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# ON SIGNED DEGREES IN SIGNED GRAPHS

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## 1. INTRODUCTION

A *signed graph* $S$ consists of a graph $G$ with a designation of its edges as either positive or negative. These were first discovered in [3]. The *(signed) degree* $\operatorname{sdeg} v$ of a vertex $v$ of $S$ is the number of positive edges incident with $v$ less the number of negative edges incident with $v$. Thus, if $v$ is incident with $d^+$ positive edges and $d^-$ negative edges, then $\operatorname{sdeg} v = d^+ - d^-$. However, in the graph $G$, $\deg v = d^+ + d^-$. Consequently, the degree of a vertex in $S$ and of the same vertex in $G$ are of the same parity.

For a vertex $v$ in a signed graph $S$, $\operatorname{sdeg} v$ may be positive, negative, or zero. For example, the signed graph $S$ of Figure 1 has two vertices of degree 2, one of degree 0, and two of degree $-1$.

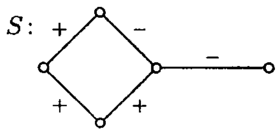


Figure 1. A signed graph $S$.

The *degree sequence* of a signed graph $S$ has the signed degrees in nonincreasing order. The degree sequence of the signed graph $S$ of Figure 1 is, therefore, 2, 2, 0, $-1$, $-1$. A finite sequence $\sigma$ of integers is *graphical* if $\sigma$ is a degree sequence of some signed graph. Certainly, then, 2, 2, 0, $-1$, $-1$ is graphical.

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It is easy to characterize degree sequences of paths and stars.

**Theorem 1.** *Let $\sigma\colon d_1, d_2, \ldots, d_p$ be a sequence of integers such that exactly two terms $d_i$ $(1 \leqslant i \leqslant p)$ are 1 or $-1$ and the remaining terms are $-2$, 0, or 2. Then $\sigma$ is the degree sequence of a signed path if and only if one of the following is satisfied:*

*(1) if there are exactly two integers $i$ $(1 \leqslant i \leqslant p)$ such that $d_i = 1$ or exactly two integers $i$ such that $d_i = -1$, then $\sigma$ contains an even number of zeros, or*

*(2) if there is one integer $i$ such that $d_i = 1$ and one integer $j$ such that $d_j = -1$, then $\sigma$ contains an odd number of zeros.*

**Theorem 2.** *Let $\sigma\colon d_1, d_2, \ldots, d_p$ be a sequence of integers such that $d_i = 1$ or $d_i = -1$ for $i = 1, 2, \ldots, p-1$. Then $\sigma$ is the degree sequence of a signed star if and only if $d_p = \sum\limits_{i=1}^{p-1} d_i$.*

A *double star* (first defined in Grossman, Harary and Klawe [1]) is a tree containing exactly two vertices that are not end-vertices. These two vertices are the *centers* of the double star. For example, the graph $G$ in Figure 2 is a double star with centers $u$ and $v$. Of course, if we designate each edge of $G$ as positive or negative, then we have a *signed double star*. One such signed double star $S$ is shown in Figure 2.

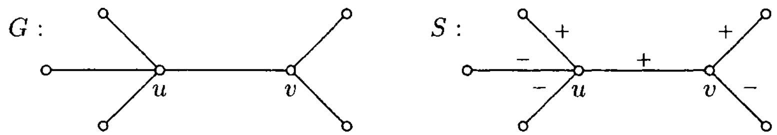


Figure 2. A double star $G$ and a signed double star $S$.

The next result gives a characterization of degree sequences of signed double stars.

