

ON SEPARABLE EXTENSIONS OF GROUP RINGS AND QUATERNION RINGS

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ABSTRACT. The purposes of the present paper are (1) to give a necessary and sufficient condition for the uniqueness of the separable idempotent for a separable group ring extension RG (R may be a non-commutative ring), and (2) to give a full description of the set of separable idempotents for a quaternion ring extension RQ over a ring R , where Q are the usual quaternions i, j, k and multiplication and addition are defined as quaternion algebras over a field. We shall show that RG has a unique separable idempotent if and only if G is abelian, that there are more than one separable idempotents for a separable quaternion ring RQ , and that RQ is separable if and only if 2 is invertible in R .

KEY WORDS AND PHRASES. Group Rings, Idempotents in Rings, Separable Algebras

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1. INTRODUCTION.

M. Auslander and O. Goldman ([1] and [2]) studied separable algebras over a commutative ring. Subsequently, the investigation of separable algebras (in particular, Brauer groups and Azumaya algebras) has attracted a lot of researchers, and rich results have been obtained (see References). K. Hirata and K. Sugano ([5]) generalized the concept of separable algebras to separable ring extensions; that is, let S be a subring of a ring T with the same identity. Then T is called a separable ring extension of S if there exists an element $\sum a_i \otimes b_i$ in $T \otimes_S T$ such that $x(\sum a_i \otimes b_i) = (\sum a_i \otimes b_i)x$ for each x in T and $\sum a_i b_i = 1$. Such an element $\sum a_i \otimes b_i$ is called a separable idempotent for T . We note that a separable idempotent takes an important role in many theorems (for example, see [6], Section 5, 6, and 7). It is easy to verify that $(1/n)(\sum g_i \otimes g_i^{-1})$ and $\sum e_{i1} \otimes e_{1i}$ ([4], Examples II and III, P. 41) are separable idempotents for a group algebra RG and a matrix ring $M_m(R)$ respectively, where $G = \{g_1, \dots, g_n\}$ with n invertible in R and e_{ij} are matrix units. We also note that the separable idempotent for a commutative separable algebra is unique ([6], Section 1, P. 722).

2. PRELIMINARIES.

Throughout, G is a group of order n , R is a ring with an identity

1. The group ring $RG = \{\sum r_i g_i / r_i \text{ in } R \text{ and } g_i \text{ in } G\}$, which is a free R -module with a basis $\{g_i\}$ and $(\sum r_i g_i)(\sum s_i g_i) = \sum t_k g_k$ where $t_k = \sum r_i s_j$ for all possible i, j such that $g_i g_j = g_k$. The ring R is imbedded in RG by $r \rightarrow r g_1$, where g_1 is the identity of G ($g_1 = 1$). The multiplication map $RG \otimes_R RG \rightarrow RG$ is denoted by π . Clearly, $\{g_i \otimes g_j / i, j =$

$1, \dots, n\}$ form a basis for $RG \otimes_R RG$. An element $\sum r_{ij}(\varepsilon_i \otimes \varepsilon_j)$ in $RG \otimes_R RG$ is called a commutant element in $RG \otimes_R RG$ if $x(\sum r_{ij}(\varepsilon_i \otimes \varepsilon_j)) = (\sum r_{ij}(\varepsilon_i \otimes \varepsilon_j))x$ for all x in RG .

3. MAIN THEOREMS.

We begin with a representation for $\pi(x)$ for a commutant element x in $RG \otimes_R RG$, and then we show that RG has a unique separable idempotent if and only if G is abelian.

LEMMA 1. Let $x = \sum r_{ij}(\varepsilon_i \otimes \varepsilon_j)$, $i, j = 1, \dots, n$, be a commutant element in $RG \otimes_R RG$. Then $\pi(x) = \sum_{i=1}^m (\sum r_{1k'_i}) n_{k'_i} C_{k'_i}$, where m is the number of conjugate classes of G , $n_{k'_i}$ is the order of the normalizer of $\varepsilon_{k'_i}$, and $C_{k'_i}$ is the sum of different conjugate elements of $\varepsilon_{k'_i}$, for some k'_i and k'_i in $\{1, \dots, n\}$.

