ON SEQUENTIALLY ADAPTIVE ASYMPTOTICALLY EFFICIENT RANK STATISTICS

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ABSTRACT

Adaptive rank statistics arising in the context of (asymptotically) efficient testing and estimation procedures for a regression (as well as two-sample location) parameter are considered. An orthonormal system based on the Legendre polynomials is incorporated in the adaptive determination of the score function, and the proposed sequential procedure is based on a suitable stopping rule. Various properties of the stopping rule and the sequentially adaptive procedure are studied. Asymptotic linearity results (in a regression parameter) of linear rank statistics are studied with reference to the Legendre polynomial system, and some improved bounds are derived in this context.

1. INTRODUCTION

Consider n independent random variables (r.v.) $X_1,\ldots,X_n$ with continuous distribution functions (d.f.) $F_1,\ldots,F_n$, respectively, all defined on the real line $E = (-\infty,\infty)$. It is assumed that

$$F_i(x) = F(x-\theta-\beta c_i) \ , \quad i = 1,\dots,n \ , \quad x \in E \ , \tag{1.1}$$

where θ and β are unknown (location and regression) parameters, F is of unspecified form, and the c_i are known (regression) constants, not all equal. For testing H_0: $\beta = 0$, against $\beta >$ (or $<$ or $\neq$) 0, typically, a rank test statistic is of the form:

$$S_n(\phi) = \Sigma_{i=1}^{n} c_{ni} a_n(R_{ni}) \ , \tag{1.2}$$

where R_{ni} = <u>rank</u> of X_i among $X_1,\dots,X_n$ $(i = 1,\dots,n)$, the <u>scores</u> $a_n(1),\dots,a_n(n)$ are (usually) defined by

$$a_n(i) = E\phi(U_{ni}) \quad \text{or} \quad \phi(EU_{ni}) \ (= \phi(\tfrac{i}{n+1})) \ , \quad 1 \leq i \leq n \ , \tag{1.3}$$

$\phi = \{\phi(u), 0<u<1\}$ is the <u>score generating function</u>, $U_{n1} < \dots < U_{nn}$ are the ordered r.v.'s of a sample of size n from the uniform (0,1) d.f., and

$$c_{ni} = (c_i-\bar{c}_n)/C_n \ , \quad 1 \leq i \leq n \ ; \quad \bar{c}_n = \tfrac{1}{n}\Sigma_{i=1}^{n} c_i \ , \quad C_n^2 = \Sigma_{i=1}^{n}(c_i-\bar{c}_n)^2 \ . \tag{1.4}$$

If F possesses an absolutely continuous probability density function (p.d.f.) f with a finite <u>Fisher information</u> $I(f) = \int(f'/f)^2 dF$ $(< \infty)$, where f' is the first derivative of f, then for a <u>local (contiguous) alternative</u> of the form

$$H_n: \ (1.1) \text{ holds with } \beta = C_n^{-1}\Delta \ , \quad \Delta \text{ fixed} \ , \tag{1.5}$$

the <u>asymptotic power</u> of the <u>one-sided size</u> α <u>test</u> $(0 < \alpha < 1)$ based on $S_n(\phi)$ is given by

$$1 - \Phi(\tau_\alpha - I^{\frac{1}{2}}(f)\rho(\phi,\phi_F)\Delta) \ , \tag{1.6}$$

where $\Phi(\cdot)$ is the standard normal d.f., $\Phi(\tau_\alpha) = 1 - \alpha$, and

$$\phi_F(u) = -f'(F^{-1}(u))/f(F^{-1}(u)) \ , \quad 0 < u < 1 \ , \tag{1.7}$$

$$\rho(\phi,\phi_F) = \langle\phi,\phi_F\rangle/\{\langle\phi,\phi\rangle\langle\phi_F,\phi_F\rangle\}^{\frac{1}{2}} \ , \tag{1.8}$$

F^{-1} is the <u>quantile function</u> corresponding to the d.f. F, $\langle a,b\rangle = \int_0^1 a(t)b(t)dt$, and it is assumed (without any loss of generality) that $\bar{\phi} = \langle\phi,1\rangle = 0$. Thus, for every $\Delta > 0$, the asymptotic power in (1.6) is maximized when $\rho(\phi,\phi_F) = 1$, i.e., $\phi = \phi_F$, so that ϕ_F is an <u>asymptotically optimal score function</u> and $S_n(\phi_F)$ is an <u>asymptotically optimal rank test statistic</u>. The same conclusion holds for the left-hand side ($\beta < 0$) and two-sided ($\beta \neq 0$) alternatives.

For the estimation of the regression parameter β, we may consider an <u>aligned rank statistic</u>

$$S_n(t,\phi) = \Sigma_{i=1}^n c_{ni} a_n(R_{ni}(t)) \ , \quad t \epsilon E \ , \tag{1.9}$$

defined as in (1.2)-(1.4), with $R_{ni}(t)$ = rank of $X_i - tc_i$ among $X_1 - tc_1, \ldots, X_n - tc_n$, for $i = 1,\ldots,n$. Then, for nondecreasing ϕ, $S_n(t,\phi)$ is nonincreasing (in t), and the so called <u>R-estimator</u> of β is defined by

$$\hat{\beta}_n(\phi) = \tfrac{1}{2}(\text{Sup}\{t: S_n(t,\phi) > 0\} + \inf\{t: S_n(t,\phi) < 0\}) \ . \tag{1.10}$$

$\hat{\beta}_n(\phi)$ is a robust, translation-invariant and consistent estimator of β, and, under quite general regularity conditions, as n increases,

$$C_n(\hat{\beta}_n(\phi) - \beta) \xrightarrow{\mathcal{D}} N(0, \{I(f)\rho^2(\phi,\phi_F)\}^{-1}) \ . \tag{1.11}$$

Thus, by reference to the Cramér-Rao information limit, we may conclude that $\hat{\beta}_n(\phi)$ is an asymptotically efficient estimator of β when $\rho(\phi,\phi_F) = 1$. Therefore, ϕ_F is asymptotically optimal for the estimation problem too.

procedure rests on a suitably posed stopping rule, and various properties of the stopping rule and the procedure are then studied in Section 3. Some additional results of general interest are presented in the concluding section.

2. SEQUENTIALLY ADAPTIVE ESTIMATOR OF ϕ_F

First, let us introduce the Legendre polynomials $\{P_k(u),\ u \in [0,1],\ k \geq 0\}$ which form a complete orthogonal system in $L_2(0,1)$. We define

$$P_k(u) = (2k+1)^{\frac{1}{2}}(-1)^k (k!)^{-1}((d^k/du^k)\{u(1-u)\}^k) \ , \quad u \in [0,1] \ , \quad k \geq 0 \ . \tag{2.1}$$

Note that $P_0 \equiv 1$, $P_1(u) = \sqrt{3}(2u-1)$, the standardized version of the classical Wilcoxon socre function (and is $\uparrow$ in u), $P_2(u) = \sqrt{5}(1-6u(1-u))$ $(= P_2(1-u))$, $P_3(u) = \sqrt{7}(20u^3-30u^2+12u-1)$ $(= -P_3(1-u))$, and so on. For $k \geq 1$, $P_k(u)$ is of bounded variation inside $[0,1]$, though not necessarily monotone. In general, we have for every $k \geq 0$, $u \in [0,1]$,

$$P_{2k}(u) = P_{2k}(1-u) \quad \text{and} \quad P_{2k+1}(u) = -P_{2k+1}(1-u) \quad . \tag{2.2}$$

Also, note that

$$\langle P_k, P_q \rangle = 0 \quad \text{for every} \quad k \neq q \ (\geq 0) \ , \quad \|P_k\|^2 = 1 \ , \quad \forall\ k \geq 0 \quad . \tag{2.3}$$

Further, by integration by parts,

$$\langle P_1, \phi_F \rangle = \sqrt{12} \int f^2(x)\,dx = \gamma_1 > 0 \quad . \tag{2.4}$$

Further, if ϕ_F is itself a polynomial of degree q (≥ 1), then it can be equivalently written in terms of $P_1, \ldots, P_q$, so that by (2.3),

$$\langle \phi_F, P_k \rangle = 0 \ , \quad \text{for every} \quad k \geq q+1 \quad . \tag{2.5}$$

