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ON SHARPNESS, APPLICATIONS AND GENERALIZATIONS OF SOME CARLEMAN TYPE INEQUALITIES

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Abstract. Carleman's inequality for Hilbert-Schmidt operators and its generalizations for Schatten-von Neumann operator ideals (see [7]) are shown to be sharp in a certain sense. Explicit classes of extremizing operators are found on which the generalized Carleman inequalities turn to asymptotic equalities. Applications are made to a priori estimation of the solutions of Fredholm and Volterra first- and second kind integral equations and to perturbation and error analysis. Some further generalizations are considered which extend the applications to singular integral equations, pseudo-differential equations and analytic functions of operator argument.

1. Introduction. In [7] upper bounds for the resolvent norm of a bounded linear operator T: $H \rightarrow H$, H Hilbert space, were obtained in terms of the so-called generalized Carleman inequalities with minimal information about the spectrum.

In this note we prove that all results obtained in [7] are *sharp* in a certain sense. Simultaneously we point out a remarkable class of extremizing operators (which we call pre-orthogonal operators) on which the sharpness assertions are attained. Sharp constants are found, too (see Section 3).

We also include some *applications* of our results (see Section 4). In particular, some important applications are obtained for the classical problems of deriving a priori estimates for the solutions of the Fredholm and Volterra operator equations. As a further development of this idea we obtain a priori estimates for expressions of the type $(\lambda I-T_1)^{-1}f_1-(\lambda I-T_2)^{-1}f_2$, which in the considered Hilbert-space case improve some corresponding earlier results in [24]. Of particular interest is the case where the compact operators are integral ones. In this case it is known that the Schatten-von Neumann quasinorms can be estimated very well by appropriate function quasinorms of the integral operator's kernel in Lebesgue and Besov spaces. These results, which, in their turn, may be regarded as generalizations of some classical results in the Hilbert-Schmidt theory, are in their final form due to Birman, Solomjak and Karadzhov (see [2], [3], [4], [12], [13], [14] and [15]). By combining these results with the main results in [7] it is possible to obtain useful a priori estimates for the solutions of classes of integral equations. In Section 4 we discuss briefly this possibility and present some examples of such applications (cf. also [8] and [9]). Another important application in this section

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is the explicit a priori estimation of regularity properties of the Fredholm resolvent's kernel in terms of regularity properties of the integral operator's kernel.

Section 5 is devoted to some possible *generalizations* of the main results in [7] and to concluding remarks. In order not to interrupt the presentations of our main results we use Section 2 to present all relevant preliminaries and a brief survey of the main results in [7] and we collect all proofs in Section 6.

2. Preliminaries.

2.1. Preliminary notation and statements. Let $L(H)$ denote the space of all bounded linear operators $T\colon H \to H$ with the usual uniform operator norm. The *approximation numbers* $s_n(T)$, $n \in N$, are defined as

$$s_n(T) = \inf\{\|T - T_n \mid L(H)\| : T_n \in L(H), \text{ rank } T_n < n\}\,.$$

There are several other equivalent ways to describe the approximation numbers in the Hilbert space case (for the general Banach space case these different descriptions give different approximation numbers in general, see [20]). Concerning general properties of s-numbers (e.g. their relations to the spectra of T and other related operators) and the operator ideals and inequalities we discuss below, we refer to [10], [17], [19], [20] and [22].

By $S_\infty = S_\infty(H)$ we denote the closed subspace of $L(H)$ consisting of all compact operators, i.e.,

$$S_\infty = S_\infty(H) = \left\{T \in L(H) : \lim_{n\to\infty} s_n = 0,\ \|T \mid S_\infty(H)\| = \|T \mid L(H)\|\right\}.$$

In the sequel we let p denote a real number such that $0 < p < \infty$. The *Schatten-von Neumann operator ideal* $S_p = S_p(H)$ is the complete quasinormed operator ideal in $S_\infty(H)$ defined by

$$S_p = \left\{T \in S_\infty(H) : \|T \mid S_p\| = \left(\sum_{n=1}^{\infty} (s_n(T))^p\right)^{1/p} < \infty\right\}.$$

See [23] and e.g. [11]; in particular S_1 and S_2 are the classical Fredholm and Hilbert-Schmidt operator ideals, respectively.

The *spectrum* of $T \in L(H)$ is denoted by $\sigma(T)$ and $C \setminus \sigma(T)$ is the corresponding *resolvent set*. In particular we remark that $T \in S_\infty$ has only a non-empty pointwise spectrum which, for infinite-dimensional H may have only 0 as a density point. Therefore the non-zero elements of $\sigma(T)$ are also referred to later on as the *eigenvalues* of T while 0 may, or may not, be an eigenvalue. Moreover, we recall that, due to Fredholm's alternative, the maximal invariant subspace corresponding to any non-zero eigenvalue of $T \in S_\infty$ is finite-dimensional and orthogonal to the maximal invariant subspace corresponding to any other eigenvalue of T.

We need the following relation between the eigenvalues and the s-numbers of operators:

LEMMA 2.1.1. *Let λ_i, $i\in \boldsymbol{N}$, denote the eigenvalues of $T\in S_p(H)$. If* $\dim H=n$, *then* $\prod_{i=1}^k |\lambda_i|\le\prod_{i=1}^k s_i$, $k=1,2,\dots$, *and if* $\operatorname{rank} T=n$, *then* $\prod_{i=1}^n|\lambda_i|=\prod_{i=1}^n s_i$.

Lemma 2.1.1 is proved, e.g., in [19, p. 232]. We also need the following generalized form of *Carleman's inequality* (see [6] or [11, pp. 1038, 1112–1114]):

THEOREM 2.1.2. *Let $T\in S_p=S_p(H)$ and let λ_k, $k=1,2,\dots$, be an enumeration of the eigenvalues of T, each repeated according to its multiplicity. Then there exists positive constants a_p and b_p depending only on p, such that*

$$\|\det_p(I-T/\lambda)(\lambda I-T)^{-1}\mid L(H)\|\le\frac{1}{|\lambda|}\exp\left(a_p\frac{\|T\mid S_p\|^p}{|\lambda|^p}+b_p\right),$$

where the generalized determinant $\det_p(I-T/\lambda)$ *is defined by*

$$\det_p(I-T/\lambda)=\prod_{k=1}^{\infty}(1-\lambda_k/\lambda)\exp(\varphi_p(\lambda_k/\lambda));\ \varphi_p(t)=0\ \textit{if}\ 0<p\le 1,\ \varphi_p(t)=\sum_{j=1}^{n}t^j/j$$

if $1<p<\infty$, $n=[p]$ *if* $p\notin\boldsymbol{N}$ *and* $n=p-1$ *if* $p\in\boldsymbol{N}$ *([p] denotes the integer part of p).*

For $T\in S_\infty(H)$, $\lambda\in\boldsymbol{C}\setminus\sigma(T)$, the operator $T(\lambda I-T)^{-1}=(\lambda I-T)^{-1}T$ will be called the *compact resolvent* of T (in contrast to the term *resolvent*, which is reserved for the operator $(\lambda I-T)^{-1}$). In the case where the compact resolvent of T is an integral operator, it is usually called the *Fredholm resolvent* of T.

REMARKS 2.1.3. (a) For the case $p=2$ we have $a_p=b_p=1/2$ and Theorem 2.1.2 coincides with the classical form of Carleman's inequality.

(b) In many sources the classical Carleman inequality is not correctly written (see e.g. [11, pp. 1023 and 1038]); the factor $|\lambda|$ in front of the exponential factor must be replaced by $1/|\lambda|$.

(c) The sharp constants a_p and b_p are not known in general. For the case of the classical Carleman inequality, $p=2$, we show in our Section 3 that the sharp constants are $a_2=1/2$ and $b_2=1/2$.

(d) It is possible to replace $n=n_p$ in Theorem 2.1.2 by any other integer $m\ge n$ but then the constants a_p and b_p must be replaced by constants $a_{p,m}$ and $b_{p,m}$, respectively, which both increase with the increase of m.

(e) For the case $0<p\le 1$ another definition of $\det_p(I-T/\lambda)$ is used in [11], namely $\det_p^*(I-T/\lambda)=\exp(\operatorname{tr}(\ln(I-T/\lambda)))$, where tr denotes the generalized trace functional and $\ln(I-T/\lambda)$ is the operator-valued logarithmic function defined by Dunford's representation (see [11, p. 1101] and [10]). However, it is easy to verify that both of $\det_p^*(I-T/\lambda)$ and $\det_p(I-T/\lambda)$ are entire functions of $\lambda\in\boldsymbol{C}$ satisfying the assumptions of Theorem 27.4.8 in [20] and by virtue of this theorem we can conclude that, in fact, $\det_p(I-T/\lambda)=\det_p^*(I-T/\lambda)$, for every $\lambda\in\boldsymbol{C}$.

(f) Some other well-known statements are also needed in the sequel. However, in order to keep this section reasonably short we do not formulate all these facts explicitly but in each such case we refer to some appropriate literature.

2.2. Generalized Carleman inequalities with minimal information about the spectrum. In this subsection the main results from [7] are summerized. In the sequel we let $d(\lambda, \sigma(T))$ denote the distance between λ and the spectrum of T and $0<p<\infty$.

