

1994

## On sequences with zero autocorrelation

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### Abstract

Normal sequences of lengths  $n = 18, 19$  are constructed. It is proved through an exhaustive search that normal sequences do not exist for  $n = 17, 21, 22, 23$ . Marc Gysin has shown that normal sequences do not exist for  $n = 24$ . So the first unsettled case is  $n = 27$ . Base sequences of lengths  $2n - 1, 2n - 1, n, n$  are constructed for all decompositions of  $6n - 2$  into four squares for  $n = 2, 4, 6, \dots, 20$  and some base sequences for  $n = 22, 24$  are also given. So T-sequences (T-matrices) of length 71 are constructed here for the first time. This gives new Hadamard matrices of orders 213, 781, 1349, 1491, 1633, 2059, 2627, 2769, 3479, 3763, 4331, 4899, 5467, 5609, 5893, 6177, 6461, 6603, 6887, 7739, 8023, 8591, 9159, 9443, 9727, 9869.

### Disciplines

Physical Sciences and Mathematics

### Publication Details

Christos Koukouvinos, Stratis Kounias, Jennifer Seberry, C. H. Yang, Joel Yang, On sequences with zero autocorrelation, *Designs, Codes and Cryptography*, 4, (1994), 327-340.

# On Sequences with Zero Autocorrelation

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**Editor:** D. Jungnickel

*Received May 3, 1993; Revised November 24, 1993*

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Base sequences of lengths  $2n-1, 2n-1, n, n$  are constructed for all decompositions of  $6n-2$  into four squares for  $n = 2, 4, 6, \dots, 20$  and some base sequences for  $n = 22, 24$  are also given. So T-sequences (T-matrices) of length 71 are constructed here for the first time. This gives new Hadamard matrices of orders 213, 781, 1349, 1491, 1633, 2059, 2627, 2769, 3479, 3763, 4331, 4899, 5467, 5609, 5893, 6177, 6461, 6603, 6887, 7739, 8023, 8591, 9159, 9443, 9727, 9869.

## 1. Introduction

Given the sequence  $A = \{a_1, a_2, \dots, a_n\}$  of length  $n$  the non-periodic autocorrelation function  $N_A(s)$  is defined as

$$N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, \quad s = 0, 1, \dots, n-1. \quad (1)$$

If  $A(z) = a_1 + a_2 z + \dots + a_n z^{n-1}$  is the associated polynomial of the sequence  $A$ , then

$$A(z)A(z^{-1}) = \sum_{i=1}^n \sum_{j=1}^n a_i a_j z^{i-j} = N_A(0) + \sum_{s=1}^{n-1} N_A(s)(z^s + z^{-s}), \quad z \neq 0. \quad (2)$$

If  $A^* = \{a_n, \dots, a_1\}$  is the reversed sequence, then

$$A^*(z) = z^{n-1} A(z^{-1}). \quad (3)$$

Base, Turyn, Golay and normal sequences are finite sequences, with zero autocorrelation function, useful in constructing orthogonal designs and Hadamard matrices [6], in communications engineering [18], in optics and signal transmission problems [7], [11], etc.

If  $A = \{a_1, \dots, a_n\}$ ,  $B = \{b_1, \dots, b_n\}$  are two binary  $(1, -1)$  sequences of length  $n$  and

$$N_A(s) + N_B(s) = 0 \quad \text{for } s = 1, \dots, n-1 \quad (4)$$

then  $A, B$  are called *Golay sequences* of length  $n$  (abbreviated  $GS(n)$ ). See [5], [6], [7].

From this definition and relation (2) we conclude that two  $(1, -1)$  sequences of length  $n$  are  $GS(n)$  if and only if

$$A(z)A(z^{-1}) + B(z)B(z^{-1}) = 2n, \quad z \neq 0. \quad (5)$$

Golay sequences  $GS(n)$  exist for  $n = 2^a 10^b 26^c$  (Golay numbers) where  $a, b, c$  are non-negative integers [5], [7], [8], [18].

The four sequences  $A, B, C, D$  of lengths  $n + p, n + p, n, n$  with entries  $+1, -1$  are called *base sequences* if:

$$\begin{aligned} N_A(s) + N_B(s) + N_C(s) + N_D(s) &= \begin{cases} 0, & s = 1, \dots, n-1 \\ 4n + 2p, & s = 0 \end{cases} \quad (6) \\ N_A(s) + N_B(s) &= 0, \quad s = n, \dots, n+p-1. \end{aligned}$$