We shall term (2.5) a finite-case. [For example, when F is a logistic d.f., $\phi_F \equiv P_1$, so that (2.5) holds with $q = 1$.] Moreover, if f is a symmetric d.f., ϕ_F is a skew-symmetric function (i.e., $\phi_F(u) + \phi_F(1-u) = 0,\ \forall\ 0 \le u < 1$), so that by (2.2),

$$\langle \phi_F, P_{2k} \rangle = 0\ ,\quad \forall\ k \ge 0\ . \tag{2.6}$$

We shall term (2.6) a <u>symmetric-case</u>. In general, we may not have a finite or symmetric representation. With respect to $\mathcal{P} = \{P_0, P_1, \ldots\}$, for $\phi_F \in L^2(0,1)$, as has been assumed, we have the <u>Fourier series</u>

$$\phi_F(u) \sim \Sigma_{k \ge 1} \gamma_k P_k(u)\ ,\quad u \in [0,1)\ ; \tag{2.7}$$

$$\gamma_k = \gamma_k(F) = \langle \phi_F, P_k \rangle\ ,\quad k \ge 1\ ;\quad \gamma_0 = 0\ . \tag{2.8}$$

[In the symmetric case, we have $\phi_F(u) \sim \Sigma_{k \ge 0} \gamma_{2k+1} P_{2k+1}(u)$, $0<u<1$.] Side by side, we define for every $m \ge 1$,

$$\phi_{F,m}(u) = \Sigma_{k \le m} \gamma_k P_k(u)\ . \tag{2.9}$$

Then, note that

$$I(f) = ||\phi_F||^2 = \Sigma_{k \ge 1} \gamma_k^2\ , \tag{2.10}$$

$$||\phi_F - \phi_{F,m}||^2 = \Sigma_{k>m} \gamma_k^2 \to 0 \quad \text{as} \quad m \to \infty\ . \tag{2.11}$$

Now the P_k are specified functions, while the γ_k are unknown. We proceed to estimate the γ_k from the sample, and then with a suitable choice of m in (2.9), we estimate ϕ_F by $\Sigma_{k \le \hat{m}} \hat{\gamma}_k P_k$, where $\hat{\gamma}_k$ and $\hat{m}$ are the estimates of γ_k and m, respectively. In this context, the asymptotic linearity results [on $S_n(t,\phi)$ in t] have been exploited fully. First, we note that the X_i are independent but not necessarily i.d. To estimate β in (1.1), we may note that $P_1(u)$ is $\uparrow$ in u, so that $S_n(t,P_1)$ is $\searrow$ in t ($\in E$). Let then

$$\tilde{\beta}_n = \tfrac{1}{2}(\sup\{t: S_n(t,P_1) > 0\} + \inf\{t: S_n(t;P_1) < 0\}) \tag{2.12}$$

be the R-estimator (based on Wilcoxon scores) of β in (1.1) and let C_n^2 be defined as in (1.4). For some (fixed) t (>0), define

In practice, F, and hence, ϕ_F are rarely known, while, in view of the asymptotically optimality results stated above, estimation of ϕ_F is of genuine interest. Note that $\langle\phi_F,\phi_F\rangle = I(f)$, so that

$$I(f) < \infty \implies \phi_F \in L_2 \quad . \tag{1.12}$$

Thus, one may naturally be interested in incorporating an orthonormal system $\{\phi_0,\phi_1,\phi_2,\ldots\}$ in a Fourier series representation of ϕ_F. If one assumes that ϕ_F is expressible as the difference of two nondecreasing, absolutely continuous and square integrable functions (inside $(0,1)$), then [viz., Hájek (1968)], for every $\varepsilon>0$, there exists a decomposition:

$$\phi_F(u) = \psi^0_{F,\varepsilon}(u) + \psi^{(1)}_{F,\varepsilon}(u) - \psi^{(2)}_{F,\varepsilon}(u) \ , \quad 0<u<1 \quad , \tag{1.13}$$

where $\psi^0_{F,\varepsilon}$ is a polynomial, and

$$\int_0^1 \{(\psi^{(1)}_{F,}(u))^2 + (\psi^{(2)}_{F,}(u))^2\}du < \varepsilon \quad . \tag{1.14}$$

Also, for a strongly unimodel p.d.f., $\log f(x)$ is convex, so that ϕ_F is $\nearrow$. Motivated by these, we are tempted to incorporate an orthogonal polynomial system, and the Legendre Polynomials provide the desired solution. A series representation of ϕ_F in terms of the Legendre polynomials $\{P_k(u),\ u \in [0,1];\ k\geq 0\}$ involves the series $\{\gamma_k = \langle\phi_F,P_k\rangle;\ k\geq 0\}$ of the Fourier coefficients. The estimation of these γ_k is greatly facilitated by making use of the Jurečková (1969) -linearity results on $S_n(t,\phi)$ (in t), as further extended by Ghosh and Sen (1972), Hušková and Jurečková (1981), Hušková (1982, 1983) and others. In the context of asymptotically efficient adaptive R-estimates of location, some alternative procedures are due to Beran (1974), Eplett (1982) and and others.

Along with the preliminary notions and basic regularity conditions, estimation of ϕ_F through the use of Legendre's polynomials is considered in Section 2. The proposed (sequential)

$$\hat{\gamma}_{n,k} = (2t)^{-1}\{S_n(\tilde{\beta}_n - tC_n^{-1}, P_k) - S_n(\tilde{\beta}_n + tC_n^{-1}, P_k)\} \quad , \quad k \geq 1 \quad . \quad (2.13)$$

By the basic results of Jurečková (1969), the $\hat{\gamma}_{n,k}$ are translation-invariant, consistent and robust estimators of the γ_k. We may remark here that for the testing problem (H_0: $\beta=0$ vs. $\beta \neq 0$), we do not need to estimate β, and hence, may consider an alternative (simpler) estimator:

$$\tilde{\gamma}_{n,k} = (2t)^{-1}\{S_n(-tC_n^{-1}, P_k) - S_n(tC_n^{-1}, P_k)\} \quad , \quad k \geq 1 \quad . \quad (2.14)$$

Note that under H_0 as well as $\{H_n\}$ in (1.5), the $\tilde{\gamma}_{n,k}$ are consistent estimators of the γ_k, though they are not translation-invariant. But, for (a fixed) $\beta \neq 0$, they may not perform that well. Hence, for the estimation problem, they are not recommended. We plan to incorporate these estimators ($\hat{\gamma}_{n,k}$ or $\tilde{\gamma}_{n,k}$) in (2.9) to estimate ϕ_F.

Note that in the symmetric case, $\hat{\gamma}_{n,2k}$ should estimate 0, for every $k \geq 1$. Moreover, even in the asymmetric case, the γ_k need not be $\searrow$ in k (≥ 1), though (2.11) ensures that the series $\Sigma\gamma_k^2$ converges, so that for every $\varepsilon > 0$, there exists an m ($= m(\varepsilon, F)$), such that

$$\Sigma_{k=m+1}^{m+r} \gamma_k^2 < \varepsilon \quad , \quad \forall \; r \geq 1 \quad . \quad (2.15)$$

Motivated by this, we choose a sequence $\{r_n\}$ of positive integers, such that $r_n \nearrow \infty$ but $n^{-1} r_n \searrow 0$ as $n \to \infty$, and another sequence $\{\varepsilon_n\}$ of positive numbers, such that $\varepsilon_n \searrow 0$ as $n \to \infty$, and let

$$K_n = \min\{k \geq K: \; \Sigma_{j=k+1}^{k+r_n} \hat{\gamma}_{n,j}^2 \leq \varepsilon_n\} \quad (2.16)$$

where K (≥ 2) is a prefixed positive integer. Typically, we choose r_n to be larger than 2. Then, our proposed (adaptive) estimator of ϕ_F is

$$\hat{\phi}_n(u) = \Sigma_{k \le K_n + r_n} \hat{\gamma}_{n,k} P_k(u) \quad , \quad u \in (0,1) \quad . \tag{2.17}$$

Also, the adaptive estimator of $I(f)$ is

$$\hat{I}_n = \Sigma_{k \le K_n + r_n} \hat{\gamma}^2_{n,k} \tag{2.18}$$

Note that by (2.7) and (2.17),

$$||\hat{\phi}_n - \phi_F||^2 = \int_0^1 \{\hat{\phi}_n(u) - \phi_F(u)\}^2 du$$

$$= \Sigma_{k \le K_n + r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2 + \Sigma_{k > K_n + r_n} \gamma_k^2 \quad , \tag{2.19}$$

where, by (2.15), as $K_n + r_n$ goes to ∞ (in probability or almost surely (a.s.)), the second term on the right hand side converges to 0 (in probability or a.s.). Hence, the stochastic behaviour of the stopping number $\{K_n\}$ and the mode of convergence of the $\hat{\gamma}_{n,k} - \gamma_k$ (to 0) play the vital role in our study. Theorems 3 and 4 (in Section 3) pertain to this study, while the main results are presented in Theorems 1 and 2 (in Section 3). In the remainder of this section, we present the basic regularity conditions.