**Theorem 3.** *Let $\sigma\colon d_1, d_2, \ldots, d_p$ be a sequence of integers with $d_i = 1$ or $d_i = -1$ for $i = 1, 2, \ldots, p-2$. Let $a$ denote the number of integers $i$ $(1 \leqslant i \leqslant p-2)$ such that $d_i = 1$ and let $b$ be the number of integers $i$ $(1 \leqslant i \leqslant p-2)$ such that $d_i = -1$. Further, let $d = a - b$. Then $\sigma$ is the degree sequence of a signed double star if and only if one of the following is satisfied:*

*(1) $d_{p-1} + d_p = d + 2$ and $-b + 1 \leqslant d_{p-1}, d_p \leqslant a + 1$,*

*(2) $d_{p-1} + d_p = d - 2$ and $-b - 1 \leqslant d_{p-1}, d_p \leqslant a - 1$.*

Proof. First let $\sigma\colon d_1, d_2, \ldots, d_p$ be the degree sequence of a signed double star $T$. So $T$ has order $p$, where $p-2$ of the vertices of $T$ are end-vertices. Since each end-vertex must have degree 1 or $-1$, we may assume, without loss of generality, that $d_i = 1$ or $d_i = -1$ for $i = 1, 2, \ldots, p-2$. Let $u$ and $v$ be the centers of $T$, say $\operatorname{sdeg} u = d_{p-1}$ and $\operatorname{sdeg} v = d_p$. Now the edge joining $u$ and $v$ is either positive or negative. First consider $uv$ as a positive edge.

Let $a$ be the number of integers $i$ $(1 \leqslant i \leqslant p-2)$ such that $d_i = 1$ and $b$ integers $i$ $(1 \leqslant i \leqslant p-2)$ such that $d_i = -1$. Then $-b+1 \leqslant d_{p-1} \leqslant a+1$. Similarly, $-b+1 \leqslant d_p \leqslant a+1$. Also, observe that if $d = a-b$, then

$$\operatorname{sdeg} u + \operatorname{sdeg} v = d_{p-1} + d_p = a - b + 2 = d + 2.$$

The proof is similar when $uv$ is a negative edge.

For the converse, let the sequence $\sigma\colon d_1, d_2, \ldots, d_p$ satisfy condition (1). We show that there exists a signed double star $T$ with degree sequence $\sigma$. Let $T''$ be the signed graph consisting of a single positive edge $uv$ (see Figure 3). Now $d_p > 0$ or $d_p \leqslant 0$. Since the proofs of these two cases are similar, we only consider the first.

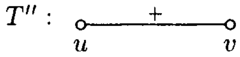


Figure 3. The signed graph $T''$ with one positive edge $uv$.

Given $d_p > 0$, we begin by adding edges to the graph $T''$ to produce a new graph $T'$ with $\operatorname{sdeg}_{T'} v = d_p$. In particular, we add $d_p - 1$ vertices to $T''$ and join each new vertex to $v$ with a positive edge. The graph $T'$ is shown in Figure 4. Note that at this time, we have one vertex of degree $d_p$, and $d_p$ vertices of degree 1. Also, since $a = p - 2 - b$, it follows that

$$a - (d_p - 1) = (p - 2 - b) - (d_p - 1) = p - (d_p + 1) - b.$$

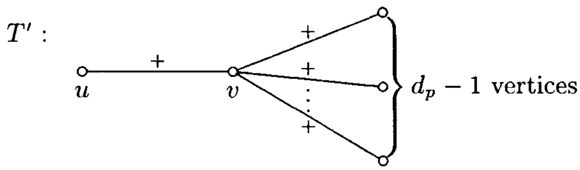


Figure 4. A signed graph $T'$ with all positive edges.

To complete the construction of $T$, we add $p-(d_p+1)$ vertices to $T'$ and join $b$ of the new vertices to $u$ by negative edges and join the remaining $p-(d_p+1)-b = a-(d_p-1)$ new vertices to $u$ by positive edges. The resulting signed graph $T$ is depicted in Figure 5.

