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On SF-rings and Regular Rings

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Abstract. A ring R is called a left (right) SF-ring if simple left (right) R-modules are flat. It is still unknown whether a left (right) SF-ring is von Neumann regular. In this paper, we give some conditions for a left (right) SF-ring to be (a) von Neumann regular; (b) strongly regular; (c) division ring. It is proved that: (1) a right SF-ring R is regular if maximal essential right (left) ideals of R are weakly left (right) ideals of R (this result gives an affirmative answer to the question raised by Zhang in 1994); (2) a left SF-ring R is strongly regular if every non-zero left (right) ideal of R contains a non-zero left (right) ideal of R which is a W-ideal; (3) if R is a left SF-ring such that $l(x)$ $(r(x))$ is an essential left (right) ideal for every right (left) zero divisor x of R, then R is a division ring.

1. Introduction

Throughout this paper, R denotes an associative ring with identity and all our modules are unitary. The symbols $J(R)$, $Z(_RR)\,(Z(R_R))$, $soc(_RR)(soc(R_R))$ respectively stand for the Jacobson radical, left (right) singular ideal and left (right) socle of R. R is *semiprimitive* if $J(R) = 0$. R is *left non-singular* if $Z(_RR) = 0$. *Right non-singular rings* are defined similarly. For any $a \in R$, $l(a)\,(r(a))$ denotes the left (right) annihilator of a. By an *ideal*, we mean a two sided ideal. As usual, a *reduced ring* is a ring without non-zero nilpotent elements. R is *left (right) duo* if every left (right) ideal of R is an ideal. R is a *left quasi duo* (*MELT*) *ring* if every maximal (maximal essential) left ideal of R is an ideal. *Right quasi duo rings* and *MERT rings* are defined similarly. R is *strongly left (right) bounded* if every non-zero left (right) ideal of R contains a non-zero ideal of R ([12]). R is a *left (right) uniform ring* if every non-zero left (right) ideal of R is essential ([10]). R is *(von Neumann) regular* if for every $a \in R$, there exists some $b \in R$ such that $a = aba$. R is *strongly regular* if for every $a \in R$, there exists some $b \in R$ such that $a = a^2b$. Clearly, R is strongly regular if and only if R is a reduced regular ring. Following [2], R is *left (right) weakly regular* if for every left (right) ideal I of R, $I = I^2$ and R

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is *weakly regular* if it is both left and right weakly regular. Clearly, a regular ring is weakly regular, but a weakly regular ring need not be regular (for example, see [2], Remark 6). Following [11], a left R-module M is *p-injective* if for every principal left ideal I of R and every left R-homomorphism $f : I \longrightarrow M$, there exists some $m \in M$ such that $f(b) = bm$ for all $b \in I$ and R is a *left P-V-ring* if every simple left R-module is p-injective. Again, following [11], a left (right) R-module M is *YJ-injective* if for each $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and every left (right) R-homomorphism from Ra^n (a^nR) to M extends to a left (right) R-homomorphism from R to M and R is a *left (right) GP-V-ring* if simple left (right) R-modules are YJ-injective. A regular ring is left (right) GP-V-ring (see, [9], Lemma 2) but a left (right) GP-V-ring need not be regular (see, [7]). Following [3], R is a *left (right) SF-ring* if simple left (right) R-modules are flat. It is well known that regular rings are left (right) SF-rings. As far as we know, the question that whether left (right) SF-rings are necessarily regular, is still open. Over the last three and a half decades, left (right) SF-rings have been studied by many authors and the regularity of left (right) SF-rings which satisfy certain additional conditions is proved (cf. for example, [3], [4], [8], [10]-[15]).
We recall the following two definitions following [11]:

Definition 1.1. An additive subgroup L of a ring R is a *weakly left ideal* of R if for every $x \in L$ and every $r \in R$ there exists a positive integer n such that $(rx)^n \in L$. The notion of a *weakly right ideal* of a ring is defined similarly.

Definition 1.2. A ring R is an $LW-ring$ ($RW-ring$) if every left (right) ideal of R is a weakly right (left) ideal of R.

Example 1.3. Let $R = UT_2(\mathbb{Q})$, the ring of upper triangular matrices over $\mathbb{Q}$. Take $L = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Q} \right\}$ and $K = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{Q} \right\}$. Then L is a left ideal and K is a right ideal of R. It is easy to see that L is not a weakly right ideal and K is not a weakly left ideal of R.