Note that $I(f) < \infty$ ensures that $\int_{-\infty}^{\infty} |f'(x)| dx \le I^{\frac{1}{2}}(f) < \infty$. We strengthen this to:

$f(x)$ and $f'(x)$ are bounded almost everywhere (a.e.) (2.20)

Next, we note that by (2.1), there exists a positive and finite constant D (independent of k), such that for every $k \ge 1$,

$$||P_k^{(j)}||_\infty = \sup_{0 \le u \le 1} |(d^j/du^j) P_k(u)| \le D k^{2j+\frac{1}{2}} \quad , \tag{2.21}$$

for $j = 0,1,2,3,4$. Therefore, by (2.7), (2.9) and (2.21), for every $m \ge 1$,

$$||\phi_F - \phi_{F,m}||_\infty \le D\{\Sigma_{k>m} k^{\frac{1}{2}} |\gamma_k|\} \quad , \tag{2.22}$$

so that whenever the series $\Sigma_{k \ge 1} k^{\frac{1}{2}} |\gamma_k|$ converges, the right hand side of (2.22) converges to 0 as $m \to \infty$; however, $\Sigma \gamma_k^2 < \infty$ does not necessarily imply that $\Sigma_{k\geq 1} k^{\frac{1}{2}} |\gamma_k| < \infty$, and, in fact, when ϕ_F is itself not a bounded function on $(0,1)$, this "max-norm" convergence may not hold. For our subsequent analysis, we do not need this stronger mode of convergence (i.e., $||\hat{\phi}_n - \phi_F||_\infty \to 0$ in probability or a.s., as $n\to\infty$), and hence, the condition that $\Sigma_{k\geq 1} k^{\frac{1}{2}} |\gamma_k| < \infty$ is also not needed for our study. However, we may note that whenever the γ_k^2 are dominated by a monotone sequence $\{\gamma_k^{*2}\}$, such that $\Sigma\gamma_k^{*2} < \infty$, then $k\gamma_k^{*2}$ (and hence, $k\gamma_k^2$) converge to 0 as k increases. Having this in mind, we use a variant form of (2.11) as a part of our assumptions:

$$k\gamma_k^2 \text{ converges to } 0 \text{ as } k\to\infty \quad . \tag{2.23}$$

[It is possible to obtain better results if we assume that $k^{\ell}\gamma_k^2 \to 0$ as $k\to\infty$, for some $\ell \geq 1$. We shall make comments on it in Section 4.] Regarding the regression constants $\{c_i\}$, following Hájek (1968), we assume that as $n\to\infty$,

$$n^{-1}C_n^2 \to C_0^2 \text{ where } 0<C_0<\infty \quad ; \tag{2.24}$$

$$\max_{1\leq k\leq n}\{C_0^{-2}(c_k-\bar{c}_n)^2\} = 0(1) \quad . \tag{2.25}$$

Finally, in (2.16), regarding the choice of $\{\varepsilon_n\}$ and $\{r_n\}$, we will consider the various possibilities:

(i) $\varepsilon_n = \varepsilon\ (>0)$ and/or $r_n = r\ (\geq 2)$, for every $n\ (\geq n_0)$, (2.26)

(ii) $\varepsilon_n \searrow 0$ and $r_n \nearrow \infty$ with $n\to\infty$. (2.27)

We shall elaborate on this later on (in Section 4).

3. THE MAIN THEOREMS

For every positive integer a, let

$$I(f,a) = \Sigma_{k\leq a}\gamma_k^2 \quad , \tag{3.1}$$

so that $I(f,a) \nearrow I(f)$ as $a\nearrow\infty$. Defining r_n, ε_n as in (2.16), we conceive of a sequence $\{a_n\}$ of positive integers, such that the following conditions hold:

$$\lim_{n\to\infty} n^{-1}(\log n)^2(r_n+a_n)^7 = 0 \quad ; \tag{3.2}$$

$$\lim_{n\to\infty} n^{-1}(\log n)^3(r_n+a_n)^{15/2} = 0 \quad ; \tag{3.3}$$

$$\lim_{n\to\infty}(r_n/\varepsilon_n a_n) = 0 \quad \text{and} \quad \lim_{n\to\infty}(a_n^{13} r_n(\log n)^4/n^2\varepsilon_n) = 0 \quad . \tag{3.4}$$

Note that (3.3) implies (3.2), and (3.4) ensures that

$$\lim_{n\to\infty}\{(r_n/\varepsilon_n a_n)(n(\log n)^2)^{-1/7}\} = 0 \quad . \tag{3.5}$$

We shall make some comments on the choice of $\{\varepsilon_n, r_n, a_n\}$ later on.

Theorem 1. Assume that the d.f. F in (1.1) has a density f satisfying (2.20). Then, under (2.27) and (3.2),

$$I(f, K+r_n) \le \hat{I}_n \le I(f, a_n+r_n) \quad \text{a.s., as} \quad n\to\infty \quad , \tag{3.6}$$

and hence,

$$\lim_{n\to\infty} \hat{I}_n = I(f) \quad \text{a.s., as} \quad n\to\infty \quad . \tag{3.7}$$

If, however, ε_n $(=\varepsilon>0)$ and r_n $(=r)$ are held fixed, then (3.7) may not hold.

Theorem 2. For the model (1.1), assume that (2.20) and $\{H_n\}$ in (1.5) hold. Then, under (2.23), (3.3) and (3.4),

$$|S_n(\hat{\phi}_n) - S_n(\phi_{F,a_n+r_n})| \to 0 \quad , \quad \text{in probability, as} \quad n\to\infty \quad , \tag{3.8}$$

and, for ϕ_{F,a_n+r_n}, the asymptotic efficiency results of Section 1 hold. Thus, $\hat{\phi}_n$ is an asymptotically efficient score function, in probability.

For the stopping variable $\{K_n\}$ in (2.16), we have the following.

Theorem 3. If, in (2.16), $\varepsilon_n = \varepsilon$ (>0), for all n, and either r_n $(=r)$ is any (fixed) positive number for every n, or $\{r_n\}$ satisfies

$$\lim_{n\to\infty}\{r_n^7(\log n)^2/n\} = 0 \quad , \tag{3.9}$$

then, for every $\varepsilon>0$, there exists a positive integer k_ε such that

$$K_n \le k_\varepsilon \quad \text{a.s., as} \quad n\to\infty \quad . \tag{3.10}$$

In the finite case, i.e., under (2.5), when $\Sigma_{k\ge m}\gamma_k^2 = 0$, $\forall\ m\ge q+1$, if $\{\varepsilon_n\}$, $\{r_n\}$ satisfy the following: for some $\eta>0$,

$$\lim_{n\to\infty}\{\varepsilon_n^{-1}(\max\{r_n^6 n^{-1+\eta},\ r_n^{14} n^{-2}(\log n)^4\}) = 0 \quad , \tag{3.11}$$

then

$$K_n \le q+1 \quad \text{a.s., as} \quad n\to\infty \quad . \tag{3.12}$$

In the general case, if (2.23), (3.2) and (3.4) hold, then

$$K_n \le a_n \quad \text{a.s., as} \quad n\to\infty \quad . \tag{3.13}$$

If, in addition, there exists a sequence $\{r_n^0\}$ of positive integers, such that

$$\Sigma_{k=r+1}^{r+r_n}\gamma_k^2 \ge 2\varepsilon_n \ , \quad \forall\ r \le r_n^0 \quad , \tag{3.14}$$

then, under the hypothesis of Theorem 1,

$$K_n > r_n^0 \quad \text{a.s., as} \quad n\to\infty \quad . \tag{3.15}$$

The proofs of these theorems rest heavily on the asymptotic behaviour of $\{\hat{\gamma}_{n,k};\ k\ge 1\}$, and towards this, the following theorem is of prime interest.

Theorem 4. Let $\{X_i;\ i\ge 1\}$ be i.i.d.r.v. with a p.d.f. f satisfying (2.20). Then, for every $s>0$, $\delta>0$ and $a>0$, there exist a $d>0$ and a positive integer n_s, such that for every $n\ge n_s$,

$$P\left\{\sup_{\substack{|t_j|\le a\log n\\ j=1,2}}\left|S_n(C_n^{-1}t_1,P_k) - S_n(C_n^{-1}t_2,P_k) + (t_1-t_2)\gamma_k\right| > du_{n,k}\right\} \le n^{-s} \ , \tag{3.16}$$

for every $k \leq k_n = o\,(n^{2/13}(\log n)^{-2})$, *where*

$$u_{n,k} = \max\{k^{5/2}n^{-\frac{1}{2}+\delta},\ k^{13/2}n^{-1}(\log n)^{3/2}\}\ ,\quad k\geq 1\quad . \qquad (3.17)$$

The proof of this theorem is presented in the Appendix.