Observe that $T$ is a double star of order $p$ containing $a$ vertices of degree 1 and $b$ vertices of degree $-1$. Also $\operatorname{sdeg} v = d_p$, and

$$\operatorname{sdeg} u = a - (d_p - 1) - b + 1 = (a - b) - d_p + 2 = d_{p-1}.$$

Thus, $T$ has the desired properties. The proof is similar when $\sigma$ satisfies condition (2). $\square$

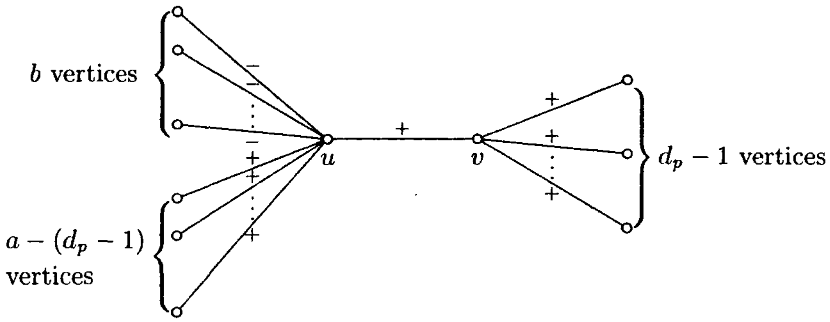

Figure 5. A signed double star $T$ with degree sequence $\sigma$.

A few facts about graphical sequences are described next.

**Theorem 4.** *If $S$ is a signed graph of order $p$ and size $q$, then*

$$k = \sum \operatorname{sdeg} v = 2q \pmod 4,$$

*and the number of positive edges of $S$ is $\frac{1}{4}(2q + k)$ while the number of negative edges of $S$ is $\frac{1}{4}(2q - k)$.*

Proof. Suppose that $S$ is obtained by designating each edge of a graph $G$ as positive or negative and $V(S) = \{v_1, v_2, \ldots, v_p\}$. Suppose, further, that $v_i$ ($1 \leqslant i \leqslant p$) is incident with $d_i^+$ positive edges and $d_i^-$ negative edges, so that $\operatorname{sdeg} v_i = d_i^+ - d_i^-$ while $\deg v_i = d_i^+ + d_i^-$. Of course, $\sum \deg v_i = 2q$.

Let $S$ have $a$ positive edges and $b$ negative edges. Then $q = a + b$, $\sum d_i^+ = 2a$, and $\sum d_i^- = 2b$. Consequently,

$$k = \sum \operatorname{sdeg} v_i = 2a - 2b = 2q - 4b$$

so that $k \equiv 2q \pmod 4$. Solving for $a$ and $b$, we have $a = \frac{1}{4}(2q+k)$ and $b = \frac{1}{4}(2q-k)$. □

**Corollary 4a.** *A necessary condition for a sequence $d_1, d_2, \ldots, d_p$ of integers to be graphical is that $\sum d_i$ is even.*

Of course, another necessary condition for $d_1, d_2, \ldots, d_p$ to be graphical is that each $|d_i| < p$. A *zero sequence* is a finite sequence every term of which is 0. Clearly, every zero sequence is graphical. If $\sigma$ is the sequence $d_1, d_2, \ldots, d_p$, then the *negative* $-\sigma$ of $\sigma$ is the sequence $-d_1, -d_2, \ldots, -d_p$. Obviously, a sequence $\sigma$ is graphical if and only if $-\sigma$ is graphical.

When considering a nonzero sequence $\sigma\colon d_1, d_2, \ldots, d_p$, we may assume, without loss of generality, that $\sigma$ is nonincreasing and $|d_1| \geqslant |d_p|$, for we may always replace

$\sigma$ by $-\sigma$ if necessary. We say that a nonzero sequence $\sigma\colon d_1, d_2, \ldots, d_p$ is a *standard sequence* if $\sigma$ is nonincreasing, $\sum d_i$ is even, $d_1 > 0$, each $|d_i| < p$, and $|d_1| \geqslant |d_p|$.