Equivalently, (6) can be replaced by

$$A(z)A(z^{-1}) + B(z)B(z^{-1}) + C(z)C(z^{-1}) + D(z)D(z^{-1}) = 4n + 2p, \quad z \neq 0 \quad (7)$$

where  $A(z), B(z), C(z), D(z)$  are the associated polynomials. Base sequences of lengths  $n + 1, n + 1, n, n$  are denoted by  $BS(2n + 1)$ . From (7) and for  $p = 1$ , if we set  $z = 1$  we obtain

$$a^2 + b^2 + c^2 + d^2 = 4n + 2 \quad (8)$$

where  $a, b, c, d$  are the sum of the elements of  $A, B, C, D$  respectively.  $BS(2n + 1)$  for all decompositions of  $4n + 2$  into four squares for  $n = 1, 2, \dots, 24$  are given in [1], [12], [13]. Also  $BS(2n + 1)$  for  $n = 25, 26, 29$  and  $n = 2^a 10^b 26^c$  (Golay numbers) are given in Yang [23]. If the sequences  $A, B, C, D$  are  $BS(2n + 1)$  and satisfy

$$N_{AC}(s-1) + N_{CA}(s) = 0, \quad s = 1, \dots, n \quad (9)$$

then they are called *Turyn sequences* (abbreviated as  $TS(2n + 1)$ ), where

$$N_{AC}(s) = \sum_{i=1}^{n-s} a_i c_{i+s}, \quad N_{CA}(s) = \sum_{i=1}^{n+1-s} c_i a_{i+s}, \quad s = 0, 1, \dots, n \quad (10)$$

are the cross-correlations.

$TS(2n + 1)$  exist for  $n \leq 7, n = 12, 14$ . They cannot exist for  $n = 10, 11, 16, 17$  and for  $n = 8, 9, 13, 15$ ,  $TS(2n + 1)$  might exist but an exhaustive machine search showed that they do not exist (see [1], [6, pp. 142–143]). Koukouvinos, Kounias and Sotirakoglou [13] developed an algorithm and proved through an exhaustive search that Turyn sequences do not exist for  $n = 18, 19, \dots, 27$ . Edmondson [3], [4] has extended this further to show they do not exist for any new order  $\leq 42$ , so the first unsettled case is 43.

*6-base sequences*, denoted  $BS(n, n, n, n; n - 1, n - 1)$  of lengths  $n, n, n, n, n - 1, n - 1$  are six sequences of  $\pm 1$  with zero non-periodic autocorrelation function.

*Definition 1.* The four sequences  $X, Y, Z, W$  of length  $n$  with entries  $0, 1, -1$  are called *T-sequences* if

$$(i) \quad |x_i| + |y_i| + |z_i| + |w_i| = 1, \quad i = 1, \dots, n \quad (11)$$

$$(ii) \quad N_X(s) + N_Y(s) + N_Z(s) + N_W(s) = \begin{cases} 0, & s = 1, \dots, n - 1 \\ n, & s = 0 \end{cases}$$

Yang [22] gives another name for T-sequences and calls them four-symbol  $\delta$ -codes. He also calls the quadruple  $Q = X + Y, R = X - Y, S = Z + W, T = Z - W$  regular  $\delta$ -code of length  $n$ , where  $X, Y, Z, W$  are T-sequences of length  $n$ .

If Williamson type matrices of size  $w$  exist and T-sequences of length  $n$  exist, then Hadamard matrices of size  $4nw$  can be constructed (Cooper and (Seberry) Wallis [2]).

If  $X(z), Y(z), Z(z), W(z)$  are the associated polynomials, then from Definition 1 and (2) we see that (ii) can be replaced by

$$X(z)X(z^{-1}) + Y(z)Y(z^{-1}) + Z(z)Z(z^{-1}) + W(z)W(z^{-1}) = n, \quad z \neq 0. \quad (12)$$

We use the notation  $(A/B)$  for the sequence  $\{a_1, b_1, \dots, a_n, b_n\}$  and  $(A, B)$  for the sequence  $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ .

## 2. On Normal Sequences

*Definition 2.* A triple  $(F; G, H)$  of sequences is said to be a set of *normal sequences* for length  $n$  (abbreviated as  $NS(n)$ ) if the following conditions are satisfied.

$$(i) \quad F = (f_k) \text{ is a } (1, -1) \text{ sequence of length } n.$$

- (ii)  $G = (g_k)$  and  $H = (h_k)$  are sequences of length  $n$  with entries  $0, 1, -1$ , such that  $G + H = (g_k + h_k)$  is a  $(1, -1)$  sequence of length  $n$ .
- (iii)  $N_F(s) + N_G(s) + N_H(s) = 0, s = 1, \dots, n - 1$ .

Condition (iii) is also equivalent to

$$F(z)F(z^{-1}) + G(z)G(z^{-1}) + H(z)H(z^{-1}) = 2n, \quad z \neq 0. \tag{13}$$

In Theorem 1 we prove that the sequences  $G$  and  $H$  of Definition 2 are *quasi-symmetric*, i.e. if  $g_k = 0$ , then  $g_{n+1-k} = 0$  and also if  $h_k = 0$ , then  $h_{n+1-k} = 0$ . This theorem means that the definition of normal sequences in [23] can be relaxed.