We may note that $\tilde{\beta}_n$ in (2.12) is based on $S_n(t,P_1)$. Thus, proceedings as in (10.3.60) through (10.3.65) of Sen (1981), we obtain that for every $s>0$, there exist positive constants c_{s1}, c_{s2} and a positive integer n_s, such that for every $n\geq s_s$,

$$P\{C_n|\tilde{\beta}_n-\beta| > c_{s1}(\log n)\} \leq c_{s2}n^{-s}\quad . \qquad (3.18)$$

From (2.13), (3.16), (3.17) and (3.18), we obtain that for every $k\leq k_n = o(n^{2/13}(\log n)^{-2})$ and $n\geq n_s$,

$$P\{|\hat{\gamma}_{n,k}-\gamma_k| > (2a)^{-1}du_{n,k}\} \leq c^*_{s2}n^{-s}\quad , \qquad (3.19)$$

where $c^*_{s2} = 1+c_{s2}$. The same inequality applies to the $\tilde{\gamma}_{n,k}$ in (2.14) when H_0 (or $\{H_n\}$) holds. In the sequel, we take $s>2$.

Now, returning to the proofs of the Theorems (1, 2 and 3), we note that by virtue of (2.16), for every $m\geq K$,

$$\{K_n>m\} = \{\Sigma_{j=k+1}^{k+r_n}\hat{\gamma}^2_{n,j} > \varepsilon_n,\ \forall K\leq k\leq m\}\quad . \qquad (3.20)$$

Also, note that for every positive integers a,b

$$|\Sigma_{j=a}^{a+b}(\hat{\gamma}^2_{n,j}-\gamma^2_j)| \leq 2\{\Sigma_{j=a}^{a+b}(\hat{\gamma}_{n,j}-\gamma_j)^2 + \Sigma_{j=a}^{a+b}\gamma_j^2\}\quad . \qquad (3.21)$$

With these, we consider first (3.13). Since by (3.4) and (3.12), $r_n = o(a_n) = o(n^{1/7})$, by letting $s' = s-1/7$ $(>13/7)$, we obtain from (2.23), (3.19) and (3.20) that for $a = a_n+1$ and $b = r_n$, the right hand side (rhs) of (3.21) is bounded from above, with probability greater than $1 - c^*_{s2}n^{-s'}$, by

$$2\{\Sigma_{j=a_n+1}^{a_n+r_n}\gamma_j^2 + \Sigma_{j=a_n+1}^{a_n+r_n}O(u^2_{n,j})\}$$

$$= 2\Sigma_{j=a_n+1}^{a_n+r_n}\{o(a_n^{-1}) + O(j^5n^{-1+2\delta}) + O(n^{-2}(\log n)^3j^{13}\}$$

$$= o(r_n/a_n) + [O(n^{-1+2\delta})][O(a_n^5 r_n)] + [O(n^{-2}(\log n)^3)][O(a_n^{13} r_n)]$$

$$= o(\varepsilon_n) + [o(\varepsilon_n)][O(n^{-1+2\delta} a_n^6)] + [o(\varepsilon_n)][O(n^{-2}(\log n)^3 a_n^{14})]$$

$$= [o(\varepsilon_n)]\{1 + O(n^{-1+2\delta} a_n^6) + O(n^{-2}(\log n)^3 a_n^{14})\} \quad , \tag{3.22}$$

as by (3.4), $(r_n/\varepsilon_n a_n) \to 0$ as $n\to\infty$. By (3.2) and (3.4), $n^{-2}(\log n)^3 a_n^{14} \to 0$ as $n\to\infty$ and $n^{-1}(\log n) a_n^7 \to 0$ as $n\to\infty$. Thus, choosing δ (>0) adequately small, we conclude that $n^{-1+2\delta} a_n^6 \to 0$ as $n\to\infty$. Hence, the rhs of (3.22) is $o(\varepsilon_n)$ as $n\to\infty$. Since, by (2.23), as $n\to\infty$, $\Sigma_{j=a_n+1}^{a_n+r_n} \gamma_j^2 = o(r_n/a_n) = o(\varepsilon_n)$, we have

$$P\{K_n > a_n \text{ for some } n \geq n_0\}$$

$$\leq \Sigma_{n\geq n_0} P\{K_n > a_n\}$$

$$\leq \Sigma_{n\geq n_0} P\{\Sigma_{j=a_n+1}^{a_n+r_n} \hat{\gamma}_{n,j}^2 > \varepsilon_n\} \qquad \text{[by (3.20)]}$$

$$\leq \Sigma_{n\geq n_0} \{O(n^{-s'})\}$$

$$= O(n_0^{-s'+1}) \to 0 \ , \quad \text{as } n_0\to\infty \ , \tag{3.23}$$

as $s' - 1 > 3/7$. Hence, (3.13) holds.

The proof of (3.11) is quite analogous. Note that under (2.5), $\gamma_k = 0$, $\forall\, k \geq q+1$. Thus, in (3.22) for $a_n = q+1$, we have the bound (with probability $> 1 - c_{s2}^* n^{-s'}$)

$$\Sigma_{j=q+2}^{q+r_n+1} \{O(u_{n,j}^2)\} = \Sigma_{j=q+2}^{q+r_n+2} \{O(j^{13} n^{-2}(\log n)^3) + O(j^5 n^{-1+2\delta})\}$$

$$= O(n^{-1+2\delta} r_n^6) + O(n^{-2}(\log n)^3 r_n^{14})$$

$$= o(\varepsilon_n) \ , \quad \text{by (3.11)}. \tag{3.24}$$

Thus, (3.12) follows from (3.24) and (3.23).

To prove (3.9), we define

$$k_\varepsilon = \min\{k \geq K: \ \Sigma_{k>m}\gamma_k^2 < \tfrac{1}{4}\varepsilon\} \ , \quad \varepsilon>0 \ . \tag{3.25}$$

Then, letting $\varepsilon_n = \varepsilon$ and r_n as in (3.9) we have by (3.19) and (3.20),

$$\begin{aligned}
&P\{K_n > k_\varepsilon, \ \text{for some} \ n \geq n_0\} \\
&\leq \Sigma_{n\geq n_0} P\{K_n > k_\varepsilon\} \\
&\leq \Sigma_{n\geq n_0} P\{\Sigma_{k=k_\varepsilon+1}^{k_\varepsilon+r_n} \hat{\gamma}_{n,k}^2 > \varepsilon\} \\
&\leq \Sigma_{n\geq n_0} P\{\Sigma_{k=k_\varepsilon+1}^{k_\varepsilon+r_n} (\hat{\gamma}_{n,k}^2 - \gamma_k^2) > \varepsilon/4\} \\
&\leq \Sigma_{n\geq n_0} P\{\Sigma_{k=k_\varepsilon+1}^{k_\varepsilon+r_n} (\hat{\gamma}_{n,k} - \gamma_k)^2 > \varepsilon/8\} \qquad \text{(by (3.21) and (3.25))} \\
&= \Sigma_{n\geq n_0} \{O(n^{-s'})\} \qquad \text{(by (3.19))} \\
&= O(n_0^{-s'+1}) \to 0 \ , \quad \text{as} \quad n_0 \to \infty \ ,
\end{aligned} \tag{3.26}$$

where we have made use of the fact that by (3.17) and (3.19), with probability $> 1 - c_{s2}^* n^{-s}$, $(\hat{\gamma}_{n,k} - \gamma_k)^2 \leq 4a^{-2}d^2u_{n,k}^2$ and $\Sigma_{k=k_\varepsilon+1}^{k_\varepsilon+r_n} u_{n,k}^2 = O(n^{-1+2\delta})\Sigma_{k=k_\varepsilon+1}^{k_\varepsilon+r_n} k^5 + O(n^{-2}(\log n)^3)\Sigma_{k=k_\varepsilon+1}^{k_\varepsilon+r_n} k^{13} =$ $O(n^{-1+2\delta}r_n^6) + O(n^{-2}(\log n)^3 r_n^{14}) \to 0$ as $n\to\infty$ (by (3.9) and letting δ (>0) adequately small). This completes the proof of (3.10).