## 2. Degree sequences of complete signed graphs

If every edge of a signed graph $S$ is designated as positive, then, in effect, $S$ is a graph and the signed degree of each vertex of $S$ is its degree. For graphs, Havel [5] and, later, Hakimi [2] independently showed that a nonincreasing nonzero sequence $\sigma\colon d_1, d_2, \ldots, d_p$ of nonnegative integers is graphical if and only if the sequence $\sigma'\colon d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_p$ is graphical. This result was also discovered and proved by the author of [4], but he found the Havel reference and hence did not publish it.

We now show that there is an analogue to the Havel-Hakimi-Harary Theorem for complete signed graphs.

**Theorem 5.** *Let $\sigma\colon d_1, d_2, \ldots, d_p$ be a standard sequence and let $r = \frac{1}{2}(d_1 + p - 1)$. Then $\sigma$ is a degree sequence of a complete signed graph if and only if $\sigma'\colon d_2 - 1, d_3 - 1, \ldots, d_{r+1} - 1, d_{r+2} + 1, d_{r+3} + 1, \ldots, d_p + 1$ is a degree sequence of a complete signed graph.*

Proof. Let $S'$ be a complete signed graph with degree sequence $\sigma'$. Label the vertices of $S'$ as $v_2, v_3, \ldots, v_p$ so that

$$\operatorname{sdeg} v_i = \begin{cases} d_i - 1, & 2 \leqslant i \leqslant r+1, \\ d_i + 1, & r+2 \leqslant i \leqslant p. \end{cases}$$

Now construct a new complete signed graph $S$ by adding a vertex $v_1$ to $S'$ and joining $v_1$ by positive edges to the vertices $v_2, v_3, \ldots, v_{r+1}$ and by negative edges to the vertices $v_{r+2}, v_{r+3}, \ldots, v_p$. Then observe that

$$\operatorname{sdeg} v_1 = r - (p - r - 1) = 2r - (p-1) = d_1.$$

Also, it is clear that $\operatorname{sdeg} v_i = d_i$ for $i = 2, 3, \ldots, p$. Hence, $S$ is a complete signed graph with degree sequence $\sigma$.

For the converse, let $\sigma$ be the degree sequence of a complete signed graph. For any complete signed graph with degree sequence $\sigma$, we may assume that the vertices are labeled $v_1, v_2, \ldots, v_p$ such that $\operatorname{sdeg} v_i = d_i$ for $i = 1, 2, \ldots, p$. Among all complete signed graphs having $\sigma$ as a degree sequence, let $S$ be one with the property that the sum $m$ of the (signed) degrees of the vertices joined to $v_1$ by positive edges is maximum. If $d^+$ denotes the number of positive edges incident with $v_1$ and $d^-$ is the number of negative edges incident with $v_1$, then $\operatorname{sdeg} v_1 = d_1 = d^+ - d^-$. Further, since $S$ is a complete signed graph, $\deg v_1 = d^+ + d^- = p - 1$. From this it follows that $d^+ = \frac{1}{2}(d_1 + p - 1) = r$. We claim that $v_1$ must be joined by positive edges to vertices having the degrees $d_2$, $d_3$, $\ldots$, $d_{r+1}$. Assume that this is not the case. Then there exist vertices $v_i$ and $v_j$ with $j > i$ such that the edge $v_1v_i$ is negative and the edge $v_1v_j$ is positive.

Since $\sigma$ is a standard sequence, $\operatorname{sdeg} v_i > \operatorname{sdeg} v_j$, or $d_i > d_j$. Hence, there exists a vertex $v_n$ of $S$ distinct from $v_1$, $v_i$, and $v_j$ such that $v_nv_i$ is a positive edge and $v_nv_j$ is a negative edge. But if we now change the signs of these edges so that $v_1v_j$ and $v_nv_i$ are negative and $v_1v_i$ and $v_nv_j$ are positive (see Figure 6), then we produce a complete signed graph with degree sequence $\sigma$ in which the sum of the degrees of the vertices joined to $v_1$ by positive edges exceeds $m$, which is a contradiction.

