ON SEQUENCES GENERIC IN THE SENSE OF PRIKRY

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I establish here a criterion for a sequence of ordinals to be generic over a transitive model of ZFC with respect to a notation of forcing first considered by Prikry in his Doctoral dissertation [2]. In Section 0 I review some notation, in Section 1 I list some facts about measurable cardinals, and in Section 2, after giving Prikry's result, I state and prove mine.

Theorem 2.2 was proved during my brief stay at Monash University in Melbourne in June 1969. I thank Professor Crossley of that organisation for his hospitality. The paper was written in my sister's house in Pakistan.

0. Notation

In general I follow that of [1], but on Formalist grounds I use "$=_{df}$" to separate definiendum from definiens even where it is fashionable to write "$\Leftrightarrow_{df}$".

Let κ be an infinite initial ordinal. I use the letters $s, t, \cdots$ for finite subsets of κ, and $S, T, S', \cdots$ for infinite. 0 is the empty set and the first ordinal.

Definition 0.1. $|s| =_{df} \max\{\alpha + 1 \mid \alpha \in s\}$.
In particular, $|s| = 0$ iff $s = 0$; $s \neq 0 \rightarrow |s| = \beta + 1$, for some β.

Definition 0.2. s *in* $S =_{df} \exists{:}\alpha < \kappa \quad s = \alpha \cap S$
("s is an initial segment of S").

Definition 0.3. $S \subseteq_f{}' T =_{df} \exists s{:}$ *in* $S \quad S - |s| \subseteq T$
("S is, apart from finitely many elements, a subset of T").

Let $\underline{\underline{F}}$ be a set of infinite subsets of κ.

Definition 0.4. $P_{\underline{\underline{F}}} =_{df} \{\langle s, S\rangle \mid |s| \leqq \min S \wedge S \in \underline{\underline{F}}\}$.
I use letters $p, q \cdots$ for elements of $P_{\underline{\underline{F}}}$.

The following partial ordering will be important:

Definition 0.5 (Prikry). $\leqq \; =_{df} \{\langle\langle s, S\rangle, \langle t, T\rangle\rangle \mid S \subseteq T \wedge t \subseteq s \wedge s - t \subseteq T\}$.

DEFINITION 0.6. $\mathbb{P}_{\underline{\underline{F}}} =_{df} \langle P_{\underline{\underline{F}}}, \leqq \cap P^2_{\underline{\underline{F}}} \rangle$.

Let
$$\Delta \subseteq P_{\underline{\underline{F}}}.$$

DEFINITION 0.7. Δ is dense in $\mathbb{P}_{\underline{\underline{F}}} =_{df} \forall p{:}\in P_{\underline{\underline{F}}}\, \exists q{:}\in \Delta\ q \leqq p$.

DEFINITION 0.8. Δ is $\leqq$-closed in $\mathbb{P}_{\underline{\underline{F}}} =_{df} \forall p{:}\in \Delta\, \forall q{:}\in P_{\underline{\underline{F}}}\, (q \leqq p \rightarrow q \in \Delta)$.

Let M be a transitive ε-model of $ZF + AC$; let κ and $\underline{\underline{F}}$ be elements of M. Then $\mathbb{P}_{\underline{\underline{F}}} \in M$. In the sequel, M may be taken to be a set or a proper class: it is left to the reader to interpret the theorems and arguments as theorem and proof schemata of ZF when appropriate.

Let $a \subseteq \kappa$, a of order type ω.

DEFINITION 0.9. $F_a = \{\langle s, S\rangle \mid S \subseteq a \subseteq s \cup S \wedge \langle s, S\rangle \in \mathbb{P}_{\underline{\underline{F}}}\}$.

DEFINITION 0.10. a is $\mathbb{P}_{\underline{\underline{F}}}$-generic over $M =_{df}$

$$\begin{aligned} &\forall \Delta{:}\ \in M\ (\Delta \text{ dense and } \leqq\text{-closed} \rightarrow \Delta \cap F_a \neq 0) \wedge \\ &\forall p, q{:}\in F_a\, \exists q'{:}\in P_{\underline{\underline{F}}}(q' \leqq p \wedge q' \leqq q) \wedge \\ &\forall p{:}\ \in F_a\, \forall q{:}\in P_{\underline{\underline{F}}}(p \leqq q \rightarrow q \in F_a). \end{aligned}$$

REMARK. The above is equivalent in ZF to all other customary definitions of genericity with respect to a partial ordering and a model of ZF.