THEOREM 2.2.1. *Assume that H is infinite-dimensional, $T\in S_p(H)$ and $\lambda\in \mathbf{C}$ is in the resolvent set of T. Then*

$$\|(\lambda I-T)^{-1}\,|\,L(H)\| \leq \frac{1}{d(\lambda, \sigma(T))} \exp\left(c_p \frac{\|T\,|\,S_p\|^p}{(d(\lambda, \sigma(T)))^p} + b_p\right), \tag{2.2.1}$$

where $c_p=\max\{2\alpha_p a_p, a_p+\beta_p\}$, a_p, b_p are the constants in Theorem 2.1.2, $\alpha_p=\max\{1, 2^{p-1}\}$, $\beta_p=0$ if $p\notin \mathbf{N}$ and $\beta_p=1/p$ if $p\in \mathbf{N}$.

In the case of finite-dimensional H we can estimate $\|T^{-1}\,|\,L(H)\|$ directly in the following way:

THEOREM 2.2.2. *Assume that H is finite-dimensional and that $T\in L(H)$ is invertible. Then*

$$\|T^{-1}\,|\,L(H)\| \leq \frac{1}{d(0, \sigma(T))} \exp\left(\frac{1}{ep} \frac{\|T\,|\,S_p\|^p}{(d(0, \sigma(T)))^p}\right). \tag{2.2.2}$$

The next result considers the case where partial information is available about $\sigma(T)$, i.e., we suppose that all eigenvalues of T with absolute values not less than $|\lambda|$ are known, together with their multiplicities. Furthermore, we consider the finite-dimensional subspace of H spanned over the invariant subspaces corresponding to the above eigenvalues of T and take the orthogonal complement. We note that the spectral radius r of the restriction of T on this orthogonal complement is strictly less than $|\lambda|$. We also suppose that r is known. Evidently this is equivalent to knowing the absolute value of one more eigenvalue of T, namely the biggest (or one of the biggest) in absolute value which is less than $|\lambda|$ (without necessarily knowing its multiplicity).

THEOREM 2.2.3. *Assume that $T\in S_p(H)$ and that $\lambda\in \mathbf{C}$ is in the resolvent set of T. Let λ_j be all points in $\sigma(T)$ with $|\lambda_j|\geq|\lambda|$ and let n_j, X_j, $j=1, 2, \ldots, k$, be the corresponding multiplicities and invariant subspaces, respectively. Moreover, let Y be the orthocomplement in H of the union of all X_j, and let r be the spectral radius of $T|_Y$ $(0\leq r<|\lambda|)$. Then*

$$\|(\lambda I-T)^{-1}\,|\,L(H)\| \leq \max\left(\left(\max_{j=1,\ldots,k} \frac{|\lambda|^{n_j-1}}{|\lambda-\lambda_j|^{n_j}}\right)\exp\left(d_p \frac{\|T\,|\,S_p\|^p}{|\lambda|^p} + b_p\right),\right.$$
$$\left.\frac{1}{|\lambda|-r} \exp\left(c_p \frac{\|T\,|\,S_p\|^p}{(|\lambda|-r)^p} + b_p\right)\right), \tag{2.2.3}$$

where c_p, b_p *are the constants from Theorem* 2.2.1, $d_p = a_p + \gamma_p$, a_p *is the constant in Theorem* 2.1.2, $\gamma_p = \sum_{k=1}^{m_p} 1/k$; $m_p = [p]$ *if* $p \notin \boldsymbol{N}$ *and* $m_p = p - 1$ *if* $p \in \boldsymbol{N}$.

As a limiting case of Theorem 2.2.3 we get the following useful estimate for the case where λ is outside of the spectral circle of T:

THEOREM 2.2.4. *Assume that* $T \in S_p(H)$, $0 < p < \infty$, *and let* $|\lambda| > r(T)$, *where* $r = r(T)$ *is the spectral radius of* T. *Then,*

$$\|(\lambda I - T)^{-1} \mid L(H)\| \leq \frac{1}{|\lambda| - r} \exp\left(c_p \frac{\|T \mid S_p\|^p}{(|\lambda| - r)^p} + b_p\right), \tag{2.2.4}$$

where c_p *and* b_p *are as in Theorem* 2.2.3.

REMARK 2.2.5. The estimate (2.2.4) can be useful because so far little is known about the behaviour of linear-operator resolvents in the ring included in the normal circle and outside the spectral circle. Recall that the expansion of the resolvent in a Neuman series is valid everywhere outside the spectral circle, but the well-known estimate (1.1) in [7] based on this expansion is only available outside the bigger normal circle.

3. Sharpness.

3.1. Definitions of the sharpness properties. In order to be able to formulate our sharpness results in suitable forms we need the following definitions and notation:

(a) Theorems 2.1.2 and 2.2.1–4 are called *exponentially sharp* (ES), if, for $p \in (0, \infty)$, $\lambda \in \boldsymbol{C}$, $\varepsilon \to 0$ and, for every Hilbert space H with infinite or sufficiently big finite dimension, there exists an operator $T = T(H, p, \lambda, \varepsilon)$, such that (i) $T \in S_p(H)$, (ii) $\lambda \in \boldsymbol{C} \setminus \sigma(T)$ and (iii) $\|(\lambda I - T)^{-1} \mid L(H)\| + \varepsilon$ is bounded from below by the same right-hand term as in the direct (upper) estimate, with embedding constants a'_p, b'_p, c'_p, depending only on p and such that $0 < a'_p \leq a_p$, $0 < b'_p \leq b_p$ and $0 < c'_p \leq c_p$.

(b) For a particular choice of $p \in (0, \infty)$, Theorems 2.1.2 and 2.2.1–4 are called *exponentially sharp with sharp constants* (ESSC), if (i) they are ES for the particular choice of p and (ii) the constants a'_p, b'_p and c'_p can be chosen equal to a_p, b_p and c_p, respectively.

(c) Theorem 2.2.3 is called *power-sharp* (PS), if, for every H, $p \in (0, \infty)$, there exists an operator $T \in S_p(H)$, $T = T(H, p)$, and $\lambda_0 \in \sigma(T)$ with $|\lambda_0| > r(T)$, such that, for some $\varepsilon > 0$, the following statements hold:

(i) $|\lambda - \lambda_0| < \varepsilon$, $\lambda \neq \lambda_0 \Rightarrow \lambda \in C \setminus \sigma(T)$ and the maximum in the right-hand side of (2.2.3) is attained on the term corresponding to λ_0,

(ii) there exists a constant $c_0 > 0$, $c_0 = c_0(H, p, T, \lambda_0, \varepsilon)$ such that $\|(\lambda I - T)^{-1} \mid L(H)\| \geq c_0 |\lambda - \lambda_0|^{-n_0}$ for every λ, $|\lambda - \lambda_0| < \varepsilon$, where n_0 is the multiplicity of λ_0.

3.2. Introductory remarks on sharpness. It is well-known that the available proofs of the generalized Carleman inequalities with complete information about the spectrum (Theorem 2.1.2) do not give sharp constants a_p and b_p. In the proofs of our main results in [7] we essentially use Theorem 2.1.2 in the case of quasinilpotent operators. Therefore, it is clear that sharp constants in our main results in [7] can be obtained only after that the sharp constants a_p, b_p in Theorem 2.1.2 are found. We intend to do this in a separate paper even for more general Schatten-von Neumann ideals than S_p (see Section 5 for more details). Here we only consider the case $p=2$ (the classical Carleman inequality) for which the sharp constants a_2, b_2 and c_2 are found below to be equal to 1/2 (see Theorem 3.4.2). For the general case $0<p<\infty$ we prove that Theorem 2.1.2, Theorem 2.2.1 and Theorem 2.2.4 are ES (see Theorem 3.4.1) and that Theorem 2.2.3 is ES or PS depending on the situation of λ with respect to $\sigma(T)$ (see Theorem 3.4.1 and Theorem 3.4.4). Finally, according to the fact that the proof of Theorem 2.2.2 is independent of Theorem 2.1.2, we can prove that the constant $(ep)^{-1}$ in this theorem is sharp for every $p\in(0, \infty)$ (see Theorem 3.4.3). In all cases concrete classes of compact operators are found for which the ES property holds. In the cases where even ESSP holds, these classes are shown to be the sharp ones on which ESSP is attained.

Before we present our sharpness statements we present the following heuristic reason why the results in Subsection 2.2 and Theorem 2.1.2 ought to have the ES property: Some of the assertions in [5] and [18] (cf. also [11, §§11.10–11.11]) show, roughly speaking, that, if the growth of the right-hand side of the estimates is less than exponential, then this implies additional properties of the invariant subspaces of T. Therefore, the right-hand sides of the estimates using minimal information about the spectral properties of T must necessarily have exponential behaviour with respect to $1/(d(\lambda, \sigma(T))$. This argument can be applied to Theorem 2.1.2, too, since it essentially coincides with Theorem 2.2.1 on quasinilpotent operators. These heuristic remarks indicate that one candidate for an appropriate operator is the "unicellular" Volterra integral operator $(Tf)(x)=\int_0^x f(\xi)d\xi$, where $H=L_2[0, 1]$, with the usual Lebesgue measure (see [5]). One shortcoming here is that $T\in S_p$ only for $p\in(1, \infty)$. However, we note that this difficulty can be overcome by considering the more general operators

$$(T_k f)(x)=\frac{1}{(k-1)!}\int_0^x (x-\xi)^{k-1} f(\xi)d\xi\,, \qquad k\in N\,.$$

By making some laborious but straightforward computations we find that these Volterra operators T_k belongs to $S_p(L_2[0, 1])$ for $p\in(1/k, \infty]$, and that, for $\lambda\in C\diagdown\{0\}$,

$$\|(\lambda I-T_k)^{-1} \mid L(L_2[0, 1])\|=O\left(|\lambda|^{-1/k}\exp\left(A_{p,k}\frac{\|T\mid S_p\|^p}{|\lambda|^{1/k}}\right)\right),$$

where $A_{p,k}$ is proportional to $\xi(pk)$ (Riemann's ξ-function). Thus, the "unicellular" operator T and its generalizations T_k have, indeed, exponential behaviour but it is still not good enough to prove the ES property of our results.

