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ON SHARP BOUNDS FOR MARGINAL DENSITIES OF PRODUCT MEASURES

GALYNA LIVSHYTS, GRIGORIS PAOURIS, PETER PIVOVAROV

Abstract. We discuss optimal constants in a recent result of Rudelson and Vershynin on marginal densities. We show that if f is a probability density on $\mathbb{R}^n$ of the form $f(x) = \prod_{i=1}^{n} f_i(x_i)$, where each f_i is a density on $\mathbb{R}$, say bounded by one, then the density of any marginal $\pi_E(f)$ is bounded by $2^{k/2}$, where k is the dimension of E. The proof relies on an adaptation of Ball's approach to cube slicing, carried out for functions. Motivated by inequalities for dual affine quermassintegrals, we also prove an isoperimetric inequality for certain averages of the marginals of such f for which the cube is the extremal case.

1. Introduction

In this note we present an alternate approach to a recent theorem of Rudelson and Vershynin on marginal densities of product measures [15]. To fix the notation, if f is a probability density on Euclidean space $\mathbb{R}^n$ and E is a subspace, the marginal density of f on E is defined by

$$\pi_E(f)(x) = \int_{E^\perp + x} f(y)dy \quad (x \in E).$$

In [15], it is proved that if $f(x) = \prod_{i=1}^{n} f_i(x_i)$, where each f_i is a density on $\mathbb{R}$, bounded by 1, then for any $k \in \{1, \ldots, n-1\}$, and any subspace E of dimension k,

$$\|\pi_E(f)\|_{L^\infty(E)}^{1/k} \leq C, \tag{1.1}$$

where C is an absolute constant.

In [15], it is pointed out that when $k = 1$, the constant C in (1.1) may be taken to be $\sqrt{2}$. This follows from a theorem of Rogozin [14], which reduces the problem to $f = \mathbb{1}_{Q_n}$ where $Q_n = [-1/2, 1/2]^n$ is the unit cube, together with Ball's theorem [1], [2] on slices of Q_n. More precisely, one can formulate Rogozin's Theorem as follows: if θ is a unit vector with linear span $[\theta]$, then

$$\left\|\pi_{[\theta]}(f)\right\|_{L^\infty([\theta])} \leq \left\|\pi_{[\theta]}(\mathbb{1}_{Q_n})\right\|_{L^\infty([\theta])} \tag{1.2}$$

for any f in the class

$$\mathcal{F}_n = \left\{ f(x) = \prod_{i=1}^{n} f_i(x_i) : \|f_i\|_{L^\infty(\mathbb{R})} \leq 1 = \|f_i\|_{L^1(\mathbb{R})}, \ i = 1, \ldots, n \right\}.$$

By definition of the marginal density and the Brunn-Minkowski inequality,

$$\begin{aligned} \left\| \pi_{[\theta]}(\mathbb{1}_{Q_n}) \right\|_{L^\infty([\theta])} &= \max_{x \in [\theta]} |Q_n \cap (\theta^\perp + x)|_{n-1} \\ &= |Q_n \cap \theta^\perp|_{n-1}, \end{aligned}$$

where $|\cdot|_{n-1}$ denotes $(n-1)$-dimensional Lebesgue measure. Ball's theorem gives $|Q_n \cap \theta^\perp|_{n-1} \leq \sqrt{2}$, which shows $C = \sqrt{2}$ works in (1.1).

Since Ball's theorem holds in higher dimensions, i.e.,

$$\max_{E \in G_{n,k}} |Q_n \cap E^\perp|_{n-k}^{1/k} \leq \sqrt{2} \quad (k \geq 1), \tag{1.3}$$

where $G_{n,k}$ is the Grassmannian of all k-dimensional subspaces of $\mathbb{R}^n$, it is natural to expect that $C = \sqrt{2}$ works in (1.1) for all $k > 1$. However, in the absence of a multi-dimensional analogue of Rogozin's result (1.2), the authors of [15] prove (1.1) with an absolute constant C via different means.

Our goal is to show that one can determine the optimal C for suitable $k > 1$ directly by adapting Ball's arguments giving (1.3), and a related estimate, to the functional setting. In particular, we prove the following theorem.

