which is a free R-module with a basis $\{g_i\}$ and $(\sum r_i g_i)(\sum s_i g_i) = \sum t_k g_k$ where $t_k = \sum r_i s_j$ for all possible i,j such that $g_i g_j = g_k$. The ring R is imbedded in RG by $r \rightarrow r g_1$, where g_1 is the identity of G ($g_1 = 1$). The multiplication map $RG \otimes_R RG \rightarrow RG$ is denoted by π. Clearly, $\{g_i \otimes g_j \ / \ i,j =$

$1,\ldots,n\}$ form a basis for $RG\otimes_R RG$. An element $\sum r_{ij}(g_i\otimes g_j)$ in $RG\otimes_R RG$ is called a <u>commutant element</u> in $RG\otimes_R RG$ if $x(\sum r_{ij}(g_i\otimes g_j)) = (\sum r_{ij}(g_i\otimes g_j))x$ for all x in RG.

3. MAIN THEOREMS.

We begin with a representation for $\pi(x)$ for a commutant element x in $RG\otimes_R RG$, and then we show that RG has a unique separable idempotent if and only if G is abelian.

LEMMA 1. Let $x = \sum r_{ij}(g_i\otimes g_j)$, $i,j = 1,\ldots,n$, be a commutant element in $RG\otimes_R RG$. Then $\pi(x) = \sum_{i=1}^{m}(\sum r_{1k_i'})n_{k_i}C_{k_i}$, where m is the number of conjugate classes of G, n_{k_i} is the order of the normalizer of g_{k_i}, and C_{k_i} is the sum of different conjugate elements of g_{k_i}, for some k_i and k_i' in $\{1,\ldots,n\}$.

PROOF. Since x is a commutant element, $g_p x = x g_p$ for each g_p in G. The coefficient of the term $g_p\otimes g_k$ in $g_p x$ is r_{1k}, and the coefficient of the same term in xg_p is r_{pq}, where $g_q g_p = g_k$. Hence $r_{1k} = r_{pq}$ whenever $g_q g_p = g_k$. Thus $x = \sum_k r_{1k}(\sum g_p\otimes g_q)$, where p,q run over $1,\ldots,n$, such that $g_q g_p = g_k$; that is, $x = \sum_k r_{1k}(\sum_p g_p\otimes g_k g_p^{-1})$. Taking $\pi(x) = \sum_k r_{1k}(\sum_p g_p g_k g_p^{-1})$. For a fixed k, $\sum_p g_p g_k g_p^{-1} = n_k C_k$ where n_k is the order of the normalizer of g_k and C_k is the sum of all different conjugate elements of g_k. Hence $\pi(x) = \sum_{k=1}^{n} r_{1k} n_k C_k$. Since conjugate classes form a partition of G, $C_i = C_j$ if and only if g_i is conjugate to g_j. Renumerating elements, we let $\{g_{k_1},\ldots,g_{k_m}\}$ be all non-conjugate elements of each other; then $\{C_{k_1},\ldots,C_{k_m}\}$ are all different elements in the set, $\{C_1,\ldots,C_n\}$. Thus $\pi(x) = \sum_{i=1}^{m}(\sum r_{1k_i'})n_{k_i}C_{k_i}$, where $r_{1k'}$ are coefficients of the same C_{k_i}, and m is the number of conjugate classes of G.

THEOREM 2. Let RG be a separable extension of R. Then, RG has a unique separable idempotent if and only if G is abelian.

PROOF. Let $x = \sum r_{ij}(g_i \otimes g_j)$ be a separable idempotent for RG. Then by the lemma, $\pi(x) = \sum_{i=1}^{m}(\sum r_{1k_i'})n_{k_i}C_{k_i}$, where C_{k_i} is the sum of all conjugate elements of g_{k_i}. Let $g_{k_1} = 1$, the identity of G. Then $C_{k_1} = 1$ and $n_{k_1} = n$, the order of G. Since $\pi(x) = 1$, $(\sum r_{1k_1'})n_{k_1}C_{k_1} = 1$ and $(\sum r_{1k_1'})n_{k_1}C_{k_1} = 1$ and $(\sum r_{1k_i'})n_{k_i}C_{k_i} = 0$ for each $i \neq 1$. Noting that $C_{k_1} = 1$, we have $\sum r_{1k_1'} = r_{11}$, and so the first equation becomes $r_{11}n = 1$. Hence the order of G, n, is invertible in R. Thus n_{k_i}, being a factor of n, is also invertible in R. But conjugate classes form a partition of G, so $(\sum r_{1k_i'})n_{k_i}C_{k_i} = 0$ implies that $\sum r_{1k_i'} = 0$ for each $i \neq 1$. This system of homogeneous equations $\sum r_{1k_i'} = 0$ in the unknowns $r_{1k_i'}$ with $i \neq 1$ has trivial solutions if and only if $n = m$, and this holds if and only if G is abelian. Since the uniqueness of the separable idempotent $(= (1/n)(\sum g_i \otimes g_i^{-1}))$ is equivalent to the existence of trivial solutions of the above system of equations, the same fact is equivalent to G being abelian.