**THEOREM 1** *If  $(F; G, H)$  are  $NS(n)$ , then*

$$\begin{aligned} g_s + g_{n-s+1} &\equiv 0 \pmod{2} \\ h_s + h_{n-s+1} &\equiv 0 \pmod{2} \end{aligned} \quad s = 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor$$

**Proof:** We have

$$\sum_{i=1}^{n-s} (f_i f_{i+s} + g_i g_{i+s} + h_i h_{i+s}) = 0, \quad s = 1, \dots, n - 1$$

but  $f_i = \pm 1$  and so  $f_i f_{i+s} = f_i + f_{i+s} - 1 + 4p, (p = 0, 1)$  but

$$\begin{aligned} 2(g_i g_{i+s} + h_i h_{i+s}) &= (g_i + h_i)(g_{i+s} + h_{i+s}) + (g_i - h_i)(g_{i+s} - h_{i+s}) \\ &= g_i + h_i + g_{i+s} + h_{i+s} + g_i - h_i + g_{i+s} - h_{i+s} - 2 + 4q, \\ &\quad (q = 0, 1) \\ &= 2(g_i + g_{i+s} - 1) + 4q. \end{aligned}$$

So we have

$$\begin{aligned} g_i g_{i+s} + h_i h_{i+s} &= g_i + g_{i+s} - 1 + 2q \equiv g_i + g_{i+s} - 1 \pmod{2} \\ &= h_i + h_{i+s} - 1 + 2r \equiv h_i + h_{i+s} - 1 \pmod{2} \end{aligned}$$

and so

$$\begin{aligned} f_i f_{i+s} + g_i g_{i+s} + h_i h_{i+s} &\equiv f_i + f_{i+s} + g_i + g_{i+s} - 2 \pmod{2} \\ &\equiv (f_i + f_{i+s} + g_i + g_{i+s}) \pmod{2} \end{aligned}$$

and

$$\sum_{i=1}^{n-s} f_i + \sum_{i=s+1}^n f_i + \sum_{i=1}^{n-s} g_i + \sum_{i=s+1}^n g_i \equiv 0 \pmod{2}, \quad s = 1, \dots, n - 1. \tag{14}$$

Similarly, if we set  $s - 1$  instead of  $s$ , we have

$$\sum_{i=1}^{n-s+1} f_i + \sum_{i=s}^n f_i + \sum_{i=1}^{n-s+1} g_i + \sum_{i=s}^n g_i \equiv 0 \pmod{2}, \quad s = 2, \dots, n. \quad (15)$$

From (14) and (15) ((15) - (14)) we obtain

$$f_{n-s+1} + f_s + g_{n-s+1} + g_s \equiv 0 \pmod{2}, \quad s = 2, \dots, n - 1. \quad (16)$$

The relation (16) is still valid and for  $s = n$ , because

$$f_1 f_n + g_1 g_n + h_1 h_n = 0$$

but  $f_s + f_{n-s+1} \equiv 0 \pmod{2}$ , and so from (16) we obtain

$$g_s + g_{n-s+1} \equiv 0 \pmod{2}, \quad s = 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor.$$

Similarly, using the same argument as before, we obtain

$$h_s + h_{n-s+1} \equiv 0 \pmod{2}, \quad s = 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor. \quad \blacksquare$$

In Theorem 2 we prove that if  $L = A + B$  is a sequence of real numbers with symmetric  $A$  and skew  $B$ , then

$$L(z)L(z^{-1}) = A(z)A(z^{-1}) + B(z)B(z^{-1}), \quad \text{for all } z \neq 0.$$

**THEOREM 2** *If  $A = \{a_1, \dots, a_n\}$ ,  $B = \{b_1, \dots, b_n\}$  are two sequences of length  $n$ , where  $n$  is even,  $A$  is symmetric i.e.  $a_i = a_{n+1-i}$ ,  $B$  is skew symmetric i.e.  $b_i = -b_{n+1-i}$  and  $L = \{l_1, \dots, l_n\}$ ,  $l_i = a_i + b_i$ , then*

$$L(z)L(z^{-1}) = A(z)A(z^{-1}) + B(z)B(z^{-1}), \quad z \neq 0.$$

**Proof:** Since  $A$  is symmetric and  $B$  is skew symmetric we have for  $n = 2m$ , writing  $X^*$  for the reverse of the sequence  $X$ ,

$$A = (A_1, A_1^*), \quad B = (B_1, -B_1^*),$$

where  $A_1 = \{a_1, \dots, a_m\}$ ,  $B_1 = \{b_1, \dots, b_m\}$ .

So we have

$$\begin{aligned} A(z) &= A_1(z) + z^m A_1^*(z) = A_1(z) + z^{2m-1} A_1(z^{-1}) \\ B(z) &= B_1(z) - z^m B_1^*(z) = B_1(z) - z^{2m-1} B_1(z^{-1}) \end{aligned}$$

and

$$\begin{aligned} A(z)B(z^{-1}) &= (A_1(z) + z^{2m-1} A_1(z^{-1})) (B_1(z^{-1}) - z^{-2m+1} B_1(z)) \\ &= A_1(z)B_1(z^{-1}) + z^{2m-1} A_1(z^{-1})B_1(z^{-1}) \\ &\quad - z^{-2m+1} A_1(z)B_1(z) - A_1(z^{-1})B_1(z) \\ &= -(A_1(z^{-1}) + z^{-2m+1} A_1(z)) (B_1(z) - z^{2m-1} B_1(z^{-1})) \\ &= -A(z^{-1})B(z) \end{aligned}$$

or

$$\begin{aligned} L(z)L(z^{-1}) &= (A(z) + B(z))(A(z^{-1}) + B(z^{-1})) \\ &= A(z)A(z^{-1}) + A(z)B(z^{-1}) + B(z)A(z^{-1}) + B(z)B(z^{-1}) \\ &= A(z)A(z^{-1}) + B(z)B(z^{-1}). \end{aligned} \quad \blacksquare$$