Theorem 1.1. *Let $1 \leq k < n$ and $f \in \mathcal{F}_n$. Then for each $E \in G_{n,k}$,*

$$\|\pi_E(f)\|_{L^\infty(E)} \leq \min \left(\left(\frac{n}{n-k} \right)^{\frac{n-k}{2}}, 2^{k/2} \right). \tag{1.4}$$

As noted in [2], if $f = \mathbb{1}_{Q_n}$, the bound $\left(\frac{n}{n-k}\right)^{(n-k)/2}$ is achieved when $n-k$ divides n and $E_0 \in G_{n,k}$ is chosen so that $Q_n \cap E_0^\perp$ is a cube of suitable volume; note that $\left(\frac{n}{n-k}\right)^{\frac{n-k}{2}} \leq e^{k/2}$. When $k \leq n/2$, the bound $2^{k/2}$ is sharp when $Q_n \cap E_0^\perp$ is a box of suitable volume. Thus for such k, Theorem 1.1 reads

$$\sup_{E \in G_{n,k}} \|\pi_E(f)\|_{L^\infty(E)} \leq \sup_{E \in G_{n,k}} \|\pi_E(\mathbb{1}_{Q_n})\|_{L^\infty(E)} \quad (f \in \mathcal{F}_n). \tag{1.5}$$

In terms of random vectors, if $X \in \mathbb{R}^n$ is distributed according to f, then the density of the orthogonal projection $P_E X$ of X onto E is simply $\pi_E(f)$. Thus if X has density $f \in \mathcal{F}_n$ and Y has density $\mathbb{1}_{Q_n}$, the density of $P_E X$ is uniformly bounded above by the value of the density of $P_{E_0} Y$ at the origin (with E_0 chosen as above).

For another probabilistic consequence, note that (1.1) implies the following small-ball probability: for each $z \in E$,

$$\mathbb{P}\left(|P_E X - z| \leq \varepsilon\sqrt{k}\right) \leq (C\sqrt{2e\pi}\varepsilon)^k \quad (\varepsilon > 0), \tag{1.6}$$

which was part of the motivation in [15].

Ball's approach to cube slicing [1], [2] has been adapted to a variety of related problems. We mention just a sample and refer the reader to the references therein; see Barthe's multidimensional version [3] of the Brascamp-Lieb inequality [4] and its normalized form; the use of the latter by Gluskin [8] for slices of products of measurable sets; Koldobsky and König [10] for problems involving measures other than volume; Brzezinski [6] for recent work on slices of products of Euclidean balls.

Recently, bounds for marginals of arbitrary bounded densities have been found by S. Dann and the second and third-named authors [7]. They obtain extremal inequalities for certain averages, e.g., for any k and $f : \mathbb{R}^n \to \mathbb{R}^+$ satisfying $\|f\|_{L^\infty(\mathbb{R}^n)} \leq 1 = \|f\|_{L^1(\mathbb{R}^n)}$ and $f(0) = \|f\|_{L^\infty(\mathbb{R}^n)}$, one has

$$\int_{G_{n,k}} \pi_E(f)(0)^n d\mu_{n,k}(E) \leq \int_{G_{n,k}} \pi_E(\mathbb{1}_{D_n})(0)^n d\mu_{n,k}(E), \tag{1.7}$$

where $\mu_{n,k}$ is the Haar probability measure on $G_{n,k}$ and D_n is the Euclidean ball in $\mathbb{R}^n$ of volume one centered at the origin.

Using an idea from the proof of Theorem 1.1, we also obtain the following strengthening of (1.7) within the class $\mathcal{F}_n$.

Proposition 1.2. *Let $1 \leq k < n$ and $f \in \mathcal{F}_n$. Then*

$$\int_{G_{n,k}} \pi_E(f)(0)^n d\mu_{n,k}(E) \leq \int_{G_{n,k}} \pi_E(\mathbb{1}_{Q_n})(0)^n d\mu_{n,k}(E). \tag{1.8}$$

The latter can be seen as a type of "average" domination of marginals of $\mathbb{1}_{Q_n}$ over those of $f \in \mathcal{F}_n$. This contrasts with the pointwise domination of Rogozin's Theorem (1.2) when $k = 1$ and the worst-case comparison in (1.5) when $k \leq n/2$ or $n - k$ divides n. As in [7], inequality (1.8) is another step in extending results about dual affine quermassintegrals (we recall the definition in §4) from convex sets to functions in order to quantify characteristics of high-dimensional probability measures.