1. Measurable cardinals

DEFINITION 1.1. $\underline{\underline{U}}$ is a two-valued measure on $\kappa =_{df} \underline{\underline{U}}$ is a non-principal ultrafilter on κ and whenever $\lambda < \kappa$ and $\langle A_i \mid i < \lambda\rangle$ is a sequence of elements of $\underline{\underline{U}}$, $\bigcap_{i<\lambda} A_i \in \underline{\underline{U}}$.

DEFINITION 1.2. Let $A \subseteq \kappa$. $[A]^n =_{df} \{s \subseteq A \mid \bar{\bar{s}} = n\}$.

$$[A]^{<\omega} =_{df} \cup \{[A]^n \mid n < \omega\}.$$

Note that $0 \in [A]^{<\omega}$.

DEFINITION 1.3. $\underline{\underline{U}}$ is a normal measure on $\kappa =_{df} \underline{\underline{U}}$ is a two-valued measure on κ and whenever $\langle A_t \mid t \in [\kappa]^{<\omega}\rangle$ is a family of elements of $\underline{\underline{U}}$ indexed by the finite subsets of κ, there is a $B \in \underline{\underline{U}}$ such that

$$\forall t{:}\in [\kappa]^{<\omega}\ \ B - |t| \subseteq A_t.$$

The following lemma verifies that that definition is equivalent to the usual definitions of normal measure.

LEMMA 1.4. *Let U be a two-valued measure on κ. $\underline{\underline{U}}$ is normal if and only if for any sequence $\langle C_\alpha \mid \alpha < \kappa\rangle$ of elements of $\underline{\underline{U}}$ such that $\forall \alpha{:} < \kappa\ C_\alpha = \cap \{C_{\beta+1} \mid \beta < \alpha\}$, $\{\alpha \mid \alpha \in C_\alpha\} \in \underline{\underline{U}}$.*

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PROOF. Suppose $\underset{=}{U}$ normal and let $\langle C_\alpha \mid \alpha < \kappa\rangle$ be a sequence of elements of $\underset{=}{U}$ such that

$$\forall\alpha\colon < \kappa \; C_\alpha = \cap\{C_{\beta+1} \mid \beta < \alpha\}.$$

Let $A_s = \cap\{C_{\nu+1} \mid \nu \in s\}$ if $s \in [\kappa]^{<\omega}$ and $s \neq 0$, and let $A_0 = C_0$. Let $B \in \underset{=}{U}$ be such that $B - |s| \subseteq A_s$, and let $\nu \in B$. If $\nu = 0$, then $\nu \in A_0$ and so $\nu \in C_0$. If $\nu = \alpha + 1$, then $\nu \in A_{\{\alpha\}}$ and so $\nu \in C_{\alpha+1} = C_\nu$. If ν is a limit ordinal, let $\beta < \nu$: then $\nu \in A_{\{\beta\}}$ and so $\nu \in C_{\beta+1}$; so

$$\nu \in \cap\{C_{\beta+1} \mid \beta < \nu\}$$

which is C_ν. Thus $B \subseteq \{\nu \mid \nu \in C_\nu\}$ which is therefore in $\underset{=}{U}$.

Contrariwise, if $\langle A_s \mid s \in [\kappa]^{<\omega}\rangle$ is a family of elements of a two-valued measure $\underset{=}{U}$ which satisfies the hypothesis of the lemma on sequences $\langle C_\alpha \mid \alpha < \kappa\rangle$, define

$$C_\alpha = \cap\{A_s \,\big|\, |s| \leqq \alpha\}.$$

Then $\forall\alpha \; C_\alpha = \cap\{C_{\beta+1} \mid \beta < \alpha\}$ as $|s|$ is never a limit ordinal, and each C_α is in $\underset{=}{U}$ as

$$\alpha < \kappa \to \{s \,\big|\, |s| \leqq \alpha\}$$

has cardinality $< \kappa$, and so, writing $B = \{\alpha \mid \alpha \in C_\alpha\}$, $B \in \underset{=}{U}$. If $s = 0$, $B - 0 = B$, and

$$\forall\alpha\colon \in B \; \alpha \in C_0 = A_0,$$

so $B - |0| \subseteq A_0$. If $s \neq 0$, let $\alpha = \max s$. Let $\beta \in B - |s|$: then $\beta \in C \subseteq A_s$ as $|s| = \alpha + 1 \leqq \beta$; so $B - |s| \subseteq A_s$.