In Theorem 3 we prove that from a given pair of Golay sequences  $GS(n)$ :  $(F, H)$ , we can obtain two sets of normal sequences,  $(F; H, O_n)$  and  $(F; L, M)$ , where  $L$  and  $M$  are respectively the symmetric and skew parts of  $H$ . Therefore, a set of normal sequences can be regarded as a generalization of a pair of Golay sequences.

**THEOREM 3** *If  $F, H$  are Golay sequences of length  $n$ , then the following two sets*

- (i)  $(F; H, O_n)$ ,
- (ii)  $(F; L, M)$

*are  $NS(n)$ , where  $L$  and  $M$  are respectively the symmetric and skew parts of  $H$ .*

**Proof:**

- (i) The sequences  $F, H$  and  $O_n$  (a sequence of  $n$  zeros) satisfy the conditions (i) and (ii) of Definition 2. Also we have

$$\begin{aligned} F(z)F(z^{-1}) + H(z)H(z^{-1}) + O_n(z)O_n(z^{-1}) &= F(z)F(z^{-1}) + H(z)H(z^{-1}) \\ &= 2n. \end{aligned}$$

- (ii) 
$$\begin{aligned} L &= (l_k), \quad \text{where } l_k = \begin{cases} h_k & \text{if } h_k = h_{n+1-k} \\ 0 & \text{if } h_k = -h_{n+1-k} \end{cases} \\ M &= (m_k), \quad \text{where } m_k = \begin{cases} h_k & \text{if } h_k = -h_{n+1-k} \\ 0 & \text{if } h_k = h_{n+1-k} \end{cases} \end{aligned}$$

i.e.  $L$  and  $M$  are respectively the symmetric and skew parts of  $H$ . Then  $H(z) = L(z) + M(z)$ ,  $H(z)H(z^{-1}) = L(z)L(z^{-1}) + M(z)M(z^{-1})$  and so we have

$$\begin{aligned} F(z)F(z^{-1}) + L(z)L(z^{-1}) + M(z)M(z^{-1}) &= F(z)F(z^{-1}) + H(z)H(z^{-1}) \\ &= 2n. \end{aligned} \quad \blacksquare$$

In Theorem 4 we prove that we can obtain sets of normal sequences  $NS(2m + 1)$ :  $(A/C; B/O_m, O_{m+1}/D)$  from Turyn sequences  $TS(2m + 1)$ :  $(A, B; C, D)$  with lengths  $m + 1, m + 1, m, m$ . Where for even  $m$   $A$  is symmetric and  $C$  skew, and for odd  $m$   $A$  is skew and  $C$  symmetric.  $TS(2m + 1)$  can be written as:

- 1. For  $m$  odd

$$\begin{aligned} A &= \{A_1, -A_1^*\}, & B &= \{b_1, B_1, -B_1^*, b_1\}, \\ C &= \left\{C_1, c_{\frac{m+1}{2}}, C_1^*\right\}, & D &= \left\{D_1, d_{\frac{m+1}{2}}, D_1^*\right\} \end{aligned}$$



where

$$A_1 = \{a_1, \dots, a_{\frac{m+1}{2}}\}, B_1 = \{b_2, \dots, b_{\frac{m+1}{2}}\},$$

$$C_1 = \{c_1, \dots, c_{\frac{m-1}{2}}\}, D_1 = \{d_1, \dots, d_{\frac{m-1}{2}}\}$$

2. For  $m$  even

$$A = \{A_1, a_{\frac{m}{2}+1}, A_1^*\}, B = \{b_1, B_1, b_{\frac{m}{2}+1}, B_1^*, -b_1\},$$

$$C = \{C_1, -C_1^*\}, D = \{D_1, -D_1^*\}$$

where

$$A_1 = \{a_1, \dots, a_{\frac{m}{2}}\}, B_1 = \{b_2, \dots, b_{\frac{m}{2}}\},$$

$$C_1 = \{c_1, \dots, c_{\frac{m}{2}}\}, D_1 = \{d_1, \dots, d_{\frac{m}{2}}\}.$$

**THEOREM 4** *If  $A, B, C, D$  are Turyn sequences of lengths  $m+1, m+1, m, m$  ( $TS(2m+1)$ ), i.e.*

- (i)  $A(z)A(z^{-1}) + B(z)B(z^{-1}) + C(z)C(z^{-1}) + D(z)D(z^{-1}) = 4m + 2$
- (ii)  $A(z)C(z^{-1}) + zC(z)A(z^{-1}) = 0$

for all  $z \neq 0$ .