For the final part of Theorem 3, i.e., (3.15), note that

$$\Sigma_{j=a}^{a+b} \hat{\gamma}_{n,j}^2 \geq \Sigma_{j=a}^{a+b} \gamma_j^2 - 2\Sigma_{j=a}^{a+b} |\gamma_j||\hat{\gamma}_{n,j} - \gamma_j| \ , \tag{3.27}$$

so that by (3.14) and (3.27)

$$\min_{K\leq r\leq r_n^0} \Sigma_{k=r+1}^{r+r_n} \hat{\gamma}_{n,k}^2 \geq 2\varepsilon_n - 2[\max_{K\leq r\leq r_n^0}\{\Sigma_{j=r+1}^{r+r_n} |\gamma_j||\hat{\gamma}_{n,j} - \gamma_j|\}] \ , \tag{3.28}$$

where by (2.23), $|\gamma_j| = o(j^{-\frac{1}{2}})$, and by (3.13), we may take $r_n^0 \le a_n$, $\forall n \ge n_0$. Hence, using (3.17), (3.19), we claim that with probability greater than $1 - c^*_{s2} n^{-s'}$ (for some $s'>7/3$), the second term on the rhs of (3.28) is bounded by $o(n^{-\frac{1}{2}+\delta} r_n (r_n + r_n^0)^2) +$ $o(n^{-1}(\log n)^2 r_n (r_n + r_n^0)^6) = o(n^{-\frac{1}{2}+\delta} \varepsilon_n a_n (r_n + a_n)^2) +$ $o(n^{-1}(\log n)^2 \varepsilon_n a_n (r_n + a_n)^6) = o(\varepsilon_n)$, by (3.2) and (3.4). Further note that

$$P\{K_n \le r_n^0, \text{ for some } n \ge n_0\}$$

$$\le \Sigma_{n \ge n_0} P\{K_n \le r_n^0\}$$

$$= \Sigma_{n \ge n_0} P\{\Sigma_{k=r+1}^{r+r_n} \hat{\gamma}_{n,k}^2 \le \varepsilon_n, \text{ for some } r: K \le r \le r_n^0\}$$

$$= \Sigma_{n \ge n_0} \{O(n^{-s'})\} \qquad \text{(by (3.23))}$$

$$= O(n_0^{-s'+1}) \to 0 \ , \quad \text{as } n_0 \to \infty \ . \tag{3.29}$$

Hence, (3.15) holds. This completes the proof of Theorem 3.

Next, we consider the proof of Theorem 1. Note that by (2.16) and (2.18),

$$\hat{I}_n \ge \Sigma_{k \le K+r_n} \hat{\gamma}_{n,k}^2 \quad , \text{ with probability } 1 \ . \tag{3.30}$$

By (2.23), (3.19) and (3.27), we have under the assumed regularity conditions, with probability $> 1 - c^*_{s2} n^{-s'}$, $s'>7/3$,

$$\Sigma_{k \le K+r_n} \hat{\gamma}_{n,k}^2 \ge I(f, K+r_n) - \{o(n^{-\frac{1}{2}+\delta} r_n^3) + o(n^{-1}(\log n)^2 r_n^7)\}$$

$$= I(f, K+r_n) - o(1) \qquad \text{(by (3.2))} \tag{3.31}$$

Thus, $\hat{I}_n \ge I(f, K+r_n)$ a.s., as $n \to \infty$. Also, by (3.13)

$$\hat{I}_n \le \Sigma_{k\le a_n+r_n}\hat{\gamma}^2_{n,k}$$

$$= I(f,\ a_n+r_n) + \Sigma_{k\le a_n+r_n}\{\hat{\gamma}^2_{n,k}-\gamma^2_k\}$$

$$\le I(f,\ a_n+r_n) + \Sigma_{j\le a_n+r_n}(\hat{\gamma}_{n,j}-\gamma_j)^2 + 2\Sigma_{j\le a_n+r_n}|\gamma_j||\hat{\gamma}_{n,j}-\gamma_j|$$

$$= I(f,\ a_n+r_n) + o(1) \quad \text{a.s., as} \quad n\to\infty \quad , \tag{3.32}$$

where in the ultimate stage, we have made use of (2.23) and (3.19) along with the bounds in (3.17) and (3.4). This completes the proof of (3.6). Under (2.27), (2.11) and (3.6) imply (3.7). For the final part of the theorem, consider the following hypothetical model:

$$\phi_F(u) = \gamma_1^{\frac{1}{2}}P_1(u) + \gamma_q^{\frac{1}{2}}P_q(u) \quad , \quad 0<u<1 \quad , \tag{3.33}$$

where q ($>K+r$) is a finite positive integer and choose γ_1 and γ_q such that $\gamma_q \ge d > \varepsilon$ (>0) and $12\gamma_1 > (2q+1)q^2(q+1)^2\gamma_q$. Then we have $\phi_F(u) \nearrow$ in u, $I(f) = \gamma_1+\gamma_q$ and $I(f,\ a+b) = \gamma_1 = I(f) - \gamma_2 \le I(f) - d$, $\forall\, a,b$: $a+b \le q-1$. As such, by (2.16) with $\varepsilon_n=\varepsilon$ (given) and $r_n=r$ (given), using (3.19), we conclude that

$$K_n = K \quad \text{a.s., as} \quad n\to\infty \quad ; \tag{3.34}$$

while $I(f,\ K+r) = I(f) - d < I(f) - \varepsilon$. Hence (3.7) does not hold. This completes the proof of Theorem 1, and points the desirability of letting ε_n and r_n, dependent on n, so that (3.7) holds.

For the proof of Theorem 2, we note that by (1.2), (1.3) and (2.19),

$$S_n(\hat{\phi}_n) = \Sigma_{i=1}^{n} c_{ni}(\Sigma_{j=1}^{K_n+r_n}\hat{\gamma}_{n,j}a_{nj}(R_{ni}))$$

$$= \Sigma_{j=1}^{K_n+r_n}\hat{\gamma}_{n,j}S_n(P_j) \quad , \tag{3.35}$$

where the $a_{nj}(i)$ is defined by (1.3) with $\phi(\cdot)$ replaced by

$P_j(\cdot)$, $1\le i\le n$ and $j\ge 1$. Similarly, for every $m\ge 1$,

$$S_n(\phi_{F,m}) = \Sigma_{j=1}^m \gamma_j S_n(P_j) \quad . \tag{3.36}$$

Keeping (3.13) in mind, we first prove the following:

Lemma 5. Under $\{H_n\}$ in (1.5) and the regularity conditions of Section 2, when (3.4) holds, defining $r_n^* = a_n + r_n$,

$$\max_{k\le r_n^*} |S_n(\hat{\phi}_{n,k}) - S_n(\phi_{F,k})| \overset{P}{\to} 0 \ , \quad \text{as} \quad n\to\infty \ , \tag{3.37}$$

where $\hat{\phi}_{n,k}(\cdot) = \Sigma_{j\le k}\hat{\gamma}_{n,j}P_j(\cdot)$, for $k\ge 1$.

Proof. Proceeding as in (10.3.60) through (10.3.64) of Sen (1981, Ch. 10) and using (2.21), it follows that under (1.5), for every k (≥ 1), for every s (>0), there exist positive constants $c_s^{(1)}, c_s^{(2)}$ and an integer n_0, such that for every $n\ge n_0$,

$$P_{H_n}\{|S_n(P_k) - \Delta\gamma_k| > c_s^{(1)} \log n\} \le c_s^{(2)} n^{-s} \quad . \tag{3.38}$$

Therefore, choosing $s>1$, we conclude that under $\{H_n\}$ in (1.5),

$$\max_{k\le r_n^*} |S_n(P_k) - \Delta\gamma_k| = 0_p(\log n) \quad . \tag{3.39}$$

Next, we note that

$$\begin{aligned} &\max_{k\le r_n^*} |S_n(\hat{\phi}_{n,k}) - S_n(\phi_{F,k})| \\ &= \max_{k\le r_n^*} |\Sigma_{j\le k}(\hat{\gamma}_{n,j} - \gamma_j) S_n(P_j)| \\ &\le \{\max_{k\le r_n^*} |S_n(P_k)|\}\{\Sigma_{k\le r_n^*} |\hat{\gamma}_{n,k} - \gamma_k|\} \quad . \end{aligned} \tag{3.40}$$

By (3.3), (3.4), (3.17), (3.19) and (3.39), with probability greater than $1 - (c_s^{(2)} + c_{s2}^*)n^{-s'}$ (where $s'>1$), the rhs of (3.40) is

$$\{0(\log n)\}\{0(n^{-\frac{1}{2}+\delta}(a_n+r_n)^{7/2}) + 0(n^{-1}(\log n)^2(a_n+r_n)^{15/2})\}$$

$$= \{O((\log n)^{15/7}(a_n+r_n)^{15/2} n^{-15(\frac{1}{2}-\delta)/7})\}^{7/15} + O(n^{-1}(\log n)^2(a_n+r_n)^{15/2})$$

$$= o(1) \text{ , when } \delta\ (>0) \text{ in (3.17) is chosen } < 1/30 \text{ .} \tag{3.41}$$

Thus, (3.40) is $o_p(1)$ as $n\to\infty$, and hence, (3.37) holds.