3.3. A remarkable subclass of compact operators. We will now define and study the basic properties of the operators for which the ES and ESSC properties of our results are attained. For a finite-rank operator T in $L(H)$ we consider the subspace $X_T=\{h\in H: Th=0\}^{\perp}$. Obviously, $\dim X_T=\operatorname{rank} T=n<\infty$.

The operator $T\in L(H)$, rank $T=n<\infty$, is called *pre-orthogonal* (PO), if the matrix $T|_{X_T}$ (with respect to any fixed basis in X_T) has $n-1$ vector-lines which are two-by-two orthogonal with respect to $\langle \cdot, \cdot \rangle_H$. In particular, T is called *pre-unitary* (PU), if the vector lengths $l_j, j=1, 2, \ldots, n-1$, of these vector-lines are all equal.

We note that, from the geometric point of view, T is PO, if and only if the n-dimensional prism with the vector-lines of the matrix of $T|_{X_T}$ as edges has $(n-1)$-dimensional rectangular support. Furthermore, T is PU, if and only if this support is an $(n-1)$-dimensional cube. Hence, Lemma 2.1.1 implies that, if T is PO, then the first n s-numbers of $T|_{X_T}$ are exactly $l_j, j=1, 2, \ldots, n-1$, and $l_h=|\det(T|_{X_T})|(\prod_{j=1}^{n-1} l_j)^{-1}$ (the "height" of the prism).

A PO operator T is called *dominant* (DPO) if $l_j\geq l_h, j=1, 2, \ldots, n-1$. (In the particular case where T is PU we adopt the abbreviation DPU.)

We note that, from the geometric point of view, the $n-1$ biggest s-numbers of $T|_{X_T}$ are all lengths of "edges", the smallest one, namely s_n, is the "height" of the prism. In the limit case when l_h is also an edge, i.e., $l_h\perp l_j, j=1, 2, \ldots, n-1$, $T|_{X_T}$ is an orthogonal operator and T is a normal operator. Therefore, the limit case cannot be useful for proving sharpness of the generalized Carleman inequalities.

LEMMA 3.3.1. *For every $\lambda\in C$ and $n\in N$, $n\geq 2$, and every $(n-1)$-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_{n-1})$, such that $\sigma_1\geq\sigma_2\geq\cdots\geq\sigma_{n-1}>|\lambda|^n(\prod_{j=1}^{n-1}\sigma_j)^{-1}$, there exists a* DPO *operator T of rank n, such that $s_j(T|_{X_T})=\sigma_j, j=1, 2, \ldots, n-1$, $s_n(T|_{X_T})=|\lambda|^n(\prod_{j=1}^{n-1}\sigma_j)^{-1}$ and $T|_{X_T}=\lambda I_{X_T}-T_1$, where $T_1: X_T\to H$ is a subdiagonal nilpotent operator.*

We note that if $\sigma_1=\sigma_2=\cdots=\sigma_{n-1}$, then T is even DPU. For such operators we also have:

LEMMA 3.3.2. *Assume that T is a* DPU *operator with a single-point spectrum $\sigma(T)=\{\lambda_0\}$, which is not normal. Then, $s_j(T)=\sigma>|\lambda_0|, j=1, 2, \ldots, n-1$; if $\lambda_0=0$, then $s_n(T)=\lambda_0=0$; if $\lambda_0\neq 0$, then $|\lambda_0|>s_n(T)$; if $n\to\infty$, then $s_n(T)\to 0$.*

Next we extend the sharpness definitions to the general case where $T\in S_p$ is not necessarily a finite-rank operator.

DEFINITION 3.3.3. Let $T\in S_p$, $0<p<\infty$. T is called PO, PU, DPO, DPU, if it is an S_p-limit of a sequence of finite-rank operators having the corresponding property.

3.4. Sharpness statements. Our sharpness statement with respect to the ES property read as follows:

THEOREM 3.4.1. *Theorems* 2.1.2, 2.2.1, 2.2.3 *and* 2.2.4 *are* ES *with embedding* *constants* $b'_p=0$, $a'_p=c'_p=(p\max\{1, 2^{p-1}\}\exp(x_0))^{-1}$, *where* $x_0>1$ *is the unique solution of the equation* $x=1+\exp(-x)$, $x\in R$.

We note that the sharp class of operators on which the ES property is realized includes quasinilpotent operators T, such that $\lambda I-T$ are DPU operators.

Next we present our sharpness results with respect to the ESSC property.

THEOREM 3.4.2. *For* $p=2$, *Theorem* 2.1.2 (*the classical Carleman inequality*) *is* ESSC *with the sharp constants* $a_2=b_2=1/2$. *If, in addition, T is a quasinilpotent operator, then also Theorems* 2.2.1, 2.2.3 *and* 2.2.4 *are* ESSC. *The class of quasinilpotent operators T on which* ESSC *is attained consists exactly of the ones such that* $\lambda I-T$ *is a* DPO *operator.*

THEOREM 3.4.3. *For all finite-dimensional H with sufficiently big dimension, and for every* $p\in(0, \infty)$, *Theorem* 2.2.2 *is* ESSC *with the sharp constant* $(ep)^{-1}$. *The class of operators on which* ESSC *is attained consists only of those which behave as* DPU *operators on the maximal T-invariant subspace corresponding to some* $\lambda_0\in\sigma(T)$ *with* $d(0, \sigma(T))=|\lambda_0|$.

We also include the following statement concerning the PS property:

THEOREM 3.4.4. *Theorem* 2.2.3 *is* PS. *The sharp class of operators on which* PS *is attained are the ones which behaves as "unicellular" operators on the maximal T-invariant subspace corresponding to a* λ_0 *in the spectrum such that* $|\lambda-\lambda_0|=d(\lambda, \sigma(T))$.

The following remarks are important for our proofs later on but are also of independent interest:

REMARK 3.4.5. If $T\in S_p$, $0<p<\infty$, $\lambda_0\in\sigma(T)$ and X_0 denotes the maximal T-invariant subspace corresponding to λ_0, then (cf. the proof of Theorem 2.2.1) $\lambda I-T$ can be represented on X_0 as $(\lambda-\lambda_0)I-(T-\lambda_0 I)$, where $T-\lambda_0 I$ is quasinilpotent. Thus, in the part concerning Theorem 2.2.1 it is sufficient to prove Theorems 3.4.1 and 3.4.2 for quasinilpotent operators. Moreover, for such operators Theorems 2.1.2 and 2.2.1 essentially coincide and we arrive at the important conclusion that the generalized Carleman inequalities with complete information about the spectrum are ES or ESSC if and only if the ones with minimal information are sharp in a respective sense.

REMARK 3.4.6. The density of finite-rank operators in S_p, $0<p<\infty$, implies that it is sufficient to prove Theorems 3.4.1 and 3.4.2 for finite-dimensional operators H (see [11, §11.9] for justifying this general approach to proofs of certain statements about S_p). Furthermore, for a finite-dimensional space it is easy to prove that the parts of Theorems 3.4.1 and 3.4.2 concerning Theorem 2.2.1 imply the corresponding parts concerning Theorems 2.2.3 and 2.2.4.

PROOF OF THE LAST STATEMENT. For every $\lambda\in C\setminus\{0\}$, we consider an operator T_0 with a single-point spectrum $\{\lambda_0\}$, such that $|\lambda|>|\lambda_0|$ and $\arg\lambda_0=\arg\lambda$. Lemma 3.3.1, applied to $\lambda-\lambda_0$, implies that T_0 can be chosen so that $\lambda I-T_0=(\lambda-\lambda_0)I-$

$(T_0-\lambda_0 I)$ is a DPU operator with an a priori prescribed value of $\sigma=s_j(\lambda I-T_0)$, $j=1,2,\dots,N-1$, $\sigma>|\lambda-\lambda_0|$. It will be proved below that the sharpness assertions of the parts of Theorems 3.4.1 and 3.4.2 concerning Theorem 2.2.1 are attained on operators of the type T_0. It remains to note that $r(T_0)=|\lambda_0|$ and $d(\lambda, \sigma(T_0))=|\lambda-\lambda_0|=|\lambda|-r(T_0)$ which completes the proof.

Summing up the observations in the Remarks 3.4.5 and 3.4.6 we find that, in order to prove Theorems 3.4.1 and 3.4.2, it is sufficient to prove the sharpness of Theorem 2.2.1 for the case where H is a finite-dimensional space and T is a nilpotent operator.

4. Applications.

4.1. Introductory remarks on applications. Theorem 2.2.2 can be applied to derive a priori estimates about the solutions of high-dimensional systems of algebraic equations and first kind Fredholm and Volterra regular and singular integral equations with special right-hand sides or degenerate kernels. Theorems 2.2.1, 2.2.3 and 2.2.4 have applications to second-kind Fredholm and Volterra regular and singular integral equations. In this paper we shall restrict ourselves to considering some model applications to the second-kind integral equations only. These applications are based on the fundamental fact that if T is an integral operator with kernel K, then, for every $q\in(0,\infty]$, the S_q-quasinorm of T can be estimated from above by an appropriate function quasinorm of K. For the cases $q=2$, $q=\infty$ these are the classical results and the general case is essentially due to Birman, Solomjak and Karadzhov. The lemmas below provide more details about these upper bounds.