Then the sequences  $F = A/C, G = B/O_m, H = O_{m+1}/D$  are  $NS(2m+1)$ , where

$$A/C = \{a_1, c_1, a_2, c_2, \dots, a_m, c_m, a_{m+1}\}$$

$$B/O_m = \{b_1, 0, b_2, 0, \dots, b_m, 0, b_{m+1}\}$$

$$O_{m+1}/D = \{0, d_1, 0, d_2, \dots, 0, d_m, 0\}$$

**Proof:**

$$F = A/C, \quad \text{so } F(z) = A(z^2) + zC(z^2)$$

$$G = B/O_m, \quad \text{so } G(z) = B(z^2)$$

$$H = O_{m+1}/D, \quad \text{so } H(z) = zD(z^2).$$

So we have

$$F(z)F(z^{-1}) + G(z)G(z^{-1}) + H(z)H(z^{-1})$$

$$= A(z^2)A(z^{-2}) + C(z^2)C(z^{-2}) + B(z^2)B(z^{-2}) + D(z^2)D(z^{-2})$$

$$+ zC(z^2)A(z^{-2}) + z^{-1}A(z^2)C(z^{-2})$$

$$= 4m + 2. \quad \blacksquare$$

Normal sequences do not exist for  $n = 2^{2a-1}(8b+7)$ ,  $a, b$  non-negative integers as the numbers  $4^a(8b+7)$  cannot be written as the sum of three squares. In particular  $n \neq 14, 30, 46, 56, 62, 78, 94, \dots$  It is known that  $NS(6)$  does not exist. Yang [23] has

given these sequences for  $n = 3, 5, 7, 9, 11, 12, 13, 15, 25, 29$  and he notes they exist for  $n = 2^a 10^b 26^c$  (Golay numbers).

$NS(n)$  have not been found for  $n \geq 17$  (except  $n = 25, 29$  and  $g$  where  $g$  is a Golay number). This means we can construct T-sequences of length  $(2n + 1)t$ , if a new set of  $NS(n)$  can be found, where  $t = 2s + 1$  is the length of base sequences ( $BS(2s + 1)$ ).

In this paper we construct  $NS(n)$  for  $n = 18, 19$ . It is also proved through an exhaustive search that  $NS(n)$  do not exist for  $n = 17, 21, 22, 23$ . Marc Gysin [9], [10] has shown that normal sequences do not exist for  $n = 24$ . Hence the first unsettled case is  $n = 27$ .

Therefore we know these sequences

- (i) exist for  $n \in \{1, 2, \dots, 5, 7, 8, \dots, 13, 15, 16, 18, 19, 20, 25, 26, 29, 32, \dots\}$ ;
- (ii) do not exist for  $n \in \{6, 14, 17, 21, 22, 23, 24, 30, 46, 56, 62, 78, 94, \dots\}$ .
- (iii)  $n \in \{27, 28, 31, \dots\}$  are undecided.

We now formally prove Theorem 5:

**THEOREM 5** *If  $(F; G, H)$  are  $NS(n)$ , then the following  $(1, -1)$  sequences  $A = (F, 1)$ ,  $B = (F, -1)$ ,  $C = G + H$ ,  $D = G - H$  of lengths  $n + 1, n + 1, n, n$  respectively are  $BS(2n + 1)$  and vice versa.*

**Proof:** We have

$$\begin{aligned}
 A(z) &= F(z) + z^n \\
 B(z) &= F(z) - z^n \\
 C(z) &= G(z) + H(z) \\
 D(z) &= G(z) - H(z)
 \end{aligned} \tag{17}$$

From (17) we obtain

$$\begin{aligned}
 &A(z)A(z^{-1}) + B(z)B(z^{-1}) + C(z)C(z^{-1}) + D(z)D(z^{-1}) \\
 &= (F(z) + z^n)(F(z^{-1}) + z^{-n}) + (F(z) - z^n)(F(z^{-1}) - z^{-n}) \\
 &\quad + (G(z) + H(z))(G(z^{-1}) + H(z^{-1})) + (G(z) - H(z))(G(z^{-1}) - H(z^{-1})) \\
 &= 2(F(z)F(z^{-1}) + G(z)G(z^{-1}) + H(z)H(z^{-1})) + 2 \\
 &= 4n + 2,
 \end{aligned}$$

i.e.  $A, B, C, D$  are  $BS(2n + 1)$ .

The converse of the theorem is immediate from the first part and need not be proved separately. ■

We carried out a computer search by appropriately modifying the algorithm given in [13]. The original searches were carried out on an IBM 4361 running VM/CMS at the University of Thessalonica, Greece in 1989.

**Results:** With the application of our algorithm and using Theorem 5, the following normal sequences NS( $n$ ) have been found for  $n = 18$  and 19.

$$\begin{aligned} n = 18. \quad F &= (+ + + + + - - + + - - + - + + - + -) \\ G &= (+00 - 0 - 00 + +00 + 0 + 00+) \\ H &= (0 + +0 - 0 + +00 + -0 - 0 - +0) \end{aligned}$$

$$\begin{aligned} n = 19. \quad F &= (+ - - + + + - - - - + + + + + + - +) \\ G &= (00 - 0 + - + 0 + - + 0 + + - 0 + 00) \\ H &= (- + 0 - 000 + 000 + 000 + 0 - +) \end{aligned}$$

where + stands for 1 and - for -1.