Example 1.4. Take $R = \left\{ \begin{pmatrix} a & a_1 & a_2 & a_3 \\ 0 & a & a_4 & a_5 \\ 0 & 0 & a & a_6 \\ 0 & 0 & 0 & a \end{pmatrix} : a,\, a_i \in \mathbb{Z}_2,\, i = 1,2,3,4,5,6 \right\}$.

Let $r \in R$. Suppose A is any left ideal of R. If $A = R$, then A is a weakly right ideal of R. If $A \neq R$ and $x \in A$, then $x = \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ 0 & 0 & x_4 & x_5 \\ 0 & 0 & 0 & x_6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ for some $x_i \in \mathbb{Z}_2$, $i = 1,2,3,4,5,6$. Therefore $(xr)^4 = 0$. This implies that A is a weakly right ideal of R. It follows that R is an LW-ring.
Suppose $B \neq R$ be any right ideal of R. It is easy to see that for every

$y \in B$, $(ry)^4 = 0$. Hence B is a weakly left ideal of R. Therefore R is an RW-ring.

Let $L = \left\{ \begin{pmatrix} 0 & b_1 & b_1 & b_2 \\ 0 & 0 & 0 & b_3 \\ 0 & 0 & 0 & b_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} : b_i \in \mathbb{Z}_2, i = 1, 2, 3, 4 \right\}$.

Then L is a left ideal of R. Now

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in L, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in R$$

and

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \notin L.$$

It follows that L is not an ideal of R and hence R is not left duo.

Also, $K = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : b_i \in \mathbb{Z}_2, i = 1, 2 \right\}$ is a right ideal of R.

However

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \notin K.$$

We see that K is not an ideal of R. Therefore R is not a right duo ring.

2. SF-rings and Weakly One Sided Ideals

In this section, as a continuation of [11], we give further characterizations of strongly regular rings via weakly one sided ideals. We also prove the regularity of right SF-rings whose maximal essential right (left) ideals are weakly left (right) ideals.

We start with the following observation:

Proposition 2.1. *The following conditions are equivalent for a ring R:*

(1) *R is an LW-ring.*

(2) *Every principal left ideal of R is a weakly right ideal.*

(3) *Every finitely generated left ideal of R is a weakly right ideal.*

Proof. It is clear that (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3).
(2) $\Longrightarrow$ (1). Let L be a left ideal of R and $a \in L$, $r \in R$. By hypothesis, Ra is a weakly right ideal. Hence there exists a positive integer n such that $(ar)^n \in Ra \subseteq L$. This proves that L is a weakly right ideal of R so that R is an LW-ring.
(3) $\Longrightarrow$ (1) can be proved similarly. □

Lemma 2.2. *Let R be a semiprimitive ring whose maximal left ideals are weakly right ideals, then R is reduced.*

Proof. Suppose $0 \neq a \in R$ such that $a^2 = 0$, then $a \notin J(R)$. So $a \notin M$ for some maximal left ideal M of R. Hence $M + Ra = R$ implying $Ma = Ra$ which yields $a = ba$ for some $b \in M$. We therefore get $x + rba = 1$ for some $x \in M$, $r \in R$. As M is a weakly right ideal and $rb \in M$, there exists a positive integer k such that $(rba)^k \in M$ and so $(1-x)^k \in M$. Using $x \in M$ in $(1-x)^k \in M$ we get $1 \in M$ which contradicts that $M \neq R$. Therefore R is reduced. □

Similarly, we can prove Lemma 2.3.

Lemma 2.3. *Let R be a semiprimitive ring whose maximal right ideals are weakly left ideals, then R is reduced.*

Lemma 2.4([4], Proposition 3.2)**.** *Let R be a left (right) SF-ring and I be an ideal of R. Then R/I is also a left (right) SF-ring.*

Lemma 2.5([4], Remark 3.13)**.** *Let R be a reduced left (right) SF-ring. Then R is strongly regular.*

Lemma 2.6([4], Theorem 4.10)**.** *A left (right) quasi duo left SF-ring is strongly regular.*

Lemma 2.7([4], Proposition 4.3)**.** *Let R be a reduced ring. Then R is left weakly regular if and only if R is right weakly regular.*

Lemma 2.8([5], Lemma 2.1)**.** *If R is a left (right) GP-V-ring, then $J(R) = 0$.*

The following lemma can be proved easily.