2. Preliminaries

The setting is Euclidean space $\mathbb{R}^n$ with the standard basis $\{e_1, \ldots, e_n\}$, usual inner product $\langle \cdot, \cdot \rangle$, Euclidean norm $|\cdot|$ and unit sphere S^{n-1}. We reserve $|\cdot|_k$ for k-dimensional Lebesgue measure; the subscript k will be omitted if the context is clear. We denote the positive reals by $\mathbb{R}^+$.

Recall that for an integrable function $g : \mathbb{R} \to \mathbb{R}^+$ its symmetric decreasing rearrangement g^* is defined by

$$g^*(x) = \int_0^\infty \mathbf{1}_{\{g>t\}^*}(x)dt,$$

where $[a, b]^* = [-\frac{b-a}{2}, \frac{b-a}{2}]$. This can be compared with the layer-cake representation of g:

$$g(x) = \int_0^\infty \mathbf{1}_{\{g>t\}}(x)dt = \int_0^{\|g\|_{L^\infty(\mathbb{R})}} \mathbf{1}_{\{g>t\}}(x)dt. \tag{2.1}$$

Then g and g^* are equimeasurable, i.e., $|\{g > t\}| = |\{g^* > t\}|$ for each $t > 0$. In particular, $\|g\|_{L^p(\mathbb{R})} = \|g^*\|_{L^p(\mathbb{R})}$ for $1 \leq p \leq \infty$.

3. Adapting Ball's arguments

We start with the following basic fact used in [1], [2], proved for completeness.

Lemma 3.1. *Let $b = (b_1, \ldots, b_n) \in S^{n-1}$ and let A be a measurable subset of $b^\perp$ with $\dim(\mathrm{span}(A)) = k \in \{1, \ldots, n-1\}$. Then for each $1 \leq i \leq n$,*

$$|b_i||A|_k \leq |P_i(A)|_k,$$

where $P_i = P_{e_i^\perp}$ is the orthogonal projection onto $e_i^\perp$.

Proof. We may assume that $P_i : b^\perp \to e_i^\perp$ is injective (otherwise $b_i = 0$ and the inequality is trivial). We may also assume that $b \neq \pm e_i$ (otherwise equality holds). Let $v_1, \ldots, v_k$ be an orthonormal basis of $\mathrm{span}(A)$ with $v_1 = \frac{1}{|P_i b|} e_i - \frac{b_i}{|P_i b|} b$ and $v_2, \ldots, v_k$ orthogonal to both e_i and b. Then $P_i(v_i) = v_i$ for $i \geq 2$, and $|P_i(v_1)| = |b_i|$. Consider the k-dimensional cube $C = \prod_{i=1}^k [0, v_i] \subset \mathrm{span}(A)$. Then $P_i C$ is a k-dimensional box in $e_i^\perp$ with the sides $|b_i|, 1, ..., 1$. Hence $|P_i(C)|_k = |b_i||C|_k$. Thus the lemma is true for coordinate cubes in $\mathrm{span}(A)$. The inequality follows by approximating A by disjoint cubes. Since $P_i|_{b^\perp}$ is injective, the images of such cubes under P_i remain disjoint. □

The first ingredient in Ball's approach is the following integral inequality [1].

Theorem 3.2. *For every $p \geq 2$,*

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left|\frac{\sin t}{t}\right|^p dt \leq \sqrt{\frac{2}{p}}. \tag{3.1}$$

The second ingredient is Ball's normalized form [2] of the Brascamp-Lieb inequality [4].

Theorem 3.3. *Let $u_1, ..., u_n$ be unit vectors in $\mathbb{R}^n$, $m \geq n$, and $c_1, ..., c_m > 0$ satisfying $\sum_1^m c_i u_i \otimes u_i = I_n$. Then for integrable functions $f_1, ..., f_m : \mathbb{R} \to [0, \infty)$,*

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(\langle u_i, x\rangle)^{c_i} dx \leq \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i\right)^{c_i}. \tag{3.2}$$

There is equality if the f_i's are identical Gaussian densities.

We will also use the following standard fact, proved for the convenience of the reader.