We note that these normal sequences of lengths 18 and 19 are unique up to equivalence.

Also with the application of our algorithm it is proved through an exhaustive search that normal sequences do not exist for  $n = 17, 21, 22, 23$ .

### 3. On Base Sequences

We give a formal proof of a well-known result.

**THEOREM** (Turyn [18]) *If  $X, Y, Z, W$  are  $(1, -1)$  sequences of lengths  $n, n, n$  and  $n - 1$  respectively and satisfy the condition*

$$N_X(s) + N_Y(s) + 2N_Z(s) + 2N_W(s) = 0, \quad s \neq 0$$

*i.e. if the sequences  $X, Y, Z, Z; W, W$  are 6-base sequences  $(6-BS(n, n, n, n; n - 1, n - 1))$  then the following  $(1, -1)$  sequences*

$$A = (Z; W), \quad B = (Z; -W), \quad C = X, \quad D = Y,$$

*of lengths  $2n - 1, 2n - 1, n, n$  respectively are base sequences  $(BS(3n - 1))$ .*

**Proof:** We have

$$\begin{aligned} A(z) &= Z(z) + z^n W(z) \\ B(z) &= Z(z) - z^n W(z) \\ C(z) &= X(z) \\ D(z) &= Y(z) \end{aligned} \tag{18}$$

Since  $X, Y, Z, Z; W, W$  are 6-BS( $n, n, n, n; n - 1, n - 1$ ) we have

$$\begin{aligned} X(z)X(z^{-1}) + Y(z)Y(z^{-1}) + 2Z(z)Z(z^{-1}) + 2W(z)W(z^{-1}) &= 6n - 2, \\ z &\neq 0. \end{aligned} \tag{19}$$

From (18) we obtain

$$\begin{aligned}
 & A(z)A(z^{-1}) + B(z)B(z^{-1}) + C(z)C(z^{-1}) + D(z)D(z^{-1}) \\
 &= (Z(z) + z^n W(z)) (Z(z^{-1}) + z^{-n} W(z^{-1})) \\
 &\quad + (Z(z) - z^n W(z)) (Z(z^{-1}) - z^{-n} W(z^{-1})) \\
 &\quad + X(z)X(z^{-1}) + Y(z)Y(z^{-1}) \\
 &= X(z)X(z^{-1}) + Y(z)Y(z^{-1}) + 2Z(z)Z(z^{-1}) + 2W(z)W(z^{-1}) \\
 &= 6n - 2, \quad z \neq 0 \quad (\text{because of (19)}),
 \end{aligned}$$

i.e.  $A, B, C, D$  are BS( $3n - 1$ ). ■

*Remark 1.* We note that from the 6 - BS(24, 24, 24, 24; 23, 23)  $X, Y, Z, Z, W, W$ , T-sequences (T-matrices)  $T_1 = (Z, 0_{47})$ ,  $T_2 = (0_{24}, W, 0_{24})$ ,  $T_3 = (0_{47}, (X + Y)/2)$ , and  $T_4 = (0_{47}, (X - Y)/2)$ , of length 71 are constructed here for the first time ( $0_t$  is the sequence of  $t$  zeros). These give new Hadamard matrices of orders 213, 781, 1349, 1491, 1633, 2059, 2627, 2769, 3479, 3763, 4331, 4899, 5467, 5609, 5893, 6177, 6461, 6603, 6887, 7739, 8023, 8591, 9159, 9443, 9727, 9869 see also [15] and [16, p. 452, pp. 481–482]).

**THEOREM 7** *Suppose there are four  $(1, -1)$  sequences  $X, Y, Z, W$  of lengths  $n, n, n, n - 1$  respectively which satisfy the condition*

$$N_X(s) + N_Y(s) + 2N_Z(s) + 2N_W(s) = 0, \quad s \neq 0.$$

*Then  $n$  can only be even or 1. If  $n$  is even,  $x + y \equiv 2 \pmod{4}$  where  $x, y$  are the sum of the elements of sequences  $X$  and  $Y$  respectively. Further*

$$x_s x_{n-s+1} + y_s y_{n-s+1} = 0, \quad s = 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor.$$

**Proof:** Write  $X = \{x_i\}$ ,  $Y = \{y_i\}$ ,  $Z = \{z_i\}$ ,  $i = 1, 2, \dots, n$ ,  $W = \{w_j\}$ ,  $j = 1, 2, \dots, n - 1$ . We consider the autocorrelation function and use the fact that if  $x, y$  are  $\pm 1$ ,  $x + y \equiv xy + 1 \pmod{4}$ .  $N(n - 1) = x_1 x_n + y_1 y_n + 2z_1 z_n = 0$ . Hence

$$x_1 + y_1 + x_n + y_n - 2 + 2z_1 z_n \equiv 0 \pmod{4}$$

$$x_1 + y_1 + x_n + y_n \equiv 0 \pmod{4} \tag{20}$$

$$N(n - 2) = x_1 x_{n-1} + x_2 x_n + y_1 y_{n-1} + y_2 y_n + 2z_1 z_{n-1} + 2z_2 z_n + 2w_1 w_{n-1}$$