THEOREM 1.5 (Scott See for example Solovay [3].). *(ZF + AC) If κ has a two-valued measure, it has a normal measure.*

THEOREM 1.6 (Rowbottom). *(ZF + AC) Let $\underset{=}{U}$ be a normal measure on κ; let $\lambda < \kappa$ and $f\colon |\kappa]^{<\omega} \to \lambda$. Then $\exists A\colon \in \underset{=}{U} \; \forall n\colon < \omega \; \forall x, y\colon \in [A]^n \; f(x) = f(y)$.*

Such an A is said to be homogeneous for f.

I sketch a proof of Rowbottom's theorem. You show first by induction on n that

(†) $$\forall f'((f'\colon [K]^n \to \lambda) \to \exists A\colon \in \underset{=}{U} \; \forall x, y\colon \in [A]^n \; f'(x) = f'(y)).$$

For $n = 0$ (†) is trivial, and for $n = 1$ it follows from the property that

$$\forall\alpha\colon < \lambda \; C_\alpha \in \underset{=}{U} \to \cap\{C_\alpha \mid \alpha < \lambda\} \in \underset{=}{U}.$$

Suppose true for $n = k$, and let $f'\colon [\kappa]^{k+1} \to \lambda$. Then for each $s \in [\kappa]^k$ there is an $A_s \in \underset{=}{U}$ such that f' is constant on $\{s \cup \{\alpha\} \mid \alpha \in A_s\}$ and $A_s \subseteq \kappa - |s|$. Let $g(s)$ be that constant value of f'. Let $A_s = \kappa$ if $s \notin [\kappa]^k$. Let $B \in \underset{=}{U}$ be such that $\forall s \; B - |s| \subseteq A_s$. Let $C \in \underset{=}{U}$ be such that

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$$\forall s, t :\in [C]^k \; g(s) = g(t).$$

(Such a C exists by the induction hypothesis.) Let $A = B \cap C$. Then f' is constant on $[A]^{k+1}$.

To prove the theorem, pick for each n an $A^{(n)} \in \underset{=}{U}$ such that f is constant on $[A^{(n)}]^n$, and let

$$A = \cap \{A^{(n)} \mid n < \omega\}.$$

2. Prikry sequences

THEOREM 2.1 (Prikry). *Let M be a transitive model of $ZF + AC$; let $\kappa \in M$, and let $\underset{=}{U} \in M$ be in M a normal measure on κ. Let a be a subset of κ of order type ω, and suppose that a is $\mathbb{P}_{\underset{=}{U}}$ generic over M. Then every cardinal in M is a cardinal in $M[a]$; a is cofinal in κ, and so κ is of cofinality ω in $M[a]$; and if $\lambda < \kappa$, $b \subseteq \lambda$ and $b \in M[a]$, then $b \in M$.*

The principal result of the paper is now stated.

THEOREM 2.2. *Let M, κ, $\underset{=}{U}$ be as in 2.1. Let $a \subseteq \kappa$ be of order type ω. Then a is $\mathbb{P}_{\underset{=}{U}}$-generic over M if and only if*

$$\forall A :\in \underset{=}{U} \;\; a \subseteq_f A.$$

Here $a \subseteq_f A$ is as defined in 0.3.

COROLLARY 2.3. *If a is $\mathbb{P}_{\underset{=}{U}}$-generic over M, so is every infinite subset of a.*

The proof of Theorem 2.2 uses Theorem 1.6, as did Prikry's proof of 2.1. For the time being I argue in the theory $ZF + AC$ with the assumption that $\underset{=}{U}$ is a normal measure on κ.