Lemma 3.4. *Let $1 \leq k < n$ and $E \in G_{n,k}$. Then there exist vectors $w_1, \dots, w_n$ in $\mathbb{R}^{n-k} = \mathrm{span}\{e_1, \dots, e_{n-k}\}$ such that $I_{n-k} = \sum_{i=1}^n w_i \otimes w_i$ and for any integrable function $f(x) = \prod_{i=1}^n f_i(x_i)$ with $f_i : \mathbb{R} \to [0, \infty)$,*

$$\pi_E(f)(0) = \int_{\mathbb{R}^{n-k}} \prod_{i=1}^n f_i(\langle y, w_i\rangle) dy. \tag{3.3}$$

Proof. Let $v_1, \cdots, v_{n-k} \in \mathbb{R}^n$ be an orthonormal basis of $E^\perp$ and let w_i be defined by

$$w_i := (\langle v_1, e_i\rangle, \cdots, \langle v_{n-k}, e_i\rangle), \quad 1 \leq i \leq n.$$

In matrix terms, if V is the $n \times (n-k)$ matrix with columns $v_1, \dots, v_{n-k}$, then $w_i = V^T e_i$, where V^T is the transpose of V. Then

$$\sum_{i=1}^n w_i w_i^T = \sum_{i=1}^n V^T e_i e_i^T V = I_{\mathbb{R}^{n-k}}$$

and

$$\begin{aligned}
\pi_E(f)(0) &= \int_{E^\perp} f(y) dy = \int_{\mathbb{R}^{n-k}} f\left(\sum_{i=1}^{n-k} y_i v_i\right) dy \\
&= \int_{\mathbb{R}^{n-k}} \prod_{i=1}^n f_i(\langle V y, e_i\rangle) dy = \int_{\mathbb{R}^{n-k}} \prod_{i=1}^n f_i(\langle y, w_i\rangle) dy.
\end{aligned}$$

□

The following two propositions extend Ball's estimates on slices of the cube to coordinate boxes. It is essential that we obtain estimates that are uniform among all such boxes. The proofs draw heavily on [2].

Proposition 3.5. *Let $1 \leq k < n$ and $H \in G_{n,n-k}$. Then there exists $\{\beta_i\}_{i=1}^n \subset [0,1]$ such that for any $z_1, \ldots, z_n \in \mathbb{R}^+$, the box $B = \prod_{i=1}^n [-z_i/2, z_i/2]$ satisfies*

$$|B \cap H| \leq \left(\frac{n}{n-k}\right)^{\frac{n-k}{2}} \prod_{i=1}^n z_i^{\beta_i}. \tag{3.4}$$

Proof. Let $w_1, \ldots, w_n$ be as in Lemma 3.4 (with k and $n-k$ interchanged). For $i = 1, \ldots, n$, let $u_i = w_i/|w_i|$ and $a_i = |w_i|$ and $\beta_i = a_i^2$. For $z_1, \ldots, z_n \in \mathbb{R}^+$, we apply (3.3) with $f = \mathbb{1}_B$ and $E = H^\perp$ to get

$$\begin{aligned} |B \cap H| &= \pi_{H^\perp}(\mathbb{1}_B)(0) \\ &= \int_{\mathbb{R}^k} \prod_{i=1}^n \mathbf{1}_{[-\frac{z_i}{2}, \frac{z_i}{2}]}(\langle y, w_i \rangle) dy \\ &= \int_{\mathbb{R}^k} \prod_{i=1}^n \mathbf{1}_{[-\frac{z_i}{2a_i}, \frac{z_i}{2a_i}]}(\langle y, u_i \rangle) dy. \end{aligned}$$

Using Theorem 3.3, the latter is at most

$$\prod_{i=1}^n \left(\int_{\mathbb{R}} \mathbf{1}_{[-\frac{z_i}{2a_i}, \frac{z_i}{2a_i}]}(t) dt \right)^{a_i^2} = \prod_{i=1}^n \left(\frac{z_i}{a_i}\right)^{a_i^2}. \tag{3.5}$$