<u>Lemma 6</u>. <u>Under</u> $\{H_n\}$ <u>in</u> (1.5) <u>and the assumed regularity conditions</u>,

$$\max_{K+r_n\le k\le r_n^*} |S_n(\phi_{F,k}) - S_n(\phi_{F,r_n^*})| \overset{p}{\to} 0 \text{ , as } n\to\infty \text{ .} \tag{3.42}$$

<u>Proof</u>. First, consider the case of H_0: $\beta=0$. Note that for every $q>k$,

$$S_n(\phi_{F,q}) - S_n(\phi_{F,k}) = \Sigma_{j=k+1}^{q}\gamma_j S_n(P_j) \text{ ,} \tag{3.43}$$

where, by the orthonormality condition in (2.3) - (2.4) and the standard moment formulae for the $S_n(P_j)$, it follows that under H_0,

$$S_n(\phi_{F,q}) - S_n(\phi_{F,k}) \sim N(0, \Sigma_{j=k+1}^{q}\gamma_j^2) \text{ ;} \tag{3.44}$$

the fourth central moment may also be easily computed and verified that

$$E_{H_0}|S_n(\phi_{F,q}) - S_n(\phi_{F,k})|^4 \le c(\Sigma_{j=k+1}^{q}\gamma_j^2)^2 \text{ , } \forall\ n\ge n_0 \text{ ,} \tag{3.45}$$

where $c\ (>3)$ is a positive constant. For every $t\in[0,1]$, we write

$$k(t) = \max\{k: \Sigma_{j>k}\gamma_j^2 \le tI(f)\} \text{ .} \tag{3.46}$$

Then, note that by (2.11) and (2.27), $\Sigma_{j>K+r_n}\gamma_j^2$ 0 as $n\to\infty$, and $\Sigma_{j>r_n^*}\gamma_j^2$ 0 as $n\to\infty$. So, for every $\delta>0$, there exists an n_0, such that for every $n\ge n_0$,

$$\delta \ge \Sigma_{j>K+r_n}\gamma_j^2 \ge \Sigma_{j>r_n^*}\gamma_j^2 \ge 0 \text{ .} \tag{3.47}$$

Thus, writing $W_n = \{W_n(t) = [I^{\frac{1}{2}}(f)]^{-1}S_n(\phi_{F,k(t)}),\ 0\le t\le 1\}$, by

Theorem 12.3 of Billingsley (1968), we conclude from (3.45) and (3.46) that W_n converges weakly to a Brownian motion W on $[0,1]$, and the implied "tightness" part of this weak convergence ensures that for every $\varepsilon>0$ and $\eta>0$, there exist a δ $(0<\delta<1)$ and an integer n_0, such that under H_0,

$$P_{H_0}\{\sup_{1-\delta\leq t\leq 1}|W_n(t)-W_n(1)|>\varepsilon\}<\eta\ ,\quad \forall\ n\geq n_0\ . \tag{3.48}$$

This ensures (3.42) under H_0. We invoke the contiguity of the probability measures under $\{H_n\}$ to that under H_0 to extend this under $\{H_n\}$ as well. Q.E.D.

Returning now to the proof of (3.8), we note that by (2.16) and (3.13),

$$K+r_n\leq K_n+r_n\leq a_n+r_n \quad \text{a.s., as} \quad n\to\infty\ . \tag{3.49}$$

As such, (3.8) follows from (3.37), (3.42) and (3.49). Also, note that (3.16), (3.17), (3.18) and (3.8) ensure that as $n\to\infty$,

$$C_n|\hat{\beta}_n(\hat{\phi}_n)-\hat{\beta}_n(\phi_{F,r_n^*})|\overset{P}{\to}0\ . \tag{3.50}$$

Further, by (1.7), (1.8), (2.7) and (2.11), we conclude that

$$\rho(\phi_F,\ \phi_{F,r_n^*})\to 1 \quad \text{as} \quad n\to\infty\ , \tag{3.51}$$

and hence, the asymptotic efficiency results in (1.6) and (1.11) are shared by $\{\phi_{F,r_n^*}\}$ as well. Combining this with (3.8) and (3.50), we conclude that $\hat{\phi}_n$ is also an asymptotically efficient (adaptive) score function. Q.E.D.

4. SOME GENERAL REMARKS

We have noted [after (2.8)] that for a symmetric F, $P_{2k}(\cdot)$, $k\geq 1$, do not provide any basis for the projection in (2.9), while $P_{2k+1}(\cdot)$, $k\geq 0$ are themselves skew-symmetric functions and form a basis for the skew-symmetric ϕ_F. This also explains why in (2.16), we need to choose $r\geq 2$. To eliminate the pathological cases, as in

(3.33), one also needs to choose $r=r_n \nearrow$ in n. However, in our main theorems (viz., Theorems 1, 2 and 3), we have a variety of conditions on (r_n, ε_n), and we need to have a closer look on these. Ideally, one would like to choose $\{r_n\}$ as a slowly increasing function of n, so that $\hat{\phi}_n$ becomes computationally less cumbrous. On the other hand, by (2.11), one would also like to choose $\{r_n\}$ adequately large, so that the residual term is negligible (up to the desired extent). Keeping these opposite pictures in mind, we may suggest the following. Note that r_n, ε_n, and a_n need to satisfy (3.2) (or (3.3)) and (3.4). Thus, if we let $r_n = O(\log n)$, then $a_n\varepsilon_n/(\log n) \to \infty$ as $n \to \infty$ and further that $a_n^{13}\varepsilon_n^{-1}(\log n)^5/n^2 \to 0$ as $n \to \infty$. In particular, if we let

$$r_n = O(\log n) \ , \quad a_n \sim n^{2/15}(\log n)^{-\frac{1}{2}} \ , \quad \varepsilon_n = n^{-2/15}(\log n)^2 \quad (4.1)$$

then note that $a_n\varepsilon_n = (\log n)^{3/2}$, so that $r_n/(a_n\varepsilon_n) \to 0$ as $n \to \infty$, and (3.3) - (3.4) hold. It is also possible to choose $r_n = O(n^{\alpha_1})$ for some α_1: $0<\alpha_1<1/15$. In that case, $a_n\varepsilon_n n^{-\alpha_1} \to \infty$ as $n \to \infty$ and (3.3) and (3.4) have to be true for $\{a_n\}$, $\{\varepsilon_n\}$. If we let $a_n \sim n^{\alpha_2}$ and $\varepsilon_n \sim n^{-\alpha_3}$, then we have $\alpha_2 - \alpha_3 - \alpha_1 > 0$ i.e., $\alpha_2 > \alpha_1 + \alpha_3$, and $\alpha_1 > 0$, $\alpha_3 > 0$. Further, by (3.4), $13\alpha_2 - \alpha_1 + \alpha_3 < 2$, and by (3.3), $15\alpha_2/2 < 1$. Such a choice may easily be made, e.g., $0 < \alpha_1 = \alpha_3 < \frac{1}{2}\alpha_2$, $\alpha_2 < 2/15$. In the above discussion, if instead of (3.3), we use (3.2), then 2/15 may be replaced by 1/7 as well. Also, in (2.23), if we make a more restrictive assumption that

$$k^{\ell}\gamma_k^2 \to 0 \quad \text{as} \quad k \to \infty \ , \quad \text{for some } \ell \geq 1 \ , \qquad (4.2)$$

then, in (3.4), we may use the condition that $(rn/\varepsilon_n a_n^{\ell}) \to 0$ as $n \to \infty$, and this in turn will reduce the order of a_n. In the extreme case, if we let $r_n = r$ (≥ 2) for every n, then we need only that $\varepsilon_n a_n \to \infty$ but $(a^{13}\varepsilon_n^{-1})/(n^2(\log n)^{-4}) \to 0$, as $n \to \infty$, and for this, we may let $a_n \sim n^{\alpha}(\log n)^{\frac{1}{2}}$, $\varepsilon_n \sim n^{-\alpha}$, where $0 < \alpha \leq 2/15$, and conclude that (3.10) and (3.12) hold. However, as has been remarked in Theorem 1, we may not desire to set $r_n = r$ for every n, and hence, such a solution will not be of much interest.