The vector-valued and tensor-product Lebesgue spaces we consider in the sequel are defined in the usual way, see e.g. [1] and [22], respectively. In the sequel of this section we assume that (Ω_j, μ_j) is a measure space, $H_j=L_2(\Omega_j, \boldsymbol{F}, \mu_j)$ is the respective L_2-space of $\boldsymbol{F}$-valued functions, $\boldsymbol{F}=\boldsymbol{R}$ or $\boldsymbol{F}=\boldsymbol{C}$, $j=0,1$; $L_2(\Omega_0\times\Omega_1, \boldsymbol{F}, \mu_0\otimes\mu_1)$ is the respective tensor-product L_2-space; $T\colon H_0\to H_1$ is defined by $Tu(\omega_1)=\int_{\Omega_0}K(\omega_0, \omega_1)u(\omega_0)d\mu_0(\omega_0)$, $\omega_1\in\Omega_1$. Moreover, we also assume that $n\in\boldsymbol{N}$, $X\subset\boldsymbol{R}^n$, $Y\subset\boldsymbol{R}^n$ are bounded open sets; $(X, \boldsymbol{F}, dx)$, $(Y, \boldsymbol{F}, dy)$ are measure spaces of $\boldsymbol{F}$-valued functions ($\boldsymbol{F}=\boldsymbol{R}$ or $\boldsymbol{F}=\boldsymbol{C}$) with respect to the standard Lebesgue measure; $H_0=L_2(X, \boldsymbol{F}, dx)$, $H_1=L_2(Y, \boldsymbol{F}, dy)$; $T\colon H_0\to H_1$ is defined by $(Tu)(y)=\int_X K(x,y)u(x)dx$, $y\in Y$.

We need the following well-known classical result for Hilbert-Schmidt integral operators (i.e. $T\in S_q$, $q=2$).

LEMMA 4.1.1. *Assume that $H_j=L_2(\Omega_j, \boldsymbol{F}, \mu_j)$, $j=0,1$. Then (see also our preceding convention) $T\in S_2(H_0, H_1)$ if and only if $K\in L_2(\Omega_0\times\Omega_1, \boldsymbol{F}, \mu_0\otimes\mu_1)$ and $\|T|S_2\|=\|K|L_2\|$.*

The next lemma is a complement of Lemma 4.1.1 holding for $q\in(0,2)$ and $q\in(2,\infty)$ and the more restrictive setting of the preceding convention.

LEMMA 4.1.2 (see [15], [16]). (a) *Let $2<q<\infty$. Then,*

$$\|T\mid S_q\| \le c(p_0, p_1)\|K\mid L_q(Y, L_{p_0 q}(X, \boldsymbol{F}, dx), dy)\|^{(1/2-1/q)/(1/p_0-1/q)}$$
$$\times \|K\mid L_q(X, L_{p_1 q}(Y, \boldsymbol{F}, dy), dx)\|^{(1/2-1/q)/(1/p_1-1/q)},$$

where $0<p_0, p_1<2$, $p_0+p_1>2$ *and* $(1/q-1/2)^2=(1/p_0-1/2)(1/p_1-1/2)$.

(b) *Let* $0<q<2$. *Then*

$$\|T\mid S_q\| \le c_q(X)\mu(X)^{1/q-1/2}\|K\mid B_q^\alpha(X, L_2(X\otimes Y, \boldsymbol{F}, dx\otimes dy))\|, \quad \frac{\alpha}{n}=\frac{1}{q}-\frac{1}{2}.$$

Moreover, $c_q(X)$ *depends on the shape of* X, *but not on its diameter and, for fixed* X, $c_q(X)\to 1$ *as* $q\to 2$; $\mu(X)$ *denotes the Lebesgue measure.*

Here, as usual, L_{pq}, $0<p, q\le\infty$, denotes the usual Lorentz spaces (see e.g. [1]) and

$$B_q^\alpha(X, L_2(X\otimes Y, \boldsymbol{F}, dx\otimes dy))=\{f\in L_2(X\otimes Y, \boldsymbol{F}, dx\otimes dy): \|f\mid B_q^\alpha\|<\infty\},$$

where

$$\|f\mid B_q^\alpha\| = \|f\mid L_2(X\otimes Y, \boldsymbol{F}, dx\otimes dy)\|$$
$$+\sum_{i=1}^{n}\left(\int_0^\infty\left(t^{-\alpha}\sup_{0<h\le t}\|\Delta_i^m(x,h)f\mid L_2(X\otimes Y, \boldsymbol{F}, dx\otimes dy)\|\right)^q\frac{dt}{t}\right)^{1/q}, \quad 0<\alpha<m,$$

$$\Delta_i^m(x,h)f=\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}f(x_1, x_2, \dots, x_{i-1}, x_i+kh, x_{i+1}, \dots, x_n, y), \quad \text{if}$$

$$(x_1, x_2, \dots, x_{i-1}, x_i+kh, x_{i+1}, \dots, x_n)\in X, \quad k=0, 1, \dots, m,$$

$$\Delta_i^m(x,h)f=0 \quad \textit{if } (x_1, x_2, \dots, x_{i-1}, x_i+kh, x_{i+1}, \dots, x_n)\notin X, \quad i=1, 2, \dots, n,$$

$$0<h<\infty.$$

4.2. A priori estimation of resolvent operator norms. The first application is to derive a priori estimates for the solutions of linear Fredholm and Volterra integral equations when insufficient or minimal information about the spectrum of the integral operator is available. These estimates are given in terms of regularity properties of the integral operator kernel and are obtained by a straightforward application of Lemmas 4.1.1 and 4.1.2 to the results in Section 2. Here we only present the following model example of this type of estimates:

COROLLARY 4.2.1. *Assume that* $H=L_2(\Omega, \boldsymbol{F}, \mu)$ *and* $T\in L(H)$ *is an integral operator with kernel* K. *Assume that* $\lambda\in \boldsymbol{C}\setminus\sigma(T)$.

(i) *If* $K\in L_2=L_2(\Omega\times\Omega, \boldsymbol{F}, \mu\otimes\mu)$, *then*

$$\|(\lambda I-T)^{-1}\mid L(H)\|\le\frac{1}{d(\lambda, \sigma(T))}\exp\left(c_2\frac{\|K\mid L_2\|^2}{(d(\lambda, \sigma(T)))^2}+b_2\right).$$

(ii) *If* $\Omega = X \subset R^n$, $d\mu = dx$ *and if there exist* p_0, p_1: $0 < p_0, p_1 < 2$, $p_0 + p_1 > 2$ *and* q *is the one and only root of the equation* $(1/q - 1/2)^2 = (1/p_0 - 1/2)(1/p_1 - 1/2)$, *which is in* $(2, \infty)$, *then*

$$\|(\lambda I - T)^{-1} \mid L(H)\| \le \frac{1}{d(\lambda, \sigma(T))} \exp\left(c_q (c(p_0, p_1))^q \frac{\|K \mid p_0, p_1, q\|^q}{(d(\lambda, \sigma(T)))^q} + b_q \right),$$

where $c(p_0, p_1)$ *is the constant in Lemma* 4.1.2(a) *and*

$$\|K \mid p_0, p_1, q\| = \|K \mid L_q(X, L_{p_0 q}(X, F, dx), dx)\|^{(1/2 - 1/q)/(1/p_0 - 1/q)} \times \|K \mid L_q(X, L_{p_1 q}(X, F, dx), dx)\|^{(1/2 - 1/q)/(1/p_1 - 1/q)}.$$

(iii) *If* $\Omega = X \subset R^n$, $d\mu = dx$, $\mu(X) < \infty$, $\alpha > 0$, *then*

$$\|(\lambda I - T)^{-1} \mid L(H)\| \le \frac{1}{d(\lambda, \sigma(T))} \exp\left(c_q c_q'(X)(\mu(X))^\alpha \frac{\|K \mid B_q^\alpha(X, L_2)\|^q}{(d(\lambda, \sigma(T)))^q} + b_q \right),$$

where q *is defined by* $\alpha/n = 1/q - 1/2$, $B_q^\alpha(X, L_2)$ *and* $c_q'(X)$ *are defined as in Lemma* 4.1.2(b).

Another similar type of estimates is to find an upper bound for the Schatten-von Neumann (quasi-)norm of the compact resolvent of a compact operator T. There follows a model example.