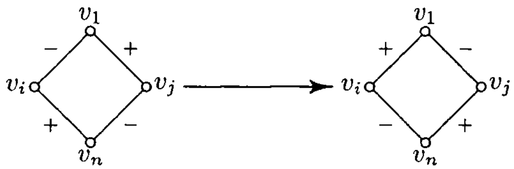


Figure 6. A change of signs of edges in $S$.

Thus, we may assume that $v_1$ is joined by positive edges to the vertices $v_2$, $v_3$, $\ldots$, $v_{r+1}$ and by negative edges to the vertices $v_{r+2}$, $v_{r+3}$, $\ldots$, $v_p$. The complete signed graph $S - v_1$ thus has $\sigma'$ as a degree sequence. □

As we will see in the next section, it is possible to generalize this idea to obtain a characterization of degree sequences of signed graphs in general. But first we illustrate the algorithm that is suggested by the previous theorem. Consider the sequence $\sigma_1\colon 3, 1, 1, -1, -1, -1$. Note that $p = 6$ and $d_1 = 3$, so $r = 4$. Applying Theorem 5 to $\sigma_1$, we obtain $\sigma_2'\colon 0, 0, -2, -2, 0$. Now if we write $\sigma_2'$ in standard form, we have $\sigma_2\colon 2, 2, 0, 0, 0$. For $\sigma_2$, then, we have $p = 5$ and $d_1 = 2$, and so $r = 3$. Applying Theorem 5 to $\sigma_2$ gives $\sigma_3'\colon 1, -1, -1, 1$. Now observe that the complete signed graph $S$ shown in Figure 7 has degree sequence $\sigma_3'$. So, by Theorem 5, it follows that $\sigma_2$ (and $\sigma_2'$) is the degree sequence of a complete signed graph. Hence, $\sigma_1$ is also the degree sequence of a complete signed graph.

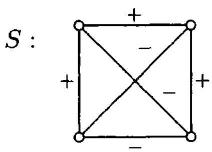


Figure 7. A complete signed graph $S$ with degree sequence $\sigma_3'$.

Now we present a necessary and sufficient condition for a sequence of integers to be graphical.

**Theorem 6.** *Let $\sigma\colon d_1, d_2, \ldots, d_p$ be a standard sequence. Then $\sigma$ is graphical if and only if there exist integers $r$, $s$ with $d_1 = r - s$ and $0 \leqslant s \leqslant \frac{1}{2}(p-1-d_1)$ such that*

$$\sigma'\colon d_2 - 1, d_3 - 1, \ldots, d_{r+1} - 1, d_{r+2}, d_{r+3}, \ldots, d_{p-s}, d_{p-s+1} + 1, \ldots, d_p + 1$$

*is graphical.*

Proof. Let $r$ and $s$ be integers with $d_1 = r - s$ and $0 \leqslant s \leqslant \frac{1}{2}(p-1-d_1)$ such that $\sigma'$ is the degree sequence of a signed graph. Then there exists a signed graph $S'$ having degree sequence $\sigma'$. We may assume that the vertices of $S'$ are labeled $v_2$, $v_3$, …, $v_p$ so that

$$\operatorname{sdeg} v_i = \begin{cases} d_i - 1, & 2 \leqslant i \leqslant r+1, \\ d_i, & r+2 \leqslant i \leqslant p-s, \\ d_i + 1, & p-s+1 \leqslant i \leqslant p. \end{cases}$$

Now we construct a new signed graph $S$ by adding a vertex $v_1$ to $S'$ and joining $v_1$ to $v_i$ $(2 \leqslant i \leqslant r+1)$ by positive edges and to $v_i$ $(p-s+1 \leqslant i \leqslant p)$ by negative edges. Then it is clear that $S$ is a signed graph with degree sequence $\sigma$.