Lemma 2.9. *Let R be a ring and I an ideal of R. For every left (right) ideal K of R such that $I \subseteq K$, K is a weakly right (left) ideal of R if and only if K/I is a weakly right (left) ideal of R/I.*

Zhang in [11] proposed the following question: Is R von Neumann regular if it is a left SF-ring whose every maximal right (left) ideal is a weakly left (right) ideal of R ? Theorem 2.10 not only gives an affirmative answer to this question but also gives generalizations of many other results established in [11].

Theorem 2.10. *Let R be a ring whose maximal left (right) ideals are weakly right (left) ideals, then the following conditions are equivalent:*

(1) *R is a left GP-V-ring.*

(2) *R is a left weakly regular ring.*

(3) *R is a left SF-ring.*

(4) *R is a right GP-V-ring.*

(5) *R is a right weakly regular ring.*

(6) *R is a right SF-ring.*

(7) *R is a regular ring.*

(8) *R is a strongly regular ring.*

Proof. Let R be a ring whose maximal left ideals are weakly right ideals.
That (8) $\Longrightarrow$ (1), (2), (3), (4), (5), (6), (7) are well known.
(1) $\Longrightarrow$ (8). R is reduced by Lemma 2.8 and Lemma 2.2. Let $a \in R$. If $l(a)+Ra \neq R$, then it must be contained in a maximal left ideal M of R. Since R is a left GP-V-ring, R/M being a simple left R-module is YJ-injective. Hence there exists a positive integer n such that $a^n \neq 0$ and every left R-homomorphism from Ra^n to R/M extends to one from R to R/M. Define $f : Ra^n \longrightarrow R/M$ by $f(ra^n) = r + M$ for every $r \in R$. As R is reduced, $l(a^n) = l(a)$ so that f is well-defined. It follows that $1+M = a^n(b+M)$ for some $b \in R$ which yields $1-a^nb \in M$. If $a^nb \notin M$, then $M + Ra^nb = R$. This gives $x + ra^nb = 1$ for some $x \in M$, $r \in R$. As M is a weakly right ideal and $ra^n \in M$, there exists some $k > 0$ such that $(ra^nb)^k \in M$, that is $(1-x)^k \in M$, whence $1 \in M$, a contradiction to $M \neq R$. Therefore $l(a)+Ra = R$. This implies that there exists some $x \in R$, $y \in l(a)$ such that $xa + y = 1$ which yields $a = xa^2$. Hence R is strongly regular.
(2) $\Longrightarrow$ (8). Let $a \in R$. If $l(a) + Ra \neq R$, then there exists a maximal left ideal M of R containing $l(a) + Ra$. As R is left weakly regular, $Ra = RaRa$ which gives $a = \sum r_i a s_i a$ for some $r_i \in R$, $s_i \in R$. So $1 - \sum r_i a s_i \in l(a) \subseteq M$. Suppose $\sum r_i a s_i \notin M$, then $r_k a s_k \notin M$ for some k. Therefore $M + Rr_k a s_k = R$ and hence $x + rr_k a s_k = 1$ for some $x \in M$, $r \in R$. As M is a weakly right ideal and $rr_k a \in M$, there exists a positive integer n such that $(rr_k a s_k)^n \in M$ so that $(1-x)^n \in M$, whence $1 \in M$. This contradicts that $M \neq R$. Thus $l(a) + Ra = R$. This implies that R is strongly regular.
(3) $\Longrightarrow$ (8). Let $\overline{R} = R/J(R)$. Then $\overline{R}$ is a semiprimitive ring. Also, by hypothesis and Lemma 2.9, every maximal left ideal of $\overline{R}$ is a weakly right ideal of $\overline{R}$. Therefore it follows from Lemma 2.2, Lemma 2.4 and Lemma 2.5 that $\overline{R}$ is strongly regular. Therefore $\overline{R}$ is left duo so that R is left quasi duo. Hence by Lemma 2.6, R is strongly regular.
(6) $\Longrightarrow$ (8) can be proved similarly.
(4) $\Longrightarrow$ (8). Let $a \in R$. R is reduced by Lemma 2.8 and Lemma 2.2 and therefore $r(a) = l(a)$. If $r(a) + Ra \neq R$, then it must be contained in some maximal left ideal L of R. We claim that $RaR \subseteq L$. If this is not true, then $ras \notin L$ for some $r \in R$, $s \in R$. Then $L + Rras = R$ which yields $b + tras = 1$ for some $b \in L$, $t \in R$. Since L is a weakly right ideal and $tra \in L$, there exists some positive integer n such