COROLLARY 4.2.2. *Assume that* H *is a Hilbert space,* $T \in S_q(H)$, $0 < q < \infty$ *and* $\lambda \in C \setminus \sigma(T)$. *Let* R_λ *be the kernel of* $\bar{R}_\lambda = T(\lambda I - T)^{-1}$. *Then*

$$\|R_\lambda \mid S_q(H)\| \le \frac{\|T \mid S_q(H)\|}{d(\lambda, \sigma(T))} \exp\left(c_q \frac{\|T \mid S_q(H)\|^q}{(d(\lambda, \sigma(T)))^q} + b_q \right).$$

Corollary 4.2.2, makes it possible to establish a remarkable relationship between regularity properties of an integral Hilbert-Schmidt operator and its Fredholm resolvent. More precisely, the following is true:

COROLLARY 4.2.3. *Assume that* $H = L_2(\Omega, F, \mu)$, $T \in L(H)$ *is an integral operator with kernel* K: $K \in L_2 = L_2(\Omega \times \Omega, F, \mu \otimes \mu)$. *Assume that* $\lambda \in C \setminus \sigma(T)$. *If* R_λ *is the kernel of* $\bar{R}_\lambda$, *then* $R_\lambda \in L_2$ *and*

$$\|R_\lambda \mid L_2\| \le \frac{\|K \mid L_2\|}{d(\lambda, \sigma(T))} \exp\left(c_2 \frac{\|K \mid L_2\|^2}{(d(\lambda, \sigma(T)))^2} + b_2 \right).$$

4.3. Perturbation analysis and error analysis. The second application (which is implied by the first one) is to make error analysis of the approximate solutions of second-kind integral equations. Before making a more detailed consideration we remark that the error estimates commonly met in numerical analysis of concrete problems usually are given in terms of regularity properties of the integral equation's solution. Our main results make it possible to derive estimates directly in terms of regularity properties of the problem's data, i.e., the integral-operator kernel, right-hand side function and scalar parameters. The first type of estimates yield upper error bounds of the type $O(h^\alpha)$ while the second one yields Ch^α (h=the length of the step of the approximation method). In other words, the first type of estimates determines only the order α of approximation while the second type yields both α and the constant C in terms of regularity properties of the problem's data. This fact is very useful and has direct applications to advanced computer-aided design (CAD) and expert systems involving the so-called "scientific computation with automatic result verification". Now we consider the equation $(\lambda I-T_0)u_0=f_0$, and estimate the error when replacing u_0 by the solution u_1 of $(\lambda I-T_1)u_1=f_1$, where $u_j\in B, f_j\in B, j=0, 1, B=L_p(X, F, dx), X\subset R^n$, $1\leq p\leq\infty$, T_j are the integral operators: $B\to B$, with $(T_ju)(\xi)=\int_X K_j(x, \xi)u(x)dx, \xi\in X, j=0, 1$; I is the identity on B; $\lambda\in C\setminus(\bigcup_{j=0}^1\sigma(T_j))$. We are now looking for an upper bound of

$$\|\varepsilon\mid B\|, \quad \text{where } \varepsilon=(\lambda I-T_1)^{-1}f_1-(\lambda I-T_0)^{-1}f_0 .$$

A general upper bound of the first kind (in terms of regularity properties of u_0 and u_1) has been obtained in [24]. This estimate has been applied in error analysis of many concrete numerical methods. We will derive a new error bound of the same type which coincides with the one in [24] for $p=1, \infty$ and improves it for $1<p<\infty$.

PROPOSITION 4.3.1. *Let* $1\leq p\leq\infty$, $\lambda\neq 0$ *and*

$$\rho_j=\left(\operatorname*{ess\,sup}_{x\in X}\int_X |R_j(x, \xi)|d\xi\right)^{1/p}\left(\operatorname*{ess\,sup}_{\xi\in X}\int_X |R_j(x, \xi)|dx\right)^{1-1/p},$$

where R_j *is the resolvent kernel of* T_j (*the kernel of the integral operator* $T(\lambda I-T)^{-1}$), $j=0, 1$; $\rho=|\lambda|^{-1}(1+\min\{\rho_0, \rho_1\})$, $\varphi=\min\{\|f_0\mid B\|, \|f_1\mid B\|\}$;

$$\Delta=\left(\operatorname*{ess\,sup}_{x\in X}\int_X |K_1(x, \xi)-K_0(x, \xi)|d\xi\right)^{1/p}\left(\operatorname*{ess\,sup}_{\xi\in X}\int_X |K_1(x, \xi)-K_0(x, \xi)|dx\right)^{1-1/p};$$

$\delta=\|f_1-f_0\mid B\|$. *If*

$$\Delta<|\lambda| \quad \textit{and} \quad \min\{\rho_0, \rho_1\}<\frac{|\lambda|-\Delta}{\Delta} \quad (\textit{i.e.}, 1-\rho\Delta>0), \tag{4.3.1}$$

then

$$\|\varepsilon\mid B\|\leq\rho\delta+\frac{\rho^2\varphi}{1-\rho\Delta}\Delta+\frac{2\rho^2}{1-\rho\Delta}\delta\Delta . \tag{4.3.2}$$

We remark that most of the concrete numerical methods for solving integral equations are equivalent to approximating T_0 by a finite-rank integral operator T_1 with a degenerate kernel. Proposition 4.3.1 provides an error estimate, which is given in terms of regularity properties of the solution, in view of the definition of ρ. This limits the efficiency of the estimate to a considerable extent. Thus, if $\{T_n\}$ is a sequence of integral operators with $T_n \to T$ in $L(H)$ as $n \to \infty$, and $\{f_n\}$ is a sequence of functions from B with $\lim_{n\to\infty} \|f_n - f \mid B\| = 0$, then (4.3.1) implies that there exists an N such that, for every $n > N$, (4.3.2) holds and, thus, $\|\varepsilon_n \mid B\| = O(\delta_n + \Delta_n + \delta_n \Delta_n)$. However, no efficient estimation of N or the O-constant can be derived.

By using our results from Section 4.2 we will now for $p=2$ derive an estimate of the second type.

PROPOSITION 4.3.2. *Let $p=2$ and let φ, Δ and δ be defined as in Proposition* 4.3.1. *Let $0 < q_j < \infty$, $j=0, 1$, let T be an integral operator with kernel K and consider $\tau_j = \|K_j\|_{q_j}$, $j=0, 1$. $\|K\|_q$, $0 < q < \infty$, denotes the function quasinorm appearing on the right-hand side of the corresponding bound for $\|T \mid S_q\|$ in Lemmas* 4.1.1 *or* 4.1.2 *and c_q and b_q are as in Theorem* 2.2.1. *Then*

$$\|\varepsilon \mid B\| \leq \rho\delta + \rho_0\rho_1\varphi\Delta + 2\rho_0\rho_1\delta\Delta ,$$

where

$$\rho_j = \frac{1}{d(\lambda, \sigma(T_j))} \exp(c_{q_j}\tau_j^{q_j} d(\lambda, \sigma(T_j))^{-q_j} + b_{q_j}) , \quad j=0, 1, \quad \rho = \min\{\rho_0, \rho_1\} .$$

Compared with Proposition 4.3.1, we claim that Proposition 4.3.2 has two major advantages, namely

(a) there are no restrictions of the type (4.3.1);

(b) for any choice of T_0 and T_1, $\|\varepsilon \mid B\| \leq c_1\delta + c_2\Delta + c_3\delta\Delta$, where c_1, c_2 and c_3 depend on λ and the data T_0 and T_1 but *not* on the resolvents.

This dependence on the data is available in an explicit way under minimal information about the operator spectra.

Furthermore, it can be shown that Proposition 4.3.2 is sharp in a certain sense related to the ES and ESSC properties defined in Section 3. For more results relevant to Propositions 4.3.1 and 4.3.2, as well as for related sharpness results, we refer to [8] and [9]. We conclude this section by stating the following perturbation estimate (which can be useful in error analysis, too):

COROLLARY 4.3.3. *Assume that $H = L_2(\Omega, \boldsymbol{F}, \mu)$, $T_j \in L(H)$ is an integral operator with kernel $K_j \in L_2 = L_2(\Omega \times \Omega, \boldsymbol{F}, \mu \otimes \mu)$, $j=0, 1$. Assume that $\lambda \in \boldsymbol{C} \setminus (\sigma(T_0) \cup \sigma(T_1))$ and denote by $R_{j,\lambda}$ the kernel of $\bar{R}_{j,\lambda} = T_j(\lambda I - T_j)^{-1}$, $j=0, 1$. Then*

$$\|R_{1,\lambda} - R_{0,\lambda} \mid L_2\| \leq \frac{|\lambda| \, \|K_1 - K_0 \mid L_2\|}{d(\lambda, \sigma(T_0)) d(\lambda, \sigma(T_1))} \exp\left(c_2\left(\frac{\|K_0 \mid L_2\|^2}{d(\lambda, \sigma(T_0))^2} + \frac{\|K_1 \mid L_2\|^2}{d(\lambda, \sigma(T_1))^2}\right) + 2b_2\right).$$

5. Generalizations and concluding remarks.

5.1. Non-separable Hilbert spaces. In [7] it was mentioned that the results from Section 2 hold true also if the Hilbert spaces under consideration are non-separable. As far as the sharpness results in Section 3 are concerned, the fact that separability of the Hilbert spaces involved is not essential, is obvious. For the reader's convinience we only mention here the following peculiarity of the non-separable case compared with the separable one: if H is non-separable and $T \in S_\infty(H)$, then the countability of $\sigma(T)$ and the fact that the maximal T-invariant subspace corresponding to any $\lambda \in \sigma(T)$, $\lambda \neq 0$, is finite-dimensional, implies that 0 is necessarily an eigenvalue of T and its corresponding T-invariant subspace is non-separable.

5.2. Generalized weighted Schatten-von Neumann ideals. Let $H_j, j=0, 1$, be Hilbert spaces, $0<p\leq\infty$ and $0\leq\alpha<\infty$. Then (see e.g. [14] and [21])

$$S_{p,\alpha}=S_{p,\alpha}(H_0, H_1)=\left\{T\in S_\infty(H_0, H_1): \|T\,|\,S_{p,\alpha}\|=\left(\sum_{n=1}^{N-1}(s_n(T)n^\alpha)^p\frac{1}{n}\right)^{1/p}<\infty\right\}$$

with the usual supremum interpretation for the case $p=\infty$. We note that $S_{p,\alpha}$ are quasi-Banach ideals in $L(H_0, H_1)$ which coincide with S_p in the special case $\alpha=1/p$, $0<p\leq\infty$. Moreover, $S_{p,\alpha}$ occur as intermediate spaces between S_p-spaces with respect to Lions-Peetre's real interpolation method (see [21] for more precise information and historical remarks concerning this statement). We are just working with a paper where we intend to prove generalized Carleman inequalities with complete information about the spectrum in terms of $S_{p,\alpha}$ and with sharp constants a_p and b_p. Clearly this will give the sharp constants in Theorem 2.1.2, too. By using these sharp results we intend to prove generalized Carleman inequalities with minimal information about the spectrum in terms of $S_{p,\alpha}$ with sharp constants b_p, c_p and d_p, thereby extending our results in subsection 2.2. We also intend to extend our applications by proving an appropriate generalization of Proposition 4.3.2 and proving in detail its sharpness (cf. also [8] and [9]).