Hence

$$x_1 + y_1 + x_2 + y_2 + x_{n-1} + y_{n-1} + x_n + y_n \equiv 2 \pmod{4}$$

and using (20) we have

$$x_2 + y_2 + x_{n-1} + y_{n-1} \equiv 2 \pmod{4}. \quad (21)$$

$N(n-3)$  gives

$$x_1 + x_2 + x_3 + y_1 + y_2 + y_3 + x_{n-2} + x_{n-1} + x_n + y_{n-2} + y_{n-1} + y_n \equiv 0 \pmod{4}$$

and hence

$$x_3 + y_3 + x_{n-2} + y_{n-2} \equiv 2 \pmod{4}.$$

Assume that for  $t > 1$ ,  $t < \lfloor \frac{n}{2} \rfloor$

$$x_t + y_t + x_{n-t+1} + y_{n-t+1} \equiv 2 \pmod{4}.$$

Then noting  $N(n-t)$ , for  $t < \lfloor \frac{n}{2} \rfloor$ , is

$$\begin{aligned} x_1 + \cdots + x_t + y_1 + \cdots + y_t + x_{n-t+1} + \cdots + x_n + y_{n-t+1} + \cdots + y_n \\ \equiv 2(t-1) \pmod{4} \end{aligned}$$

we consider  $N(n-t-1)$  which gives, after noting that the  $Z$  and  $W$  sequences contribute  $2t+1$  terms each twice or a total of  $2 \pmod{4}$ ,

$$x_1 + \cdots + x_{t+1} + y_1 + \cdots + y_{t+1} + x_{n-t} + \cdots + x_n + y_{n-t} + \cdots + y_n \equiv 2t \pmod{4}.$$

Using the assumption we have

$$x_{t+1} + y_{t+1} + x_{n-t} + y_{n-t} \equiv 2 \pmod{4} \quad (22)$$

for  $t > 1$ ,  $t \leq \lfloor \frac{n}{2} \rfloor$ .

Now suppose  $n = 2k+1$  then  $N(k+1)$  where  $t = k$  is

$$\begin{aligned} x_1 + \cdots + x_k + y_1 + \cdots + y_k + x_{k+2} + \cdots + x_{2k+1} + y_{k+2} + \cdots + y_{2k+1} \\ \equiv 2(k-1) \pmod{4}. \end{aligned}$$

then, as before, we get

$$x_k + y_k + x_{k+2} + y_{k+2} \equiv 2 \pmod{4}.$$

However for  $n = 2k+1$  and  $t = k+1$  we have  $N(k)$  is

$$\begin{aligned} x_1 + \cdots + x_{k+1} + y_1 + \cdots + y_{k+1} + x_{k+1} + \cdots + x_{2k+1} + y_{k+1} + \cdots + y_{2k+1} \\ \equiv 2 \pmod{4} \end{aligned}$$

which gives, under the assumption,

$$x_{k+1} + y_{k+1} + x_{k+1} + y_{k+1} \equiv 2 \pmod{4},$$

which is impossible. So  $n$  is not odd and  $> 1$ .

For  $n = 2k$  we see that (22) means exactly three of  $x_{t+1}, y_{t+1}, x_{n-t}, y_{n-t}$  are one and one is minus one or exactly three are minus one and one is one, in either case we have, by induction,

$$x_s x_{n-s+1} + y_s y_{n-s+1} = 0, \quad s = 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor.$$

The remark that for  $n$  even,  $x + y \equiv 2 \pmod{4}$  where  $x, y$  are the sum of the elements of sequences  $X$  and  $Y$  respectively follows immediately from the equations (20) and (22). ■

*Remark 2.* BS( $3n - 1$ ) for all decompositions of  $2(3n - 1)$  into four squares for  $n = 2, 4, \dots, 20$  are given in Table 1. Some BS( $3n - 1$ ) for  $n = 22, 24$  are also given in Table 1. These results were found by computer search using an appropriately modified

*Table 1.* Sequences  $X, Y, Z, W$  which give base sequences BS( $3n - 1$ ) where  $A = Z, W, B = Z, -W, C = X, D = Y$  of lengths  $2n - 1, 2n - 1, n, n$ .