Conversely, let $\sigma$ be the degree sequence of a signed graph. Then there exists a signed graph $T$ having degree sequence $\sigma$. We may assume that the vertices of $T$ are labeled $v_1$, $v_2$, …, $v_p$ so that $\operatorname{sdeg} v_i = d_i$ for $i = 1, 2, \ldots, p$. So in $T$, if $v_1$ is incident to $d^+$ positive edges and $d^-$ negative edges, then $d_1 = \operatorname{sdeg} v_1 = d^+ - d^-$ and the degree of $v_1$ in the underlying graph of $T$ is $d^+ + d^-$, that is, $\deg v_1 = d^+ + d^-$.

We may assume that the vertices of all signed graphs having degree sequence $\sigma$ are labeled $v_1$, $v_2$, …, $v_p$ so that $\operatorname{sdeg} v_i = d_i$ for $i = 1, 2, \ldots, p$. Now consider the collection of all signed graphs having degree sequence $\sigma$ such that $\deg v_1 = d^+ + d^-$. Among all such signed graphs, choose $S$ to be one with the property that the sum of the degrees of the vertices joined to $v_1$ by positive edges is maximum. Let $r = d^+$ and $s = d^-$. We claim that $v_1$ must be joined by positive edges to the vertices of $S$ having degrees $d_2$, $d_3$, …, $d_{r+1}$. For assume that this is not the case. Then there exist vertices $v_i$ and $v_j$ with $i < j$ such that $v_1v_j$ is positive and either (1) $v_1v_i$ is negative or (2) $v_1$ and $v_i$ are not adjacent in $S$. Also, we know that $d_i > d_j$. The proofs of these two cases are similar, so we consider only (1).

First, note that if there exists a vertex $v_n$ such that $v_nv_i$ is positive and $v_nv_j$ negative, then we may relabel the edges so that $v_1v_i$ and $v_nv_j$ are positive and $v_1v_j$

and $v_n v_i$ are negative. (See Figure 8.) This results in a signed graph in which the sum of the degrees of the vertices joined to $v_1$ by positive edges exceeds that in $S$, which contradicts our choice of $S$. Hence, we may assume that there is no such vertex $v_n$ in $S$.

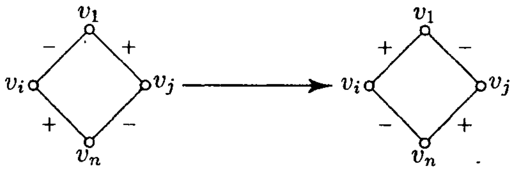


Figure 8. A relabeling of the edges in the signed graph $S$.

Next, assume that $v_i$ is not incident to any positive edges. Then since $d_i > d_j$, there exist at least two negative edges $v_n v_j$ and $v_k v_j$ such that $v_n$ and $v_k$ are not adjacent to $v_i$. This situation is shown in Figure 9. If the edges are changed as shown also in Figure 9, then again a contradiction is produced. So $v_i$ must be incident to at least one positive edge.

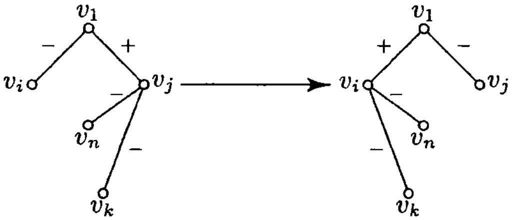


Figure 9. Constructing a new signed graph with degree sequence $\sigma$.