It should be noted that Lemmas 4.1.1 and 4.1.2 are proved in [14], [15] and [16] in the more general case with $S_{p,\alpha}$, $0<p\leq\infty$, $0\leq\alpha<\infty$, when $H_0=L_2(X, \boldsymbol{F}, \rho)$, $H_1=L_2(Y, \boldsymbol{F}, \tau)$, where $X\subset R^n$ is a bounded or unbounded domain with the cone property, Y is arbitrary, ρ and τ are σ-finite measures, $(Tu)(y)=\int_X K(x, y)u(x)\rho(dx)$ and $\boldsymbol{F}=\boldsymbol{R}$ or $\boldsymbol{C}$.

In [14] and [15] there are also improvements of the general results in the particular cases when ρ and/or τ are absolutely continuous $\rho(dx)=|a(x)|^2dx$, $a\in L_{p_0}(X, \boldsymbol{F}, dx)$ and/or $\tau(dy)=|b(y)|^2dy$, $b\in L_{p_1}(Y, \boldsymbol{F}, dy)$, $0<p_0, p_1\leq\infty$, and/or T is a convolution operator $(K(x, y)=K(x-y))$.

5.3. Different domain and codomain of T. Instead of considering the case $H_0=H_1=H$ as here we can have different Hilbert spaces H_0 and H_1. Such a generalization of Theorem 2.2.2 makes sense for arbitrary finite-dimensional H_0 and H_1. In order to generalize the other results in Subsection 2.2 in this way there is the natural restriction $H_0\subset H_1$ (in the sense of set inclusion). Then I is the canonical imbedding of H_0 into H_1. In the applications this can be very useful for extending from compact integral operators over to singular integral operators and, more generally, to pseudodifferential operators. This is possible because an operator may act as a pseudodifferential one between a certain couple of Hilbert spaces and as a compact one with respect to another couple. In applications, if the first couple consists of Lebesgue spaces, then the second one should contain a Sobolev or a Besov space, maybe with a negative smoothness index (see e.g. [25]).

5.4. Analytic functions of a compact operator. Besides the operator resolvent $(\lambda I-T)^{-1}$, by using Dunford's representation (see, e.g., [10, Ch. 7]), we can extend our results to an arbitrary $f(T)$, which is analytic in a neighbourhood of $\sigma(T)$ with a sufficiently regular boundary. This approach implies several interesting generalizations of which we only present the following simple model example:

PROPOSITION 5.4.1. *Assume that* $0<p<\infty$, $T\in S_p$, f *is analytic in a neighbourhood* U *of* $\sigma(T)$, *the boundary* ∂U *consists of a finite number of rectifiable Jordan curves with a positive orientation in the usual sense of the theory of functions of a complex variable. Assume that* $f|_{\partial U}\in H_1(\partial U)$, *where* $H_1(\partial U)=\{g: g \text{ is analytic in } U \text{ and } \int_{\partial U}|g(z)||dz|<\infty\}$. *Then*

$$\|f(T)\,|\,L(H)\|\le\frac{\|f\,|\,H_1(\partial U)\|}{d(\partial U,\sigma(T))}\exp\left(c_p\frac{\|T\,|\,S_p\|^p}{d(\partial U,\sigma(T))^p}+b_p\right),$$

where c_p, b_p *are as in Theorem* 2.2.1 *and for compact sets* $A\subset C$, $B\subset C$ *we define, as usual,* $d(A,B)=\min\{|a-b|: a\in A, b\in B\}$.

5.5. A generalization of Theorem 2.2.3. A useful generalization of Theorem 2.2.3 with respect to applications is to loosen the assumption that λ_j should be all points in $\sigma(T)$ with $|\lambda_j|\ge|\lambda|$ by taking instead all points λ_j in $\sigma(T)$ with $|\lambda_j|\ge\tau|\lambda|$, where $\tau: 0<\tau\le 1$. Then, with minor modifications in the proof we arrive at the conclusion that (2.2.3) still holds true with $\gamma_p=\gamma_p(t)=\tau^{-p}\sum_{k=1}^{m_p}\tau^k/k$ and $r=\tau\sigma(T)$.

5.6. Negative results. It is important to note that the generalized (or classical) Carleman inequality of both types considered (with complete and with minimal information about the spectrum) can be expected to provide sharpness of the estimate neither uniformly with respect to the choice of T, nor uniformly in $\lambda\in C\setminus\sigma(T)$. In order to prove this assertion in the part concerning T, we take T to be a normal operator; then $a_p=b_p=c_p=0$. The part concerning λ follows e.g., for $p=2$, by comparing the classical Carleman inequality for a quasinilpotent operator T (where we know, by Theorem 3.4.2, that $a_2=b_2=1/2$ are the sharp constants) to the standard upper estimate and observe that, as $\lambda\to\infty$, the latter is better.

5.7. Detailed partial information about the spectrum. Let T be compact, $\lambda_0\in\sigma(T)$, and $n_0\in N$ be the multiplicity of λ_0 (in other words, complete information is available about λ_0). Let X_0 be the maximal invariant subspace corresponding to λ_0 ($\dim X_0=n_0$). Assume further that a Jordan canonical form of $T|_{X_0}$ is known. Let v_i, $i=1,2,\ldots,\mu_0$, be the ranks of the different Jordan cells appearing in the canonical form. Here $\mu_0\in N$, $1\le\mu_0\le n_0$, $\sum_{i=1}^{\mu_0}v_i=n_0$. We say that *detailed information about* λ_0 *is available* if μ_0 and v_i, $i=1,2,\ldots,\mu_0$, are known. Similarly, the notion of *detailed* *partial information about* $\sigma(T)$ is a similar modification of the one of partial information about $\sigma(T)$ (cf. [7]). We note that n_0 is often called the "algebraic" multiplicity, while ν_i are sometimes referred to as its "geometric" multiplicities. Obviously the two notions coincide if and only if $T|_{X_0}$ is a "unicellular" operator, i.e., $\mu_0=1$.

If the notion of "partial information" about $\sigma(T)$ is replaced by the one of "detailed partial information" about $\sigma(T)$ for the corresponding part of the spectrum in the conditions of Theorem 2.2.3 (or its generalization in Subsection 5.5), then (2.2.3) still holds true in the improved form whenever each term of the kind $|\lambda|^{n_j-1}/|\lambda-\lambda_j|^{n_j}$ is replaced by $\max_{i=1,2,\ldots,\mu_j}(|\lambda|^{\nu_{ij}-1}/|\lambda-\lambda_j|^{\nu_{ij}})$, where $\sum_{i=1}^{\mu_j}\nu_{ij}=n_j$, $1\le\mu_j\le n_j$. If T is not an "unicellular" operator on every maximal T-invariant subspace, then the above is obviously an essential improvement. This fact can most easily be seen if, for some j, $n_j>1$ and $T|_{X_j}$ is a normal operator, i.e., $\mu_j=n_j$, $\nu_i=1$, $i=1,2,\ldots,\mu_j$. Furthermore, in a similar way we find that the sharpness assertion in Theorem 3.4.4 holds true in a stronger form, in the sense that, in the definition of PS, the sentence "...there exists an operator $T\in S_p(H)$, $T=T(H,p)$, and $\lambda_0\in\sigma(T)$ with $|\lambda_0|>r(T)$, such that there exists $\varepsilon>0$, such that..." is replaced by the stronger "...and for every $T\in S_p(H)$ and every $\lambda_0\in\sigma(T)$ with $|\lambda_0|>r(T)$, there exists $\varepsilon>0$, such that...".

6. Proofs.

PROOF OF LEMMA 3.3.1. Let e_j, $j=1,2,\ldots,n$ be an orthonormal basis in H, $\dim H=n\ge 2$. First we let $\lambda\ne 0$. Consider $g_\nu=\lambda e_{n-\nu}$, $\nu=2,3,\ldots,n-1$, $g_1=\lambda e_{n-1}+\alpha e_n$, $\alpha\in C$. Obviously g_ν, $\nu=1,2,\ldots,n-1$, are linearly independent. By performing a Gram-Schmidt orthogonalization procedure we now obtain the vectors a_μ, $\mu=1,2,\ldots,n-1$, such that (i) $a_{\mu_1}\perp a_{\mu_2}$ for $\mu_1\ne\mu_2$; (ii) $|a_\mu|=\sigma_\mu$, $\mu=1,2,\ldots,n-1$; (iii) $a_\mu=\lambda e_\mu+\sum_{\nu=\mu+1}^{n}\alpha_{\mu\nu}e_\nu$, $\alpha_{\mu\nu}\in C$, $\mu=1,2,\ldots,n-1$. Next we define T as the unique operator whose matrix in the basis e_ν, $\nu=1,2,\ldots,n$, has a_μ, $\mu=1,2,\ldots,n-1$ and λe_n as vector-lines. Now let $\lambda=0$. Then we can define T for example in the following way: $\sum_{\nu=1}^{n-1}\sigma_\nu\exp(i\theta_\nu)\langle e_\nu,\cdot\rangle e_{\nu+1}$, for any choice of $\theta_\nu\in(-\pi,\pi]$, $\nu=1,2,\ldots,n-1$. It remains to prove the assertion for $s_n(T)$ but it follows at once by observing that $\det(T)=\lambda^n$ and applying Lemma 2.1.1. The proof is complete.