The theorem tells us that there are many separable idempotents for a separable group ring RG when G is non-abelian. Also, we remark that if RG is a separable extension of R, the order of G is invertible in R from the proof of the theorem. Next, we discuss another popular separable ring extension, a quaternion ring extension RQ, where RQ = $\{r_1+r_i i+r_j j+r_k k$ / i, j, and k are usual quaternions$\}$. (RQ,+.) is a ring extension of R under the usual addition and multiplication similar to quaternion algebras over a field. Now we characterize a separable idempotent for a separable quaternion ring extension RQ.

THEOREM 3. Let RQ be a separable quaternion ring extension. Then a commutant element $x = \sum r_{st}(s\otimes t)$, $s,t = 1,i,j,k$, in $RQ\otimes_R RQ$ is a separable idempotent for RQ if and only if $r_{11} = 1/4$.

PROOF. Since x is a commutant element in $RQ\otimes_R RQ$, $ix = xi$. The coefficients of the term $1\otimes 1$ on both sides are $-r_{i1}$ and $-r_{1i}$, so $r_{i1} = r_{1i}$. Since $jx = xj$, the coefficients of the term $k\otimes 1$ on both sides are $-r_{i1} = -r_{kj}$, so $r_{i1} = r_{kj}$. Also, $kx = xk$, so the coefficients of the term $j\otimes 1$ on both sides are $-r_{i1} = r_{jk}$. Hence $r_{1i} = r_{i1} = r_{kj} = -r_{jk}$. Similarly, by comparing coefficients of other terms, we have $r_{11} = -r_{ii} = -r_{jj} = -r_{kk}$, $r_{1j} = r_{j1} = -r_{ki} = r_{ik}$ and $r_{1k} = r_{k1} = -r_{ij} = r_{ji}$. In other words, $r_{st} = r_{pq}$ if $ts = qp$, and $r_{st} = -r_{pq}$ if $ts = -qp$. Thus

$$x = r_{11}(1\otimes 1-i\otimes i-j\otimes j-k\otimes k)+r_{1i}(1\otimes i+i\otimes 1-j\otimes k+k\otimes j)+r_{1j}(1\otimes j+j\otimes 1-k\otimes i+i\otimes k)+ r_{1k}(1\otimes k+k\otimes 1-i\otimes j+j\otimes i)\ldots\ldots(*)$$

But then $\pi(x) = r_{11}4+r_{1i}0+r_{1j}0+r_{1k}0 = 4r_{11}$. Consequently, x is a separable idempotent if and only if $r_{11} = 1/4$ (for $\pi(x) = 1$).

COROLLARY 4. Let RQ be a quaternion ring extension of R. Then RQ is separable if and only if 2 is invertible in R.

PROOF. The necessity is immediate from the theorem. The sufficiency is clear since the element x with $r_{11} = 1/4$, $r_{1i} = r_{1j} = r_{1k} = 0$ as given in (*) in Theorem 3 is a separable idempotent for RQ.

REMARK. It is easy to see that every x of the form (*) in Theorem 3 with r_{11}, r_{1i}, r_{1j} and r_{1k} in the center of R is a commutant element in $RQ\otimes_R RQ$. Hence, from the proof of Theorem 3, the complete set of commutant elements is: $C = \{\sum r_{st}(s\otimes t) \mid r_{st} = r_{pq}$ if $qp = ts$, and $r_{st} = -r_{pq}$ if $qp = -ts\}$. Also, the complete set of separable idempotents for

RQ is a subset of C such that $r_{11} = 1/4$ and r_{1i}, r_{1j}, r_{1k} are in the center of R. Thus there are many separable idempotents.

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