Length	Sums of Squares	Sequences
$n$	$2t = 2(3n - 1)$ $= a^2 + b^2 + c^2 + d^2$	
2	$0^2 + 0^2 + 1^2 + 3^2$	$X = Y = (+-), Z = (++) , W = (+)$
4	$0^2 + 2^2 + 3^2 + 3^2$	$X = (- + ++), Y = (- + - +), Z = (- + +-), W = (+ + +)$
6	$1^2 + 1^2 + 4^2 + 4^2$	$X = (+ + - + ++), Y = (+ - + + ++),$ $Z = (- - + - ++), W = (- + + + -)$
	$1^2 + 2^2 + 2^2 + 5^2$	$X = (+ - - + ++), Y = (+ - + + - +),$ $Z = (- + + + - +), W = (+ + + + -)$
8	$0^2 + 1^2 + 3^2 + 6^2$	$X = (- + + + + + + +), Y = (+ - - - + + + -),$ $Z = (- + - + + + + -), W = (+ - + + - - +)$
10	$1^2 + 2^2 + 2^2 + 7^2$	$X = (- - - - + + + + + +), Y = (- + - + + - + - + + +),$ $Z = (+ + + - - + + - + +), W = (- + + + - + - + +)$
	$0^2 + 0^2 + 3^2 + 7^2$	$X = (- - - + - + + + + -), Y = (+ - - - + + + - + +),$ $Z = (- + - - + + - + + +), W = (+ + - + - + + + +)$
12	$1^2 + 2^2 + 4^2 + 7^2$	$X = (- - + + + - + - + + + +),$ $Y = (- + + - - + + - + - + + +),$ $Z = (+ + + + - - + + - + - + +),$ $W = (- + + + + + + - - + -)$
	$2^2 + 4^2 + 5^2 + 5^2$	$X = (- - + + + + + - + - + + +),$ $Y = (- + + + - - + - - + + + +),$ $Z = (- - + - + + - + + + - -),$ $W = (+ + + + - + + + - + -)$
14	$1^2 + 4^2 + 4^2 + 7^2$	$X = (- - - + + + + + - + + + - + +),$ $Y = (- + + - - + - + + + + + - + +),$ $Z = (+ + + + - - - + + - + + - + +),$ $W = (+ + + - + + + - - + - + -)$
	$4^2 + 4^2 + 5^2 + 5^2$	$X = (- + + + - + - + + + - + + - + +),$ $Y = (+ + - - + + + + - + - + - + +),$ $Z = (- - - - + + + + - + + - + +),$ $W = (+ + + - + + + + - + - + -)$
	$1^2 + 3^2 + 6^2 + 6^2$	can be obtained by changing the signs of odd elements of each sequence in $2t = 82 = 1^2 + 4^2 + 4^2 + 7^2$

Table 1. Continued.

Length	Sums of Squares	Sequences
$n$	$2t = 2(3n - 1)$ $= a^2 + b^2 + c^2 + d^2$	
16	$2^2 + 4^2 + 5^2 + 7^2$	$X = (+ + - + + + + - + + - - - + -)$ , $Y = (- + - + - - - + + + - + + - +)$ , $Z = (+ - + + + - - - + + + + - +)$ , $W = (- + + + - + + + - - + - - + -)$
	$0^2 + 3^2 + 6^2 + 7^2$	can be obtained by changing the signs of odd elements of each sequence in
18	$1^2 + 1^2 + 2^2 + 10^2$	$2t = 94 = 2^2 + 4^2 + 5^2 + 7^2$ $X = (+ + + + - - + + + + + - + + + -)$ , $Y = (+ - + - + + - - - + + + - + - + -)$ , $Z = (- - - + - + + - + + - + - - + + + -)$ , $W = (+ - - - - - + + - + + + - + + + -)$
	$1^2 + 4^2 + 5^2 + 8^2$	can be obtained by changing the signs of odd elements of each sequence in
	$2^2 + 2^2 + 7^2 + 7^2$	$2t = 106 = 1^2 + 1^2 + 2^2 + 10^2$ $X = (+ + + + + - - - + - - + + + - - -)$ , $Y = (- - + - + + - - + + + + - - + + - +)$ , $Z = (- + - - + + + - + + - + + + - - -)$ , $W = (+ + + + - + + - + - + + - + - + -)$
20	$3^2 + 3^2 + 6^2 + 8^2$	$X = (+ + + - + + + + + - - + + - + - + -)$ , $Y = (- - - + + + - + + + + - + - + + + +)$ , $Z = (- + - + - + - - + - + + + + - - + + -)$ , $W = (+ + + + - - - + - - + + - + - - + +)$
	$0^2 + 1^2 + 6^2 + 9^2$	can be obtained by changing the signs of odd elements of each sequence in
22	$0^2 + 0^2 + 7^2 + 9^2$	$2t = 118 = 3^2 + 3^2 + 6^2 + 8^2$ $X = (- - - - + + + - + + - + - - + + + - + - -)$ , $Y = (- - - + - - - + + + - - + + + + + - - + -)$ , $Z = (- - + + - + + + + + - + + - + - + + - +)$ , $W = (+ + - + + - - - + - + - + - - + + + - -)$
24	$1^2 + 2^2 + 4^2 + 11^2$	$X = (+ + + + - - - + - + + - + + - + - + + - - -)$ , $Y = (+ + + + - + + - - + + + - + - - + - - - + + -)$ , $Z = (+ - - + + + - + + + - - - + + - + + - + + +)$ , $W = (+ + + + - + + - + + - + - + - - - - + +)$

algorithm from that described in [13]. The results for  $n = 4, 6, 8$  were first given in [18]. We recall that  $BS(3n - 1)$  can be used to construct  $T$ -sequences of lengths  $3n - 1$  and Hadamard matrices of order  $4(3n - 1)$ .

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