We may also remark that for the given model (1.1), under H_n in (1.5), or for the estimation problem, the X_i are not i.i.d. Hence, for the study of the properties of the $\hat{\gamma}_{n,j}$ in (2.13), involving the estimator $\tilde{\beta}_n$ in (2.12), noting that the residuals $X_i - \hat{\beta}_n c_i$ are not necessarily independent or identically distributed random variables, we are not in a position to adapt directly the results of Beran (1974) and Eplett (1982), among others. The Jurečková-linearity results and contiguity of probability measures provide the necessary tools in this respect. In this context, it is not necessary to use the R-estimator in (2.12). We may as well use any other estimator for which (3.18) holds. Further, one may also consider a general regression model where in (1.1), βc_i is replaced by $\underset{\sim}{\beta}' \underset{\sim}{c}_i$, $i \geq 1$, $\underset{\sim}{\beta}' = (\beta_1, \ldots, \beta_q)$ is a q (≥ 1)-vector and the $\underset{\sim}{c}_i$ are specified q-vectors too. Again an estimator $\hat{\underset{\sim}{\beta}}_n$ of $\underset{\sim}{\beta}$ satisfying (3.11) (in the Euclidean norm) may be used, and the R-estimators of Jurečková (1971) are particularly appealing in this context.

In the finite case, i.e., under (2.5), using (3.12) and the asymptotic (joint) normality of the $\gamma_{n,j}$, $j \leq q+1$, one may study the asymptotic behaviour of $n^{\frac{1}{2}}(\hat{\phi}_n(u) - \phi_F(u))$, $0<u<1$; however, this becomes quite complicated in the infinite case.

APPENDIX: PROOF OF THEOREM 4.

Note that for each k (≥ 1), $S_n(c_n^{-1}t, P_k)$ is expressible as a difference of two nonincreasing functions in t ($\in E$). Hence, if we let $t_{nj} = n^{-1}j$, $j = 0, \pm 1, \ldots, \pm[a \log n]$, then the supremum (over $t_1, t_2 \in (-a \log n,\ a \log n)$) in (3.16) can be dominated by the maximum (over the t_{nj}) on the grid-points (plus a non-stochastic remainder term $O(n^{-1})$). Hence, to prove (3.16), it suffices to choose an $s^* > s+1$ and verify that the probability inequality in (3.16) (without the supremum over t_1, t_2) holds with s being replaced by s^*, uniformly in $(t_1, t_2) \in (-a \log n,\ a \log n)$. For the sake of the simplicity of this proof, we take $t_2 = -t_1 = t$

and $a=1$, i.e., $t \in (-\log n, \log n)$. Then, by the Taylor expansion, we have, for every $k \geq 1$, real t,

$$S_n(-tC_n^{-1}, P_k) - S_n(tC_n^{-1}, P_k) - 2t\gamma_k = \Sigma_{j=0}^{3}(j!)^{-1}T^*_{nj,k}(t) \quad (A.1)$$

where

$$T^*_{n0,k}(t) = \Sigma_{i=1}^{n} c_{ni}\left\{P_k\left(\frac{E[R_{ni}(-t/c_n)|X_i]}{n+1}\right) - P_k\left(\frac{E[R_{ni}(t/c_n)|X_i]}{n+1}\right)\right\} - 2t\gamma_k \ , \quad (A.2)$$

and, for $j = 1,2,3$, $T^*_{nj,k}(t) = T_{nj,k}(-t) - T_{nj,k}(t)$, where for every real u,

$$T_{nj,k}(u)$$

$$= \Sigma_{i=1}^{n} c_{ni} P_k^{(j)}\left(\frac{E[R_{ni}(u/c_n)\ X_i]}{n+1}\right)(n+1)^{-j}\{R_{ni}(u/c_n) - E[R_{ni}(u/C_n)|X_i]\}^j \ , \quad (A.3)$$

for $j = 1,2$, and

$$T_{n3,k}(j) = \Sigma_{i=1}^{n} c_{ni} P_k^{(3)}\left(\frac{\alpha_n E[R_{ni}(u/C_n)|X_i] + (1-\alpha_n)R_{ni}(u/C_n)}{n+1}\right)$$

$$\times (n+1)^{-3}\{R_{ni}(u/C_n) - E[R_{ni}(u/C_n)|X_i]\}^3 \ , \quad (A.4)$$

with $\alpha_n \in (0,1)$. Note that

$$|T_{n3,k}(u)| \leq 4(\sup_{0\leq t\leq 1}|P_k^{(3)}(t)|)\Sigma_{i=1}^{n}|c_{ni}|(n+1)^{-3}\{|R_{ni}(0)-E[R_{ni}(0)|X_i]^3$$

$$+ |R_{ni}(u/C_n)-R_{ni}(0)-E[R_{ni}(u/C_n)-R_{ni}(0)|X_i]|^3\} \quad (A.5)$$

Since $R_{ni}(0) - E[R_{ni}(0)|X_i] = n[F_n(X_i)-F(X_i)] - [1-F(X_i)]$, we have

$$\max_{1\leq i\leq n}|(n+1)^{-1}\{R_{ni}(0)-E[R_{ni}(0)\ X_i]\}| \leq (n+1)^{-1} + \sup\{|F_n(x)-F(x)|: x\in E\}. \quad (A.6)$$

Further, by the Dvoretzky, Kiefer and Wolfowitz inequality, for every $c>0$ and $n\geq 1$,

$$P\{\sup_x |F_n(x)-F(x)| \geq c(n^{-1}\log n)^{\frac{1}{2}}\} \leq 2e^{-2c^2 \log n} \quad . \qquad \text{(A.7)}$$

Therefore, by (2.21), (2.24) - (2.25) and (A.5) - (A.7), we have with probability greater than $1 - 2n^{-2c^2}$,

$$(\Sigma_{i=1}^n |c_{ni}|(n+1)^{-3}|R_{ni}(0) - E[R_{ni}(0)|X_i]|^3) \sup_{0\leq t\leq 1} |P_k^{(3)}(t)|$$

$$\leq Dk^{13/2} n^{\frac{1}{2}} [n^{-3/2} c^3 (\log n)^{3/2} + 0(n^{-3})]$$

$$= 0(n^{-1} k^{13/2} (\log n)^{3/2}) \quad , \qquad \text{(A.8)}$$

where c (>0) can be so chosen that for the chosen value of s in (3.16), $2c^2 > s*$. Next, we note that

$$E\{(R_{ni}(c_n^{-1}u) - R_{ni}(0) - E[R_{ni}(C_n^{-1}u) - R_{ni}(0)|X_i])^2|X_i\}$$

$$= \Sigma_{j=1}^n \text{Var}\{I(X_j \leq X_i - u(c_{ni}-c_{nj}) - I(X_j\leq X_i)|X_i\}$$

$$\leq \Sigma_{j=1\,(j\neq i)}^n E\{(I(X_j \leq X_i - u(c_{ni}-c_{nj})) - I(X_j\leq X_i))^2|X_i\}$$

$$= \Sigma_{j=1\,(j\neq i)}^n 0(|u||c_{ni}-c_{nj}|) = 0(|u|n^{\frac{1}{2}}) \quad . \qquad \text{(A.9)}$$

Since for a given X_i, the $I(X_j \leq X_i - u(c_{ni}-c_{nj})) - I(X_j\leq X_i)$ $(1\leq j\leq n)$ are independent bounded r.v., by using the exponential inequality [viz. Theorem 2 of Hoeffding (1963)], we obtain that

$$P\{|R_{ni}(C_n^{-1}u) - R_{ni}(0) - E[R_{ni}(C_n^{-1}u) - R_{ni}(0)|X_i]| \geq n^{\frac{1}{4}}(\log n)^{2/3}|u|^{\frac{1}{2}}|X_i\}$$

$$\leq 2\exp\{-\tfrac{1}{2}(\log n)^{4/3}[1+o(1)]\}$$

$$= \{2n^{-\frac{1}{2}(\log n)^{1/3}}\}\{1+o(1)\} \quad , \qquad \text{(A.10)}$$

where $\frac{1}{2}(\log n)^{1/3}$ can be made larger than $s*+1$, by choosing n adequately large. Since the unconditional probability can be obtained by integrating over X_i, we have on noting that $|u| \leq \log n$, with probability $\geq 1-n^{-s*}$,

$$\max_{1\le i\le n} |R_{ni}(C_n^{-1}u)-R_{ni}(0)-E[R_{ni}(C_n^{-1}u)-R_{ni}(0)|X_i] = O(n^{\frac{1}{4}}(\log n)^{2/3}(\log n)^{\frac{1}{2}}),$$

and hence,

$$\Sigma_{i=1}^n |c_{ni}|(n+1)^{-3}|R_{ni}(C_n^{-1}u) - R_{ni}(0) - E[R_{ni}(C_n^{-1}u) - R_{ni}(0)|X_i]|^3$$

$$= O(n^{\frac{1}{2}})O(n^{-3}n^{3/4}(\log n)^2(\log n)^{3/2})$$

$$= O(n^{-7/4}(\log n)^2(\log n)^{3/2})$$

$$= O(n^{-1}(\log n)^{3/2})O(n^{-3/4}(\log n)^2)$$

$$= o(n^{-1}(\log n)^{3/2}) \quad . \qquad (A.12)$$

Combining (A.8), (A.12) and (2.21), we conclude that for every s^* ($\ge s+1$), there exist positive numbers d^* and n^*, such that for every $n\ge n^*$,

$$P\{|T^*_{n3,k}(t)| > d^* n^{-1}k^{13/2}(\log n)^{3/2}\} \le \tfrac{1}{4}n^{-s^*} \quad . \qquad (A.13)$$