that $(tras)^n \in L$, that is $(1-b)^n \in L$, whence $1 \in L$ which contradicts that $L \neq R$. Therefore $RaR \subseteq L$ and so $r(a) + RaR \subseteq L \neq R$. Hence there exists a maximal right ideal M of R such that $r(a) + RaR \subseteq M$. Since R is a right GP-V-ring, R/M is YJ-injective. Thus there exists a positive integer m such that $a^m \neq 0$ and every right R-homomorphism from a^mR to R/M extends to one from R to R/M. Define $f : a^mR \longrightarrow R/M$ by $f(a^mr) = r + M$ for every $r \in R$. As R is reduced, $r(a^m) = r(a)$ so that f is a well-defined. Therefore we get $1 + M = (b + M)a^m$ for some $b \in R$ which yields $1 - ba^m \in M$. But $ba^m \in RaR \subseteq M$, whence $1 \in M$. This contradiction shows that $r(a) + Ra = R$. Therefore $x + ya = 1$ for some $x \in r(a)$ and $y \in R$ which yields $a = aya$. Thus R is regular. As R is reduced also, R is strongly regular.

(7) $\Longrightarrow$ (2) is well known.

(5) $\Longrightarrow$ (2). Since R is right weakly regular, it is semiprimitive. Thus by Lemma 2.2 and Lemma 2.7, R is left weakly regular.

We can similarly prove the theorem for a ring R whose maximal right ideals are weakly left ideals. □

Corolary 2.11([11], Theorem 2)**.** *A ring R is strongly regular if and only if R is a left P-V-ring and every maximal left ideal of R is a weakly right ideal of R.*

Corolary 2.12([11], Theorem 3)**.** *The following conditions are equivalent:*

(1) *R is a strongly regular ring.*

(2) *R is an LW left SF-ring.*

(3) *R is an LW right SF-ring.*

Corolary 2.13([11], Theorem 6)**.** *The following conditions are equivalent:*

(1) *R is a strongly regular ring.*

(2) *R is an LW left GP-V-ring.*

(3) *R is an RW right GP-V-ring.*

Lemma 2.14([12], Theorem 1)**.** *An MERT right SF-ring is regular.*

Lemma 2.15([8], Proposition 2.2)**.** *An MERT left SF-ring is regular.*

Theorem 2.16. *If R is a right SF-ring whose maximal essential right ideals are weakly left ideals, then R is regular.*

Proof. Let $\overline{R} = R/soc(R_R)$. Then by Lemma 2.4, $\overline{R}$ is a right SF-ring. Let T be a maximal right ideal of $\overline{R}$. Then $T = M/soc(R_R)$ for some maximal right ideal M of R such that $soc(R_R) \subseteq M$. It is clear that M is an essential right ideal of R. By hypothesis, M is a weakly left ideal of R. Thus by Lemma 2.9, T is a weakly left ideal of $\overline{R}$. This implies that $\overline{R}$ is a ring whose maximal right ideals are weakly left ideals. Therefore by Theorem 2.10, $\overline{R}$ is strongly regular so that R is an MERT ring. Hence by Lemma 2.14, R is regular. □

Similarly, considering $\overline{R} = R/soc(_RR)$ and using Lemma 2.4, Lemma 2.9, Theorem 2.10 and the dual of Lemma 2.15, we can prove the following theorem:

Theorem 2.17. *If R is a right SF-ring whose maximal essential left ideals are weakly right ideals, then R is regular.*

3. SF-rings and W-ideals

Following [15], a left ideal L of a ring R is called a weak-ideal ($W-ideal$) if for every $0 \neq a \in L$, there exists some $n > 0$ such that $a^n \neq 0$ and $a^nR \subseteq L$. A right ideal K of R is defined similarly to be a $W-ideal$. By ([15], Example 1.2), the ring R of Example 1.4 is a ring in which $\{$ideals of $R\} \subsetneq \{$W-ideals of $R\}$.

In this section, we study the strong regularity of left (right) SF-rings via W-ideals.

For ease of reference, we first quote the following two lemmas.