As in [2, Proof of Proposition 4], we use the bound

$$\prod_{i=1}^n a_i^{-a_i^2} \leq \left(\frac{n}{n-k}\right)^{\frac{n-k}{2}},$$

from which the lemma follows. □

Proposition 3.6. *Let $1 \leq k < n$ and $H \in G_{n,n-k}$. Then there exists $\{\beta_j\}_{j=1}^n \subset [0,1]$ such that for any $z_1, \ldots, z_n \in \mathbb{R}^+$, the box $B = \prod_{j=1}^n [-z_j/2, z_j/2]$ satisfies*

$$|B \cap H| \leq 2^{k/2} \prod_{j=1}^n z_j^{\beta_j}. \tag{3.6}$$

Proof. Assume first that all unit vectors $b = (b_1, \ldots, b_n) \in H^\perp$, satisfy $b_i \leq \frac{1}{\sqrt{2}}$ for each $i = 1, \ldots, n$. Let $\widetilde{P} = P_{H^\perp}$ be the orthogonal projection onto $H^\perp$. For $i = 1, \ldots, n$, let $u_i = \frac{\widetilde{P} e_i}{|\widetilde{P} e_i|}$ and $a_i = |\widetilde{P} e_i|$. Note that

$$(i) \sum_{i=1}^{n} a_i^2 u_i \otimes u_i = I_{H^\perp}, \qquad (ii) \sum_{i=1}^{n} a_i^2 = k. \tag{3.7}$$

By our assumption, all $a_i \le \frac{1}{\sqrt{2}}$, since a_i is the i-th coordinate of the unit vector u_i in $H^\perp$.

Assume for the time being that $z_1, \ldots, z_n$ are fixed and satisfy $|B| = \prod_{j=1}^n z_j = 1$. Let $X = (X_1, \ldots, X_n)$ be a random vector with density $\mathbb{1}_B$ and $Y = (Y_1, \ldots, Y_n)$ be a random vector with density $\mathbb{1}_{Q_n}$. The characteristic function $\Phi : H^\perp \to \mathbb{R}$ of $\widetilde{P}X$ satisfies

$$\begin{aligned}
\Phi(w) &= \mathbb{E}\exp\left(i\langle w, \widetilde{P}X\rangle\right) \\
&= \mathbb{E}\exp\left(i\sum_{j=1}^{n} X_j a_j \langle w, u_j\rangle\right) \\
&= \mathbb{E}\exp\left(i\sum_{j=1}^{n} Y_j z_j a_j \langle w, u_j\rangle\right) \\
&= \prod_{j=1}^{n} \frac{2\sin\frac{1}{2} z_j a_j \langle w, u_j\rangle}{z_j a_j \langle w, u_j\rangle}.
\end{aligned}$$

By the Fourier inversion formula,

$$\begin{aligned}
|B \cap H| &= \pi_{H^\perp}(\mathbb{1}_B)(0) \\
&= \frac{1}{(2\pi)^k} \int_{H^\perp} \Phi(w)\,dw \\
&= \frac{1}{\pi^k} \int_{H^\perp} \prod_{j=1}^{n} \frac{\sin z_j a_j \langle w, u_j\rangle}{z_j a_j \langle w, u_j\rangle} dw \\
&\le \frac{1}{\pi^k} \int_{H^\perp} \prod_{j=1}^{n} \left|\frac{\sin z_j a_j \langle w, u_j\rangle}{z_j a_j \langle w, u_j\rangle}\right| dw \\
&= \frac{1}{\pi^k} \int_{H^\perp} \prod_{j=1}^{n} \Phi_j(\langle w, u_j\rangle)\,dw,
\end{aligned}$$

where $\Phi_j : \mathbb{R} \to [0, \infty)$ is defined by $\Phi_j(t) = \left|\frac{\sin z_j a_j t}{z_j a_j t}\right|$. Consequently, Theorem 3.3 with $c_j = \frac{1}{a_j^2}$ implies that $|B \cap H|$ is at most

$$\frac{1}{\pi^k} \prod_{j=1}^{n} \left(\int_{\mathbb{R}} \Phi_j(t)^{\frac{1}{a_j^2}} dt\right)^{a_j^2} = \prod_{j=1}^{n} \left(\frac{1}{\pi} \int_{\mathbb{R}} \Phi_j(t)^{\frac{1}{a_j^2}} dt\right)^{a_j^2} \tag{3.8}$$

(cf. (3.7)). Finally, we use Theorem 3.2:

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{R}}\Phi_j(t)^{\frac{1}{a_j^2}}\,dt &= \frac{1}{\pi}\int_{\mathbb{R}}\left|\frac{\sin z_j a_j t}{z_j a_j t}\right|^{\frac{1}{a_j^2}}dt\\ &= \frac{1}{z_j a_j}\left(\frac{1}{\pi}\int_{\mathbb{R}}\left|\frac{\sin t}{t}\right|^{\frac{1}{a_j^2}}dt\right)\\ &\le \frac{\sqrt{2}}{z_j}.\end{aligned}$$

In summary, for any $z_1,\dots,z_n\in\mathbb{R}^+$ with $|B|=\prod_{j=1}^n z_j=1$, we have

$$|B\cap H|\le\prod_{j=1}^n\left(\frac{\sqrt{2}}{z_j}\right)^{a_j^2}=2^{k/2}\prod_{j=1}^n z_j^{-a_j^2}. \tag{3.9}$$

For an arbitrary box $B=\prod_{j=1}^n[-\frac{z_j}{2},\frac{z_j}{2}]$, we get via scaling that

$$|B\cap H|\le 2^{k/2}\prod_{j=1}^n z_j\prod_{j=1}^n z_j^{-a_j^2}=2^{k/2}\prod_{j=1}^n z_j^{\beta_j}, \tag{3.10}$$

where $\beta_j=1-a_j^2$. Note that we assumed $a_j\le\frac{1}{\sqrt{2}}$, so in fact $\beta_j\in[\frac{1}{2},1]$.

Suppose now that there exists a unit vector $b\in H^\perp$ such that $b_i\ge\frac{1}{\sqrt{2}}$ for some i. By induction, assume the lemma is true for all dimensions at most $n-1$ and for all k. For $z_1,\dots,z_n\in\mathbb{R}^+$, note that the cylinder

$$C=\left\{x\in\mathbb{R}^n:\,|x_j|\le\frac{z_j}{2}\ \ \forall j\ne i\right\}$$

satisfies $|B\cap H|\le|C\cap H|$. By Lemma 3.1,

$$|C\cap H|\le\frac{1}{b_i}|P_i(C\cap H)|\le\sqrt{2}|\widetilde{B}\cap\widetilde{H}|, \tag{3.11}$$

where $\widetilde{B}$ is an $(n-1)$ dimensional box with sides $\{z_j\}_{j\ne i}$ and $\widetilde{H}$ is a $(k-1)$-codimensional subspace in $\mathbb{R}^{n-1}$. If $k=1$, then

$$|\widetilde{B}\cap\widetilde{H}|=|\widetilde{B}|=\prod_{j\ne i}z_j,$$

and thus

$$|B\cap H|\le\sqrt{2}\prod_{j\ne i}z_j,$$

hence the lemma holds. If $k\ge 2$, we use the inductive hypothesis: there exists $\{\beta_j\}_{j\ne i}\subset[0,1]$ such that

$$|\widetilde{B}\cap\widetilde{H}|\le 2^{(k-1)/2}\prod_{j\ne i}z_j^{\beta_j}.$$

Using (3.11), we conclude that

$$|B \cap H| \le 2^{k/2} \prod_{j \neq i} z_j^{\beta_j}.$$

□

4. Proofs of Theorem 1.1 and Proposition 1.2

Proof of Theorem 1.1. By translating if necessary, we may assume that

$$\|\pi_E(f)\|_{L^\infty(E)} = \pi_E(f)(0). \tag{4.1}$$

Let $w_1, \ldots, w_n$ be as in Lemma 3.4. Using (3.3), the rearrangement inequality of Rogers [13] and Brascamp-Lieb-Luttinger [5], and the layer-cake representation (2.1), we have

$$\begin{aligned} \pi_E(f)(0) &= \int_{\mathbb{R}^{n-k}} \prod_{i=1}^n f_i(\langle y, w_i \rangle) dy \\ &\le \int_{\mathbb{R}^{n-k}} \prod_{i=1}^n f_i^*(\langle y, w_i \rangle) dy \\ &= \int_0^1 \cdots \int_0^1 \int_{\mathbb{R}^{n-k}} \prod_{i=1}^n \mathbf{1}_{\{f_i^* > t_i\}}(\langle x, w_i \rangle) dx dt_1 \ldots dt_n. \end{aligned}$$