PROOF. Since x is a commutant element, $\varepsilon_p x = x \varepsilon_p$ for each ε_p in G . The coefficient of the term $\varepsilon_p \otimes \varepsilon_k$ in $\varepsilon_p x$ is r_{1k} , and the coefficient of the same term in $x \varepsilon_p$ is r_{pq} , where $\varepsilon_q \varepsilon_p = \varepsilon_k$. Hence $r_{1k} = r_{pq}$ whenever $\varepsilon_q \varepsilon_p = \varepsilon_k$. Thus $x = \sum_k r_{1k} (\sum \varepsilon_p \otimes \varepsilon_q)$, where p, q run over $1, \dots, n$, such that $\varepsilon_q \varepsilon_p = \varepsilon_k$; that is, $x = \sum_k r_{1k} (\sum_p \varepsilon_p \otimes \varepsilon_k \varepsilon_p^{-1})$. Taking $\pi(x) = \sum_k r_{1k} (\sum_p \varepsilon_p \varepsilon_k \varepsilon_p^{-1})$. For a fixed k , $\sum_p \varepsilon_p \varepsilon_k \varepsilon_p^{-1} = n_k C_k$ where n_k is the order of the normalizer of ε_k and C_k is the sum of all different conjugate elements of ε_k . Hence $\pi(x) = \sum_{k=1}^n r_{1k} n_k C_k$. Since conjugate classes form a partition of G , $C_i = C_j$ if and only if ε_i is conjugate to ε_j . Renumerating elements, we let $\{\varepsilon_{k_1}, \dots, \varepsilon_{k_m}\}$ be all non-conjugate elements of each other; then $\{C_{k_1}, \dots, C_{k_m}\}$ are all different elements in the set, $\{C_1, \dots, C_n\}$. Thus $\pi(x) = \sum_{i=1}^m (\sum r_{1k'_i}) n_{k'_i} C_{k'_i}$, where $r_{1k'_i}$ are coefficients of the same $C_{k'_i}$, and m is the number of conjugate classes

of G .

THEOREM 2. Let RG be a separable extension of R . Then, RG has a unique separable idempotent if and only if G is abelian.

PROOF. Let $x = \sum r_{ij}(g_i \otimes g_j)$ be a separable idempotent for RG . Then by the lemma, $\pi(x) = \sum_{i=1}^m (\sum r_{1k_i'}) n_{k_i} C_{k_i}$, where C_{k_i} is the sum of all conjugate elements of g_{k_i} . Let $g_{k_1} = 1$, the identity of G . Then $C_{k_1} = 1$ and $n_{k_1} = n$, the order of G . Since $\pi(x) = 1$, $(\sum r_{1k_1'}) n_{k_1} C_{k_1} = 1$ and $(\sum r_{1k_1'}) n_{k_1} C_{k_1} = 1$ and $(\sum r_{1k_i'}) n_{k_i} C_{k_i} = 0$ for each $i \neq 1$. Noting that $C_{k_1} = 1$, we have $\sum r_{1k_1'} = r_{11}$, and so the first equation becomes $r_{11}n = 1$. Hence the order of G , n , is invertible in R . Thus n_{k_i} , being a factor of n , is also invertible in R . But conjugate classes form a partition of G , so $(\sum r_{1k_i'}) n_{k_i} C_{k_i} = 0$ implies that $\sum r_{1k_i'} = 0$ for each $i \neq 1$. This system of homogeneous equations $\sum r_{1k_i'} = 0$ in the unknowns $r_{1k_i'}$ with $i \neq 1$ has trivial solutions if and only if $n = m$, and this holds if and only if G is abelian. Since the uniqueness of the separable idempotent $(= (1/n)(\sum g_i \otimes g_i^{-1}))$ is equivalent to the existence of trivial solutions of the above system of equations, the same fact is equivalent to G being abelian.

The theorem tells us that there are many separable idempotents for a separable group ring RG when G is non-abelian. Also, we remark that if RG is a separable extension of R , the order of G is invertible in R from the proof of the theorem. Next, we discuss another popular separable ring extension, a quaternion ring extension RQ , where $RQ = \{r_1 + r_2i + r_3j + r_4k \mid i, j, \text{ and } k \text{ are usual quaternions}\}$. $(RQ, +, \cdot)$ is a ring extension of R under the usual addition and multiplication similar to quaternion algebras over a field. Now we characterize a separable idem-

potent for a separable quaternion ring extension RQ .