Lemma 3.1([4], Lemma 3.14)**.** *Let L be a left ideal of R. Then R/L is a flat left R-module if and only if for every $a \in L$, there exists some $b \in L$ such that $a = ab$.*

Lemma 3.2([13], Lemma 9)**.** *Let R be a left or a right SF-ring. If $R/Z(_RR)$ is a reduced ring, then R is strongly regular.*

We now state and prove the main result of this section as follows:

Theorem 3.3. *The following conditions are equivalent for a ring R:*

(1) *R is strongly regular.*

(2) *R is a left SF-ring such that every non-zero left ideal of R contains a non-zero left ideal of R which is a W-ideal.*

(3) *R is a left SF-ring such that every non-zero right ideal of R contains a non-zero right ideal of R which is a W-ideal.*

(4) *R is a right SF-ring such that every non-zero left ideal of R contains a non-zero left ideal of R which is a W-ideal.*

(5) *R is a right SF-ring such that every non-zero right ideal of R contains a non-zero right ideal of R which is a W-ideal.*

Proof. It is well known that (1) $\Longrightarrow$ (2), (3), (4) and (5).
(2) $\Longrightarrow$ (1). Suppose $0 \neq a \in Z(_RR)$, then $l(a)$ is an essential left ideal of R. By hypothesis, there exists a non-zero left ideal I of R which is a W-ideal and $I \subseteq Ra$. Let $0 \neq b \in I$, then there exists some $n > 0$ such that $b^n \neq 0$ and $b^nR \subseteq I \subseteq Ra$. We claim that $Ra + r(b^nR) = R$. If this is not true, then there exists a maximal left ideal M of R such that $Ra + r(b^nR) \subseteq M$. Then by Lemma 3.1, there exists some $c \in M$ such that $a = ac$, that is $1 - c \in r(a) \subseteq r(b^nR) \subseteq M$, whence $1 \in M$, a contradiction. Therefore $Ra + r(b^nR) = R$ which implies $xa + y = 1$ for some $x \in R$, $y \in r(b^nR)$ yielding $b^nxa = b^n$. As $xa \in Z(_RR)$, $l(xa) \cap Rb^n \neq 0$. Let $0 \neq zb^n \in l(xa) \cap Rb^n$. Then $zb^n = zb^nxa = 0$, a contradiction to $zb^n \neq 0$.

Therefore $Z({}_RR) = 0$.

We now prove that R is reduced. Suppose $0 \neq a \in R$ such that $a^2 = 0$. Then $a \notin Z({}_RR)$. So there exists a non-zero left ideal L of R such that $l(a) \oplus L$ is an essential left ideal of R. By hypothesis, there exists a non-zero left ideal T of R which is a W-ideal and $T \subseteq L$. Let $0 \neq b \in T$. There exists a positive integer n such that $b^n \neq 0$ and $b^nR \subseteq T$. Then

$$b^nRa \subseteq b^nR \cap Ra \subseteq T \cap l(a) \subseteq L \cap l(a) = 0.$$

This implies $b^nR \subseteq L \cap l(a) = 0$ implying $b^n = 0$. This contradiction shows that R is reduced and hence by Lemma 2.5, R is strongly regular.
Similarly (5) $\Longrightarrow$ (1).
(3) $\Longrightarrow$ (1). Suppose $a \notin Z(R_R)$ such that $a^2 \in Z(R_R)$, then there exists a non-zero right ideal K of R such that $r(a) \oplus K \subseteq r(a^2)$. By hypothesis, there exists a non-zero right ideal I of R such that $I \subseteq K$ and I is a W-ideal. Let $0 \neq b \in I$. There exists some $n > 0$ such that $b^n \neq 0$ and $Rb^n \subseteq I \subseteq K$. Hence

$$aRb^n \subseteq r(a) \cap Rb^n \subseteq r(a) \cap K = 0.$$

Therefore $Rb^n \subseteq r(a) \cap K = 0$ which implies that $b^n = 0$, a contradiction to $b^n \neq 0$. Thus $R/Z(R_R)$ is reduced and therefore by dual of Lemma 3.2, R is strongly regular.
Similarly (4) $\Longrightarrow$ (1). □

Corollary 3.4([12], Theorem 3)**.** *The following conditions are equivalent:*

(1) *R is a strongly regular ring.*

(2) *R is a strongly left bounded left SF-ring.*

(3) *R is a strongly right bounded left SF-ring.*

(4) *R is a strongly left bounded right SF-ring.*

(5) *R is a strongly right bounded right SF-ring.*

4. SF-rings and Division Rings

In this section, we give some conditions for a left SF-ring to be a division ring. Our first main result of this section is the following:

Theorem 4.1. *Let R be a left SF-ring such that $l(x)$ is an essential left ideal of R for every right zero divisor x of R, then R is a division ring.*