Consider next $T^*_{n0,k}(t)$. Note that $E[R_{ni}(u/C_n)|X_i] = 1 - F(X_i) + \Sigma_{j=1}^n F(X_i-u(c_{ni}-c_{nj}))$, where $\max\{|c_{ni}-c_{nj}|: 1\le i \ne j\le n\} = O(n^{-\frac{1}{2}})$ and $||P_k^{(1)}||_\infty \le Dk^{5/2}$. Hence for every $t \in (-\log n, \log n)$, we may rewrite

$$T^*_{n0,k}(t) = \Sigma_{i=1}^n c_{ni}\{U_{ni}-EU_{ni}\} + O(n^{-\frac{1}{2}}k^{5/2}\log n) \qquad (A.14)$$

where the

$$U_{ni} = P_k\left(\frac{1-F(X_i)+\Sigma_{j=1}^n F(X_i+t(c_{ni}-c_{nj}))}{n+1}\right) - P_k\left(\frac{1-F(X_i)+\Sigma_{j=1}^n F(X_i-t(c_{ni}-c_{nj}))}{n+1}\right) \qquad (A.15)$$

($1\le i\le n$) are independent (bounded by $2(2k+1)^{\frac{1}{2}}$) r.v. and

$$\mathrm{Var}(U_{ni}) \le EU_{ni}^2 = O(n^{-1}(\log n)^2k^5) \quad , \qquad (A.16)$$

uniformly in $1\le i\le n$ and $t \in (-\log n, \log n)$. For any (arbitrary) positive integer m, we have

$$E\{\Sigma_{i=1}^{n} c_{ni}(U_{ni}-EU_{ni})\}^{2m}$$

$$= (\Sigma_{i=1}^{n} c_{ni}^{2} E(U_{ni}-EU_{ni})^{2})^{m} + \text{lower order term}$$

$$= O(n^{-m}(\log n)^{2m} k^{5m}) \qquad (A.17)$$

Consequently, for every $d>0$ and $\delta>0$,

$$P\{|\Sigma_{i=1}^{n} c_{ni}(U_{ni}-EU_{ni})| > dn^{-\frac{1}{2}+\delta} k^{5/2}\}$$

$$\leq (dn^{-\frac{1}{2}+\delta} k^{5/2})^{-2m} O(n^{-m}(\log n)^{2m} k^{5m})$$

$$= O(n^{-2\delta m}(\log n)^{2m})$$

$$= O(n^{-s^*}), \quad \text{when} \quad 2\delta m > s^* \quad . \qquad (A.18)$$

From (A.14) and (A.18), we obtain on noting that $n^{-\frac{1}{2}}\log n = o(n^{-\frac{1}{2}+\delta})$, for every $\delta>0$, that for every s^* ($> s+1$), there exists an n^*, such that for every $n \geq n^*$, $t \in (-\log n, \log n)$,

$$P\{|T^*_{n0,k}(t)| > k^{5/2} n^{-\frac{1}{2}+\delta}\} \leq \tfrac{1}{4} n^{-s^*} \quad . \qquad (A.19)$$

Finally, we consider the case of $T^*_{n1,k}(t)$ and $T^*_{n2,k}(t)$. We deal only with $T^*_{n2,k}(t)$, as the other case follows more simply. Using the expression for $E[R_{ni}(u/C_n)|X_i]$ as in after (A.13), we may rewrite $T_{n2,k}(u)$ in (A.3) as

$$(n+1)^{-2}\Bigg\{\sum_{1\leq i \neq j \neq \ell \leq n} c_{ni} P_k^{(2)}\left(\frac{1-F(x)+\Sigma_{r=1}^{n} F(X_i-u(c_{ni}-c_{nr}))}{n+1}\right)$$

$$\times [I(X_j \leq X_i - u(c_{ni}-c_{nj}) - F(X_i-u(c_{ni}-c_{nj}))]$$

$$\times [I(X_\ell \leq X_i - u(c_{ni}-c_{nj}) - F(X_i-u(c_{ni}-c_{n\ell})]$$

$$+ \sum_{1\leq i \neq j \leq n} c_{ni} P_k^{(2)}\left(\frac{1-F(x)+\Sigma_{r=1}^{n} F(X_i-u(c_{ni}-c_{nr}))}{n+1}\right)$$

$$\times [I(X_j \leq X_i - u(c_{ni}-c_{nr})) - F(X_i-u(c_{ni}-c_{nr}))]^2\Bigg\}$$

$$= \sum_{1\le i \ne j \ne \ell \le n} A_n^{(u)}(X_i, X_j, X_\ell; i,j,\ell) + \sum_{1\le i \ne j \le n} B_n^{(u)}(X_i, X_j; i,j)$$

$$= A_n^{(u)} + B_n^{(u)}, \text{ say} . \tag{A.20}$$

It is easy to show that $EA_n^{(u)} = 0$, $\forall u \in E$, and using (2.20), (2.21) and (2.24) - (2.25), we can show that for every $u, u' \in (-\log n, \log n)$

$$E[B_n^{(u)} - B_n^{(u')}] = 0(n^{-1}k^{13/2}(\log n)) . \tag{A.21}$$

Thus, we may write

$$T^*_{n2,k}(t) = [A_n^{(-t)} - A_n^{(t)}] + [B_n^{(-t)} - EB_n^{(-t)} - B_n^{(t)} + EB_n^{(t)}] + 0(n^{-1}k^{13/2}\log n). \tag{A.22}$$

Parallel to (A.17), here also, we may compute, for an arbitrary m (≥ 1), $E[A_n^{(-t)} - A_n^{(t)}]^{2m}$ and $E[B_n^{(-t)} - B_n^{(t)} - EB_n^{(-t)} + EB_n^{(t)}]^{2m}$. Using the conventional technique (as in the case of higher order moments of Hoeffding's (1948) U-statistics -- but with some straightforward modifications) along with (2.20) - (2.21) and (2.24) - (2.25), it follows readily that

$$E[A_n^{(-t)} - A_n^{(t)}]^{2m} = 0(n^{-2m}k^{9m}(\log n)^{2m}) \tag{A.23}$$

$$E[B_n^{(-t)} - B_n^{(t)} - EB_n^{(-t)} + EB_n^{(t)}]^{2m} = 0(n^{-2m}k^{9m}(\log n)^{2m} + n^{-3m}k^{13m}(\log n)^{2m}). \tag{A.24}$$

Hence, proceeding as in (A.18), we claim that for all k: $k = o(n^{2/13}(\log n)^{-2})$, for every $d>0$, $\delta>0$,

$$P\{|A_n^{(-t)} - A_n^{(t)} + B_n^{(-t)} - B_n^{(t)} - EB_n^{(-t)} + EB_n^{(t)}| > u_{n,k}\}$$

$$\le u_{n,k}^{-2m}\{0(n^{-2m}k^{gm}(\log n)^{2m} + n^{-3m}k^{13m}(\log n)^{2m}\}$$

$$= o(n^{-s^*}) , \tag{A.25}$$

where the $u_{n,k}$ are defined in (3.17) and m is chosen adequately large, so that the last step follows by using (3.17). Therefore, parallel to (A.19), we have here

$$P\{|T^*_{n2,k}(t)| > u_{n,k}\} \leq \tfrac{1}{4}n^{-s^*} \quad , \tag{A.26}$$

for every $n \geq n^*$. With a similar inequality for $T^*_{n1,k}(t)$, we obtain from (A.1), (A.13), (A.19) and (A.26) that for every s^* (> s+1), there exists an n^*, such that

$$P\{|S_n(-tC_n^{-1}, P_k) - S_n(tC_n^{-1}, P_k) - 2t\gamma_k| > u_{n,k}\} \leq n^{-s^*} \quad , \tag{A.27}$$

for every $n \geq n^*$, and uniformly in $t \in (-\log n, \log n)$. The proof for Theorem 4 is complete.

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