THEOREM 3. Let RQ be a separable quaternion ring extension. Then a commutant element $x = \sum r_{st}(s\otimes t)$, $s, t = 1, i, j, k$, in $RQ \otimes_R RQ$ is a separable idempotent for RQ if and only if $r_{11} = 1/4$.

PROOF. Since x is a commutant element in $RQ \otimes_R RQ$, $ix = xi$. The coefficients of the term $1\otimes 1$ on both sides are $-r_{i1}$ and $-r_{1i}$, so $r_{i1} = r_{1i}$. Since $jx = xj$, the coefficients of the term $k\otimes 1$ on both sides are $-r_{i1} = -r_{kj}$, so $r_{i1} = r_{kj}$. Also, $kx = xk$, so the coefficients of the term $j\otimes 1$ on both sides are $-r_{i1} = r_{jk}$. Hence $r_{1i} = r_{i1} = r_{kj} = -r_{jk}$. Similarly, by comparing coefficients of other terms, we have $r_{11} = -r_{ii} = -r_{jj} = -r_{kk}$, $r_{1j} = r_{j1} = -r_{ki} = r_{ik}$ and $r_{1k} = r_{k1} = -r_{ij} = r_{ji}$. In other words, $r_{st} = r_{pq}$ if $ts = qp$, and $r_{st} = -r_{pq}$ if $ts = -qp$. Thus

$$x = r_{11}(1\otimes 1 - i\otimes i - j\otimes j - k\otimes k) + r_{1i}(1\otimes i + i\otimes 1 - j\otimes k + k\otimes j) + r_{1j}(1\otimes j + j\otimes 1 - k\otimes i + i\otimes k) + r_{1k}(1\otimes k + k\otimes 1 - i\otimes j + j\otimes i) \dots \dots \dots (*)$$

But then $\mathcal{N}(x) = r_{11}^4 + r_{1i}^0 + r_{1j}^0 + r_{1k}^0 = 4r_{11}$. Consequently, x is a separable idempotent if and only if $r_{11} = 1/4$ (for $\mathcal{N}(x) = 1$).

COROLLARY 4. Let RQ be a quaternion ring extension of R . Then RQ is separable if and only if 2 is invertible in R .

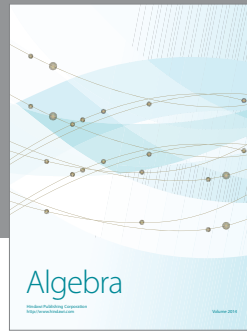
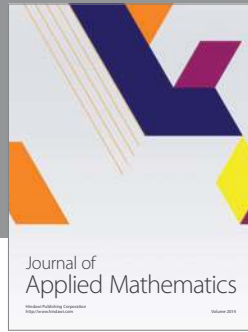
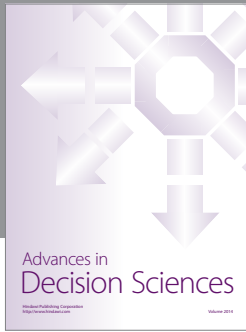
PROOF. The necessity is immediate from the theorem. The sufficiency is clear since the element x with $r_{11} = 1/4$, $r_{1i} = r_{1j} = r_{1k} = 0$ as given in (*) in Theorem 3 is a separable idempotent for RQ .

REMARK. It is easy to see that every x of the form (*) in Theorem 3 with r_{11} , r_{1i} , r_{1j} and r_{1k} in the center of R is a commutant element in $RQ \otimes_R RQ$. Hence, from the proof of Theorem 3, the complete set of commutant elements is: $C = \{ \sum r_{st}(s\otimes t) / r_{st} = r_{pq} \text{ if } qp = ts, \text{ and } r_{st} = -r_{pq} \text{ if } qp = -ts \}$. Also, the complete set of separable idempotents for

RQ is a subset of C such that $r_{11} = 1/4$ and r_{1i}, r_{1j}, r_{1k} are in the center of R . Thus there are many separable idempotents.

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