Write $dt = dt_1 \ldots dt_n$ and $M = \left(\frac{n}{n-k}\right)^{\frac{n-k}{2}}$. Since each f_i^* is symmetric and decreasing, the set $\{f_i^* > t_i\}$ is a symmetric interval. Consequently, we apply Proposition 3.5 with $z_i := |\{f_i^* > t_i\}|$, $i = 1, \ldots, n$, to get

$$\begin{aligned} \pi_E(f)(0) &\le M \int_{[0,1]^n} \prod_{i=1}^n |\{f_i^* > t_i\}|^{\beta_i} dt \\ &\le M \prod_{i=1}^n ||f_i^*||_{L^1(\mathbb{R})}^{\beta_i} \\ &\le M; \end{aligned}$$

here we used Fubini's theorem, Hölder's inequality and the fact that

$$\int_0^1 |\{f_i^* > t_i\}| dt_i = \|f_i^*\|_{L^1(\mathbb{R})} = \|f_i\|_{L^1(\mathbb{R})} = 1, \quad i = 1, \ldots, n. \tag{4.2}$$

Repeating the latter argument with $M = 2^{k/2}$, using Proposition 3.6, concludes the proof of the theorem. □

Before proving Proposition 1.2, we recall the following notion, proposed by Lutwak: if K is a convex body in $\mathbb{R}^n$, and $1 \leq k < n$, the dual affine quermassintegrals of K are defined by

$$\widetilde{\Phi}_k(K) = \frac{\omega_n}{\omega_k}\left(\int_{G_{n,k}} |K \cap E|^n d\mu_{n,k}(E)\right)^{\frac{1}{n}}, \tag{4.3}$$

where ω_n is the volume of the Euclidean ball in $\mathbb{R}^n$ of radius one; see [11], [12] for further background. Grinberg [9] proved that

$$\widetilde{\Phi}_k(K) = \widetilde{\Phi}_k(SK) \tag{4.4}$$

for each volume-preserving linear transformation S; see [7] for a generalization of the latter invariance property for functions.

Proof of Proposition 1.2. Let $w_1, \dots, w_n$ be as in Lemma 3.4. As in the proof of Theorem 1.1,

$$\begin{aligned}
\pi_E(f)(0) &\leq \int_0^1 \cdots \int_0^1 \int_{\mathbb{R}^{n-k}} \prod_{i=1}^n \mathbf{1}_{\{f_i^* > t_i\}}(\langle x, w_i \rangle) dx dt_1 \dots dt_n \\
&= \int_{[0,1]^n} |B(t) \cap E^{\perp}| dt,
\end{aligned}$$

where $B(t)$ is the origin-symmetric box with side-lengths $|\{f_i^* \geq t_i\}|$, $i = 1, \dots, n$. Thus

$$\begin{aligned}
&\int_{G_{n,k}} \pi_E(f)(0)^n d\mu_{n,k}(E) \\
&\quad \leq \int_{G_{n,k}} \left(\int_{[0,1]^n} |B(t) \cap E^{\perp}| dt\right)^n d\mu_{n,k}(E) \\
&\quad = \int_{G_{n,k}} \left(\int_{[0,1]^n} \left(\prod_{j=1}^n |\{f_j^* > t_j\}|\right)^{\frac{n-k}{n}} |\widetilde{B}(t) \cap E^{\perp}| dt\right)^n d\mu_{n,k}(E),
\end{aligned}$$

where $\widetilde{B}(t) = B(t)/|B(t)|^{1/n}$. Using Hölder's inequality (twice), along with (4.2) and (4.4), we get

$$\begin{aligned}
\int_{G_{n,k}} \pi_E(f)(0)^n d\mu_{n,k}(E) &\leq \int_{[0,1]^n} \int_{G_{n,k}} |\widetilde{B}(t) \cap E^{\perp}|^n d\mu_{n,k}(E) dt \\
&= \int_{G_{n,k}} |Q_n \cap E^{\perp}|^n d\mu_{n,k}(E).
\end{aligned}$$

□

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Galyna Livshyts: School of Mathematics, Georgia Institute of Technology, Atlanta GA 30319, glivshyts6@math.gatech.edu

Grigoris Paouris: Department of Mathematics (Mailstop 3368), Texas A&M University, College Station, TX 77843-3368, grigoris@math.tamu.edu

Peter Pivovarov: Mathematics Department, University of Missouri, Columbia, MO 65211, pivovarovp@missouri.edu