DEFINITION 2.4. Let Δ be a dense, $\leqq$-closed subset of $P_{\underset{=}{U}}$. s a (finite) subset of κ. T captures $\langle s, \Delta \rangle =_{df} \; |s| \leqq \min T \wedge \exists n :< \omega (\forall t (t \in [T]^n \rightarrow \langle s \cup t, T - |t| \rangle \in \Delta))$.

LEMMA 2.5. $(ZF + AC)$ *Let Δ be a dense $\leqq$-closed subset of $P_{\underset{=}{U}}$.*

$$\forall s :\subseteq \kappa \; \exists T :\in \underset{=}{U} \; (T \text{ captures } \langle s, \Delta \rangle).$$

PROOF. Let Δ, s be given. To each $t \subseteq \kappa - |s|$ pick $A_t \in \underset{=}{U}$ such that

$$(\exists A :\in \underset{=}{U} \; \langle s \cup t, A \rangle \in \Delta) \rightarrow \langle s \cup t, A_t \rangle \in \Delta.$$

Let $A_t = \kappa$ if $t \nsubseteq \kappa - |s|$. By the normality of $\underset{=}{U}$ there is a $B' \in \underset{=}{U}$ such that

$$\forall t :\in [\kappa]^{<\omega} \; B' - |t| \subseteq A_t :$$

let $B = B' \cap (\kappa - |s|)$. Then $B \in \underset{=}{U}$ and

$$(*) \;\; \forall t :\subseteq B \; ((\exists A :\in \underset{=}{U} \; \langle s \cup t, A \rangle \in \Delta) \rightarrow \langle s \cup t, B - |t| > \in \Delta),$$

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for if $t \cup B$ and $\exists A : \in \underline{\underline{U}}\ \langle s \cup t, A \rangle \in \Delta$ then $\langle s \cup t, A_t \rangle \in \Delta$; $B - |t| \subseteq A_t$; and so $\langle s \times t, B - |t| \in \Delta$ as Δ is $\leqq$-closed.

Define a map $f : [\kappa]^{<\omega} \to 3$ by

$$f(t) = 0 \text{ if } t \nsubseteq B;$$

$$f(t) = 1 \text{ if } t \subseteq B \text{ and } \langle s \cup t, B - |t| \rangle \in \Delta;$$

$$f(t) = 2 \text{ if } t \subseteq B \text{ and } \langle s \cup t, B - |t| \rangle \notin \Delta.$$

Let $C \in \underline{\underline{U}}$ be homogeneous for f, and let $T = C \cap B$. Then $T \in \underline{\underline{U}}$.

As Δ is dense,

$$\exists t : \subseteq T\ \exists T' \subseteq T(|t| \leqq \min T' \text{ and } \langle s \cup t, T' \rangle \in \Delta).$$

Fix such a t. Let $n = \bar{\bar{t}}$. As $T \subseteq B$, by (*) $\langle s \cup t, B - |t| \rangle \in \Delta$, and so $f(t) = 1$. That T captures $\langle s, \Delta \rangle$ remains to be seen.

Let $t' \subseteq T$ and $\bar{\bar{t}}' = n$. As T is homogeneous for f, $f(t') = f(t) = 1$, so

$$\langle s \cup t', B - |t'| \rangle \in \Delta;$$

as Δ is $\leqq$-closed and $T - |t'| \subseteq B - |t'|$,

$$\langle s \cup t', T - |t'| \rangle \in \Delta.$$

Proof of Theorem 2.2. Suppose a $\mathbb{P}_{\underline{\underline{U}}}$-generic over M, and let $A \in \underline{\underline{U}}$. Let

$$\Delta = [\langle s, S \rangle | s \in \underline{\underline{U}} \wedge S \subseteq A\}.$$

Δ is dense, $\leqq$-closed and in M, so there is an

$$\langle s, S \rangle \in \Delta \cap F_a : s \subseteq a \subseteq s \cup S;$$

so $a \subseteq_f S \subseteq A$ and hence $a \subseteq_f A$.