We claim that, in fact, there must exist at least one vertex $v_m$ such that $v_m v_i$ is positive and $v_m$ is not adjacent to $v_j$. Assume, instead, that whenever $v_i$ is joined to a vertex by a positive edge, then $v_j$ is also joined to this vertex by a positive edge. Since $d_i > d_j$, we have the situation illustrated in Figure 9, which as we have seen, contradicts the choice of $S$. Hence, as claimed, there exists a vertex $v_m$ in $S$ such that $v_m v_i$ is positive and $v_m$ is not adjacent to $v_j$. In a similar manner, it is possible to show that there exists a vertex $v_n$ such that $v_n v_j$ is negative and $v_n$ is not adjacent to $v_i$. Hence, we now have the situation shown in Figure 10. Changing the edges as illustrated in Figure 10, we again contradict the choice of $S$. Thus, $v_1$ must be joined by positive edges to the vertices of degrees $d_2, d_3, \ldots, d_{r+1}$ in $S$.

Next, we claim that $v_1$ is joined by negative edges to the vertices of degrees $d_{p-s+1}$, $d_{p-s+2}, \ldots, d_p$. Actually, the proof of this fact is similar to the argument just given and so we omit it. In conclusion, observe that the signed graph $S - v_1$ has degree sequence $\sigma'$. □

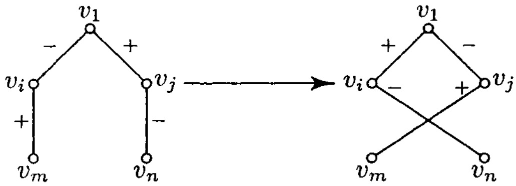


Figure 10. Constructing another signed graph with degree sequence $\sigma$.

To illustrate this theorem, consider the sequence $\sigma_1: 5, 4, 3, 0, -1, -1, -2$. Then $0 \leqslant s \leqslant \frac{1}{2}$, so we must choose $s = 0$ and $r = 5$. Applying Theorem 6, we obtain the sequence $\sigma_2: 3, 2, -1, -2, -2, -2$. Now observe that there are two possible choices for $s$, namely 0 or 1. If we choose $s = 1$ and $r = 4$, then we obtain the sequence $\sigma_3': 1, -2, -3, -3, -1$. Putting $\sigma_3'$ into standard form gives $\sigma_3: 3, 3, 2, 1, -1$. Now, choosing $s = 0$ and $r = 3$, we obtain $\sigma_4: 2, 1, 0, -1$. Finally, if we choose $s = 0$ and $r = 2$, then we have $\sigma_5: 0, -1, -1$, which is clearly graphical. Hence, by Theorem 6, it follows that $\sigma_1$ is also graphical.

Next, consider the sequence $\sigma_1 : 3, 0, -1, -2$. The only choice of $s$ is $s = 0$ in which case we have $r = 3$. Applying Theorem 6, we obtain $\sigma_2: \ -1, -2, -3$. Clearly $\sigma_2$ is not graphical as there is no signed graph of order 3 containing a vertex of degree $-3$. Hence $\sigma_1$ is not graphical.

In both examples, we chose $s \neq 0$ only once. In fact, we would have come to the same conclusion had we chosen $s = 0$ in this one instance. This could lead us to believe that one can always choose $s = 0$ in Theorem 6. The discussion that follows addresses this question.

Let $\sigma: d_1, d_2, \ldots, d_p$ be a standard sequence, and let $\sigma'$ be the sequence $d_2 - 1$, $d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_p$. Motivated by the Havel-Hakimi-Harary Theorem, we say that $\sigma'$ is obtained from $\sigma$ by *procedure* $H$.