Proof. Suppose $a^2 \in Z({}_RR)$ such that $a \notin Z({}_RR)$. If $Rr(a) + Z({}_RR) = R$, then $a = ba + \sum r_i t_i a$, where $b \in Z({}_RR)$, $r_i \in R$, $t_i \in r(a)$, $t_i a \neq 0$ for each i. Now, for each i, $(t_i a)^2 = t_i(at_i)a = 0$ which implies $t_i a \in l(t_i a)$ so that $l(t_i a) \neq 0$. By hypothesis, $l(t_i a)$ is an essential left ideal of R, that is $t_i a \in Z({}_RR)$. Therefore it follows that $a \in Z({}_RR)$ which contradicts that $a \notin Z({}_RR)$. Hence $Rr(a) + Z({}_RR) \neq R$ and so there exists a maximal left ideal M of R such that $Rr(a) + Z({}_RR) \subseteq M$. Since R is a left SF-ring and $a^2 \in Z({}_RR) \subseteq M$, by Lemma 3.1, there exists some $c \in M$

such that $a^2 = a^2c$, that is $a - ac \in r(a) \subseteq M$, whence $a \in M$. Hence again by Lemma 3.1, there exists some $d \in M$ such that $a = ad$. Then $1 - d \in r(a) \subseteq M$ so that $1 \in M$, contradicting $M \neq R$. Therefore $R/Z({}_RR)$ is reduced and hence by Lemma 3.2, R is strongly regular. Since a strongly regular ring is left non-singular, it follows that $Z({}_RR) = 0$ and therefore by hypothesis, $l(w) = 0$ for all $0 \neq w \in R$. Let $0 \neq u \in R$. If $Ru \neq R$, let L be a maximal left ideal of R such that $Ru \subseteq L$. As R is a left SF-ring, by Lemma 3.1, there exists some $v \in L$ such that $u = uv$, that is $u \in l((1 - v))$. Since $u \neq 0$, it follows that $1 - v = 0$, that is $v = 1$. Therefore $1 \in L$ which is a contradiction to $L \neq R$. Thus $Ru = R$. This proves that R is a division ring. □

Corollary 4.2. *The following conditions are equivalent for a ring R:*

(1) *R is a division ring.*

(2) *R is a left uniform, left SF-ring.*

Lemma 4.3([8], Lemma 1.2 (2))**.** *If R is a left SF-ring, then $Z(R_R) \subseteq J(R)$.*

We now give another main result of this section.

Theorem 4.4. *Let R be a left SF-ring such that $r(x)$ is an essential right ideal of R for every left zero divisor x of R, then R is a division ring.*

Proof. Let $a^2 \in J(R)$ such that $a \notin J(R)$. If $Rr(a) + J(R) = R$, then $a = ba + \sum r_it_ia$ where $b \in J(R)$, $r_i \in R$, $t_i \in r(a)$ and $t_ia \neq 0$ for each i. Now for each i, $(t_ia)^2 = t_i(at_i)a = 0$ which implies $t_ia \in r(t_ia)$ so that $r(t_ia) \neq 0$. By hypothesis, $t_ia \in Z(R_R)$. Then by Lemma 4.3, $t_ia \in J(R)$. Therefore it follows that $a \in J(R)$ which is a contradiction. Hence $Rr(a)+J(R) \neq R$ so that there exists a maximal left ideal M of R such that $Rr(a) + J(R) \subseteq M$. Since $a^2 \in J(R) \subseteq M$, by Lemma 3.1, there exists some $c \in M$ such that $a^2 = a^2c$. Following the proof of Theorem 4.1, we get a contradiction. This proves that $R/J(R)$ is reduced. Hence by Lemma 2.4 and Lemma 2.5, $R/J(R)$ is strongly regular so that R is left quasi duo. Therefore by Lemma 2.6, R is strongly regular. This yields $Z(R_R) = 0$, since a strongly regular ring is right non-singular. Then by hypothesis, it follows that $r(w) = 0$ for all $0 \neq w \in R$. Let $0 \neq u \in R$. If $Ru \neq R$, there exists a maximal left ideal L of R such that $Ru \subseteq L$. Since R is left SF-ring, there exists some $v \in R$ such that $u(1-v) = 0$. Then it follows that $u = 0$ or $v = 1$ which is a contradiction. Therefore $Ru = R$ and R is a division ring. □

Corollary 4.5([10], Theorem 6)**.** *The following conditions are equivalent for a ring R:*

(1) *R is a division ring.*

(2) *R is a left uniform, right SF-ring.*

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