Now suppose that $\forall A : \in \underline{\underline{U}}\ a \subseteq_f A$ and let

$$F_a = \{\langle s, S \rangle | S \in \underline{\underline{U}} \wedge s \subseteq a \subseteq s \cup S\},$$

as in Definition 0.9. It must now be shown that F_a has the three properties listed in Definition 0.10.

(iii) Let $\langle s, S \rangle \in F_a$, and $\langle s, S \rangle \leqq \langle s', S' \rangle \in P_{\underline{\underline{U}}}$. Then

$$s' \subseteq s \subseteq a \subseteq s \cup S \subseteq s' \cup S',$$

so $\langle s', S' \rangle \in F_a$.

(ii) Let $\langle s, S \rangle$ and $\langle s', S' \rangle \in F_a$. $s \cup s' \subseteq a$ and

$$a \subseteq (s \cup S) \cap (s' \cup S'),$$

so

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$\langle s \cup s', S \cap S' \rangle \leqq \langle s, S \rangle$, $\langle s \cup s', S \cap S' \rangle \leqq \langle s', S' \rangle$, and

$$\langle s \cup s', S \cap S' \rangle \in P_{\underset{=}{U}}.$$

(i) Let $\Delta \in M$, Δ dense and $\leqq$-closed. Working in M and using Lemma 2.5, pick for each $s \subseteq \kappa$ a $T_s \in \underset{=}{U}$ that captures $\langle s, \Delta \rangle$. There is a $B \in \underset{=}{U}$ such that $\forall s\, B - |s| \subseteq T_s$. $a \subseteq_f B$; so let s *in* a be such that $a - |s| \subseteq B$. Then $\bar{a} - |s| \subseteq T_s$; as T_s captures $\langle s, \Delta \rangle$, there is an n such that in M,

$$t \in |T_s]^n \rightarrow \langle s \cup t, T_s - |t| \rangle \in \Delta.$$

Let t' be the set of the first n elements of $a - |s|$. Then $\langle s \cup t', T_s - |t'| \rangle \in \Delta \cap F_a$.

Finally let me derive the lemma used by Prikry in his proof of Theorem 2.1 from Lemma 2.5, to which it is a kin.

LEMMA 2.6 (Prikry). *Let $\mathfrak{A}$ be a sentence of the language of forcing and* $\langle s, S \rangle \in P_{\underset{=}{U}}$. *Then*

$$\exists S' \subseteq S(S' \in \underset{=}{U} \wedge (\langle s, S' \rangle \Vdash \mathfrak{A} \vee \langle s, S' \rangle \Vdash \neg \mathfrak{A})).$$

PROOF. Let $\Delta = \{\langle t, T \rangle | \langle t, T \rangle \Vdash \mathfrak{A} \vee \langle t, T \rangle \Vdash \neg \mathfrak{A}\}$. As Δ is dense and $\leqq$-closed there are by Lemma 2.5 an $S'' \subseteq S$ and an $n \in \omega$ such that

$$\forall t : \in [S''|^n \; \langle s \cup t, S'' - |t| \rangle \in \Delta.$$

Define $f : [S'']^n \rightarrow 2$ by

$$f(t) = 0 \text{ if } \langle s \cup t, S'' - |t| \rangle \Vdash \mathfrak{A}$$
$$= 1 \text{ if } \langle s \cup t, S'' - |t| \rangle \Vdash \neg \mathfrak{A}.$$

Let $S' \subseteq S''$ be homogeneous for f. If neither $\langle s, S' \rangle \Vdash \mathfrak{A}$ nor $\langle s, S' \rangle \Vdash \neg \mathfrak{A}$, there are s', s'', T', $T'' \subseteq S'$ with

$$T', T'' \in \underset{=}{U}, \langle s \cup s', T' \rangle \Vdash \mathfrak{A}, \langle s \cup s'', T'' \rangle \Vdash \neg \mathfrak{A}$$

and, it may be assumed, $\min\{\bar{\bar{s}}', \bar{\bar{s}}''\} \geqq n$. Let t' be the first n element of s' and t'' of s''. Then $f(t') = 0$ and $f(t'') = 1$ (for S'' captures $\langle s, \Delta \rangle$), which contradicts the homogeneity of S'.

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