PROOF OF LEMMA 3.3.2. Let $\dim H=n$. Since T is not normal, we have $|\lambda_0|=r(T)\ll\|T|L(H)\|=s_1(T)=\sigma=s_\nu(T)$, $\nu=1,2,\ldots,n-1$. Then, in view of the fact that $\det(T)=\lambda_0^n$, Lemma 2.1.1 implies that $s_n(T)=\lambda_0=0$ or $|\lambda_0|>s_n(T)\ge 0$. Moreover, for the case $\lambda_0\ne 0$ we use Lemma 2.1.1 again and find that $s_n(T)=(|\lambda_0|/\sigma)^{n-1}|\lambda_0|\to 0$ since $|\lambda_0|<\sigma$ and the proof is complete.

PROOF OF THEOREM 3.4.1. According to our discussion in Subsection 3.4 we may, without loss of generality, assume that $2\le\dim H=N<\infty$ and that T is nilpotent. Fix $\lambda\in C$ and let $T=T_{N,k}$ be a nilpotent operator obtained by using Lemma 3.3.1 with $\sigma_j=(1-x_0/(k+1))^{-(k+1)}|\lambda|$, $j=1,2,\ldots,N-1$, $k\in N$. Note that if T_N is the same type of operator with $\sigma_j=\exp(x_0)$, $j=1,2,\ldots,N-1$, then $T_{N,k}\to T_N$ as $k\to\infty$, in $S_p(H)$,

$0<p\leq\infty$, $s_\nu(T_{N,k})\to s_\nu(T_N)$, $\nu=1,2,\ldots,N$, $k\to\infty$; $(\lambda I-T_{N,k})^{-1}\to(\lambda I-T_N)^{-1}$ as $k\to\infty$, in $L(H)$; $\sigma(T_{N,k})=\sigma(T_N)=\{\lambda\}$. In the same way as in the proof of Theorem 2.2.2 we obtain, for $t=\sigma^{1/(k-1)}$,

$$\text{(6.1)}\qquad \|(\lambda I-T_{N,k})^{-1}\mid L(H)\| = \frac{1}{|\lambda|}\left(\left(\frac{k}{k+1}\,t^p|\lambda|^{-p/(k+1)}\right)\left(1+\frac{\sum_{j=1}^{N-1}\sigma^p}{k(N-1)t^{(k+1)p}}\right)\right)^{(k+1)/(N-1)/p}$$

In (6.1) there is equality because due to the relation $s_\nu(\lambda I-T)=\sigma$, $\nu=1,2,\ldots,N-1$, and the special choice of t, the inequality between the geometric and the power means turn into an equality. Next we use well-known properties of s-numbers (see [11], [19] and [20]) to obtain that $\sigma=s_\nu(\lambda I-T_{N,k})\geq s_\nu(T_{N,k})-|\lambda|$, $\nu=1,2,\ldots,N-1$. Therefore, by using the elementary (form of the quasitriangle-) inequality $a^p+b^p\geq\min(1,2^{1-p})(a+b)^p$, $a,b,p>0$, we find that $\sigma^p\geq\min\{1,2^{1-p}\}s_\nu(T_{N,k})^p-|\lambda|^p$. Hence, (6.1) implies that

$$\text{(6.2)}\qquad \|(\lambda I-T_{N,k})^{-1}\mid L(H)\|\geq\frac{1}{|\lambda|}\left(\left(\frac{k}{k+1}\,t^p|\lambda|^{-p/(k+1)}\right)\left(1-\frac{|\lambda|^p}{k}\,t^{-(k+1)p}\right)\right.$$
$$\left.\times\left(1+\min\{1,2^{1-p}\}t^{-(k+1)p}\left(1-\frac{|\lambda|^p}{k}\,t^{-(k+1)p}\right)^{-1}\frac{\sum_{j=1}^{N-1}s_j(T_{N,k})^p}{k(N-1)}\right)\right)^{(k+1)(N-1)/p}=:R_{N,k}\,.$$

Thus, according to the definitions of $T_{N,k}$ and x_0, we obtain

$$\|(\lambda I-T_N)^{-1}\mid L(H)\|\geq\lim_{k\to\infty}R_{N,k}=\frac{1}{|\lambda|}\exp\left(\frac{N-1}{p}(-1+x_0-\exp(-x_0))\right)$$
$$\text{(6.3)}\qquad \times\exp\left(\frac{N-1}{p}\min\{1,2^{1-p}\}(N-1)^{-1}\exp(-x_0)\,\frac{\|T_N\mid S_p\|^p-s_N(T_N)^p}{|\lambda|^p}\right)$$
$$=\frac{1}{|\lambda|}\exp\left(p^{-1}\min\{1,2^{1-p}\}\exp(-x_0)\,\frac{\|T_N\mid S_p\|^p-s_N(T_N)^p}{|\lambda|^p}\right).$$

Moreover, by Lemma 3.3.2, $s_N(T_N)\to 0$ as $N\to\infty$ and we find that (6.2) and (6.3) imply the statement in the theorem.

PROOF OF THEOREM 3.4.2. In view of our comments in Subsection 3.4, it is sufficient to consider the case $\dim H=n<\infty$ and study in sufficient detail the proof of Theorem 2.1.2 (see [11, Ch. 11, Theorem 6. 15]) for the particular case with a nilpotent operator T. We adopt the notation in [11] and note that for the particular case with a nilpotent operator essential simplifications of the proof in [11] can be done. More precisely, the integer parameter N can now be taken equal to zero. Then, it is not necessary to study the auxiliary equality (i); since $T_N = T$ (ii) turns into a trivial equality; the identity (iv) continues to hold true when $N=0$. Moreover, tracing the proof and origin of (iii) back to [11, Ch. 11, Th. 6.12 and Lemma 6.13] we find that (iii) is based upon Hadamard's inequality and it turns into an equality for a nilpotent operator T if and only if T has the additional property that $\lambda I - T$ is PO. Next we note that, if in Lemma 3.3.1 the condition about dominance of the PO operator T is omitted, then the conclusion in the lemma holds on to be true without any restrictions whatsoever on $\sigma_1, \sigma_2, \dots, \sigma_{N-1}$, except the natural one $\sigma_j \geq 0$, $j=1, 2, \dots, n-1$. According to the observations above we find that the final inequality

$$\left\| \left(I - \frac{T}{\lambda} \right)^{-1} \mid L(H) \right\| \leq \left(1 - \frac{1}{n} \right)^{-(n-1)/2} \left(1 + \frac{\| T/\lambda \mid S_p \|^p}{n} \right)^{(n-1)/2}$$

in the proof in [11] turns into an equality exactly on those nilpotent operators T, for which $\lambda I - T$ is PO. The proof in the case concerning Theorem 2.1.2 now follows by letting n increase to infinity. The proof of the part concerning Theorems 2.2.1, 2.2.3 and 2.2.4 follows by using the fact (see [7]) that if T is a quasinilpotent operator, then $c_p = a_p$, $p \in (0, \infty)$. The proof is complete.

Proof of Theorem 3.4.3. We adopt the notation in the proof of Theorem 2.2 (see [7]). Note that $\dim H = N < \infty$. Obviously it is sufficient to consider the case of an operator with a single-point spectrum $\{\lambda\}$. Let $k \in N$ be as in the proof of Theorem 2.2. In the proof of this theorem there is only one inequality and it is the one between the geometric and the power mean. It can be seen that this inequality turns into an equality on an operator $T = T_{N,k}$ if and only if $T_{N,k}$ is DPU with $s_j(T_{N,k}) = \sigma = ((k+1)/k)^{(k+1)/p} |\lambda|$, $j = 1, 2, \dots, N-1$. The DPU operator T_N with $s_j(T_N) = \exp(1/p) |\lambda|$, $j = 1, 2, \dots, N-1$, as $k \to \infty$, is clearly the limit of $T_{N,k}$ in $S_p(H)$, $0 < p \leq \infty$; $\sigma(T_{N,k}) = \sigma(T_N) = \{\lambda\}$; $s_\nu(T_{N,k}) \to s_\nu(T_N)$, $\nu = 1, 2, \dots, N$; $T_{N,k}^{-1} \to T_N^{-1}$ in $L(H)$. Now, as in the proof of Theorem 2.2 we obtain

$$\| T_N^{-1} \mid L(H) \| = \frac{1}{|\lambda|} \exp\left(\frac{\| T_N \mid S_p \|^p - s_N(T_N)^p}{ep |\lambda|^p} \right),$$

and the proof is completed by using Lemma 3.3.2 and arguing in the same way as in the proof of Theorem 3.4.1.

Proof of Theorem 3.4.4. Let $T \in S_p(H)$ be such that there exists $\lambda_0 \in \sigma(T)$ with $|\lambda_0| > r(T)$, let ε_1, $0 < \varepsilon_1 < |\lambda_0| - r(T)$ be such that, if $|\lambda - \lambda_0| < \varepsilon_1$, $\lambda \neq \lambda_0$, then $\lambda \in C \setminus \sigma(T)$ and the maximum on the right-hand side of (2.2.3) is attained on the term corresponding to λ_0. Finally, let X_0 be the maximal T-invariant subspace corresponding to λ_0 and let T behave on X_0 as a "unicellular" operator, i.e., it has a Jordan canonical representation which contains one Jordan cell only with rank $n_0 = \dim X_0$ – (the multiplicity of λ_0). Then $(\lambda I - T)^{-1}$ has a pole of order n_0 at λ_0. Therefore, $(\lambda - \lambda_0)^{n_0} (\lambda I - T)^{-1}$ has a removable singularity there and converges in $L(H)$ as $\lambda \to \lambda_0$ to an operator which is invertible on X_0 and, thus, is not identical to zero on H. Due to the continuity of $\|\cdot\,|\,L(H)\|$ and of $(\lambda-\lambda_0)^{n_0}(\lambda I-T)^{-1}$ in $L(H)$ with respect to λ in a neighbourhood of λ_0, we obtain

$$\lim_{\lambda\to\lambda_0}\|(\lambda-\lambda_0)^{n_0}(\lambda I-T)^{-1}\,|\,L(H)\|>0$$

and there exists $\varepsilon_0>0$ and $c_0=c_0(\varepsilon_0)$, such that $\|(\lambda I-T)^{-1}\,|\,L(H)\|\geq c_0|\lambda-\lambda_0|^{n_0}$, for every $\lambda: |\lambda-\lambda_0|<\varepsilon_0$. We choose $\varepsilon=\min\{\varepsilon_0,\varepsilon_1\}$ and the proof follows.

Proof of Corollary 4.2.1. A straightforward combination of Lemmas 4.1.1 and 4.1.2(a) and (b) with the bound obtained in Theorem 2.2.1.

Proof of Corollary 4.2.2. Invoking the well-known property of s-numbers that $s_n(T_1T_2)\leq s_n(T_1)\|T_2\,|\,L(H)\|$, $n\in N$, for $T_j\in L(H)$, $j=1,2$ (see e.g. [11] or [20]), we find $s_n(\bar{R}_\lambda)\leq s_n(T)\|(\lambda I-T)^{-1}\,|\,L(H)\|$, $n\in N$, and the proof follows by applying Theorem 2.2.1.

Proof of Corollary 4.2.3. The proof follows by using Corollary 4.2.2 and applying Lemma 4.1.1 to the operators T and $\bar{R}_\lambda$.

Proof of Proposition 4.3.1. According to the well-known Riesz convexity theorem (see e.g. [1, §1.1] or [10, §6.10]), we have that $\|T_1-T_0\,|\,L(B)\|\leq\Delta$. Denote

$$\begin{aligned}
&T_-=T_0 \text{ if } \rho_0\leq\rho_1, && T_-=T_1 \text{ if } \rho_0>\rho_1;\\
&T_+=T_1 \text{ if } \rho_0\leq\rho_1, && T_+=T_0 \text{ if } \rho_0>\rho_1;\\
&f_-=f_0 \text{ if } \|f_0\,|\,B\|\leq\|f_1\,|\,B\|, && f_-=f_1 \text{ if } \|f_0\,|\,B\|>\|f_1\,|\,B\|;\\
&f_+=f_1 \text{ if } \|f_0\,|\,B\|\leq\|f_1\,|\,B\|, && f_+=f_0 \text{ if } \|f_0\,|\,B\|>\|f_1\,|\,B\|.
\end{aligned}$$

Note that $\|T_+-T_-\,|\,L(B)\|=\|T_1-T_0\,|\,L(B)\|=\Delta$ and $\|f_+-f_-\,|\,B\|=\|f_1-f_0\,|\,B\|=\delta$. In view of the identity $(\lambda I-T_-)^{-1}=\lambda^{-1}(I+T_-(\lambda I-T_-)^{-1})$ and by using the Riesz convexity theorem once more, we obtain that

$$\|(\lambda I-T_-)^{-1}\,|\,L(B)\|\leq\rho. \tag{6.4}$$

Moreover, it is easily seen that

$$\|\varepsilon\,|\,B\|\leq\min\{\|(\lambda I-T_i)^{-1}\,|\,L(B)\|\delta+\|(\lambda I-T_1)^{-1}-(\lambda I-T_0)^{-1}\,|\,L(B)\|\,\|f_j\,|\,B\|: i,j=0,1,i\neq j\}. \tag{6.5}$$

If we denote the right-hand side in (6.5) by μ, then

$$\begin{aligned}
&|\mu-\|(\lambda I-T_-)^{-1}\,|\,L(B)\|\delta-\|(\lambda I-T_1)^{-1}-(\lambda I-T_0)^{-1}\,|\,L(B)\|\varphi|\\
&\leq\|(\lambda I-T_+)^{-1}-(\lambda I-T_-)^{-1}\,|\,L(B)\|\delta\\
&\quad+\|(\lambda I-T_1)^{-1}-(\lambda I-T_0)^{-1}\,|\,L(B)\|\,\|f_+-f_-\,|\,B\|\\
&=2\|(\lambda I-T_+)^{-1}-(\lambda I-T_-)^{-1}\,|\,L(B)\|\delta.
\end{aligned} \tag{6.6}$$

Next, by using (4.3.1) and (6.4) to apply a well-known perturbation estimate (see [10, §7.6]), we find that

$$\|(\lambda I-T_+)^{-1}-(\lambda I-T_-)^{-1} \mid L(B)\| \le \frac{\rho^2\Delta}{1-\rho\Delta}. \tag{6.7}$$

The proof follows by combining (6.5), (6.6) and (6.7).

PROOF OF PROPOSITION 4.3.2. Theorem 2.2.1 and Lemmas 4.1.1 and 4.1.2 yield $\|(\lambda I-T_j)^{-1} \mid L(B)\| \le \rho_j, j=0, 1$. Let T_+, T_-, f_+ and f_- be as in the proof of Proposition 4.3.1. We use (6.5) and (6.6) again but with new estimations of the resolvent norms. Obviously, (6.4) holds true with the new value of ρ. Hilbert's identity

$$(\lambda I-T_1)^{-1}-(\lambda I-T_0)^{-1}=(\lambda I-T_1)^{-1}(T_1-T_0)(\lambda I-T_0)^{-1} \tag{6.8}$$

yields

$$\|(\lambda I-T_+)^{-1}-(\lambda I-T_-)^{-1} \mid L(B)\| = \|(\lambda I-T_1)^{-1}-(\lambda I-T_0)^{-1} \mid L(B)\| \le \rho_0\rho_1\Delta,$$

which, together with (6.4)–(6.6), completes the proof.

PROOF OF COROLLARY 4.3.3. The identities $\bar{R}_{j,\lambda}=\lambda(\lambda I-T_j)^{-1}-I$, $j=0, 1$, and Hilbert's identity (6.8) yield

$$\bar{R}_{1,\lambda}-\bar{R}_{0,\lambda}=\lambda(\lambda I-T_1)^{-1}(T_1-T_0)(\lambda I-T_0)^{-1}. \tag{6.9}$$

Next we use (6.9) and the well-known property of s-numbers that

$$s_n(V_1V_2V_3) \le \|V_1 \mid L(H)\| s_n(V_2) \|V_3 \mid L(H)\|, \qquad n\in N$$

for $V_k \in L(H)$, $k=1, 2, 3$ (see e.g. [11], [20] and the proof of Corollary 4.2.2) to obtain that

$$s_n(\bar{R}_{1,\lambda}-\bar{R}_{0,\lambda}) \le |\lambda| \|(\lambda I-T_1)^{-1} \mid L(H)\| s_n(T_1-T_0) \|(\lambda I-T_0)^{-1} \mid L(H)\|, \quad n\in N. \tag{6.10}$$

Summing up the squares in (6.10) for $n \in N$ yields

$$\|\bar{R}_{1,\lambda}-\bar{R}_{0,\lambda} \mid S_2\| \le |\lambda| \|(\lambda I-T_1)^{-1} \mid L(H)\| \, \|T_1-T_0 \mid S_2\| \, \|(\lambda I-T_0)^{-1} \mid L(H)\|,$$

whence the proof follows by using Theorem 2.2.1 and applying Lemma 4.1.1 to $R_{1,\lambda}-R_{0,\lambda}$ and T_1-T_0 (see Corollary 4.2.3).

PROOF OF PROPOSITION 5.4.1. First we note that, according to the assumptions, f admits a Dunford representation along ∂U. By using this fact, by applying the integral triangle inequality for vector-valued functions, the Hölder inequality and the identity

$$\max_{\mu\in\partial U} \frac{1}{d(\mu, \sigma(T))} \exp\left(c_p \frac{\|T \mid S_p\|^p}{d(\mu, \sigma(T))^p}+b_p\right) = \frac{1}{d(\partial U, \sigma(T))} \exp\left(c_p \frac{\|T \mid S_p\|^p}{d(\partial U, \sigma(T))^p}+b_p\right),$$

we obtain

$$\|f(T)\,|\,L(H)\| = \left\| \frac{1}{2\pi i} \int_{\partial U} f(\mu)(\mu I - T)^{-1} d\mu \,|\, L(H) \right\| \leq \frac{1}{2\pi} \int_{\partial U} |f(\mu)|\, \|(\mu I - T)^{-1} \,|\, L(H)\| d\mu$$

$$\leq \frac{1}{2\pi} \frac{1}{d(\partial U, \sigma(T))} \exp\left(c_p \frac{\|T\,|\,S_p\|^p}{d(\partial U, \sigma(T))^p} + b_p \right) \int_{\partial U} |f(\mu)|\, d\mu \,,$$

and the proof is complete.

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