Let $\sigma_1$ be a given sequence in standard form, and let $\sigma_2'$ be the sequence obtained from $\sigma_1$ by procedure $H$. Furthermore, let $\sigma_2$ be the standard sequence obtained from $\sigma_2'$ by rearranging the terms of $\sigma_2'$ or $-\sigma_2'$, as necessary. When this is done, we will say that we have *standardized* $\sigma_2'$ (to obtain $\sigma_2$). This process will be referred to as *standardization*. Continuing in this manner, we can construct a maximum number of sequences $\sigma_1, \sigma_2, \ldots, \sigma_k$, where for $i = 1, 2, \ldots, k - 1$, the sequence $\sigma_{i+1}$ is obtained from $\sigma_i$ by procedure $H$ and standardization. Thus $\sigma_i$ is standard for $1 \leqslant i \leqslant k - 1$ and $\sigma_k$ is not standard. Although $\sigma_k$ is not standard, we will write it uniquely in nonincreasing order whose first term in absolute value is at least as large as the last term in absolute value. Then we say that $\sigma_1, \sigma_2, \ldots, \sigma_k$ is the $H$*-sequence* corresponding to $\sigma_1$. Now since $\sigma_k$ is not standard, either $\sigma_k$ is a zero sequence or $\sigma_k$ contains a term whose absolute value equals the number of terms in $\sigma_k$. The first case is addressed in our next result.

**Theorem 7.** *Let $\sigma_1$ be a standard sequence and let $\sigma_1$, $\sigma_2$, …, $\sigma_k$ be the $H$-sequence corresponding to $\sigma_1$. If $\sigma_k$ is a zero sequence, then $\sigma_1$ is graphical.*

Proof. We begin by showing that if $\sigma_{i+1}$ is graphical for some $i \in \{1, 2, \ldots, k-1\}$, then $\sigma_i$ is graphical. Suppose that $\sigma_i$ is the sequence $a_1$, $a_2$, …, $a_p$. Then, performing procedure $H$ on $\sigma_i$, we obtain the sequence $\sigma'_{i+1} : a_2-1, a_3-1, \ldots, a_{a_1+1}-1, a_{a_1+2}, \ldots, a_p$. So $\sigma_{i+1}$ is either a rearrangement of $\sigma'_{i+1}$ or a rearrangement of $-\sigma'_{i+1}$. Since $\sigma_{i+1}$ is graphical, $\sigma'_{i+1}$ is graphical, that is, there exists a signed graph $S'$ with degree sequence $\sigma'_{i+1}$. Let the vertices of $S'$ be labeled so that

$$\operatorname{sdeg} v_i = \begin{cases} a_i - 1, & 2 \leqslant i \leqslant a_1 + 1, \\ a_i, & a_1 + 2 \leqslant i \leqslant p. \end{cases}$$

We now construct a new signed graph $S$ by adding a vertex $v_1$ to $S'$ and the positive edges $v_1 v_i$ for $i = 2, 3, \ldots, a_1 + 1$. Then $S$ is a signed graph having degree sequence $\sigma_i$. Hence, $\sigma_i$ is graphical.

Finally, since $\sigma_k$ is a zero sequence and $\sigma_k$ is graphical, it follows, by the previous argument, that all sequences preceding $\sigma_k$ are graphical. In particular, $\sigma_1$ is graphical. □

As an example, consider the sequence $\sigma_1 \colon 3, 0, 0, 0, -3$. Observe that the $H$-sequence corresponding to $\sigma_1$ is $\sigma_1$, $\sigma_2$, $\sigma_3$, where

$$\begin{aligned} &\sigma_1 : 3, 0, 0, 0, 0, -3, \\ &\sigma_2 : 3, 1, 1, 1, 0, \\ &\sigma_3 : 0, 0, 0, 0. \end{aligned}$$

Since $\sigma_3$ is a zero sequence, $\sigma_1$ is graphical.

On the basis of Theorem 7, then, if we are given a standard sequence $\sigma_1$ and construct its corresponding $H$-sequence $\sigma_1$, $\sigma_2$, …, $\sigma_k$, it follows that $\sigma_1$ is graphical if $\sigma_k$ is a zero sequence. But what if $\sigma_k$ is a nonzero sequence? Then $\sigma_k$ must contain a term whose absolute value equals the number of terms in $\sigma_k$. Of course, $\sigma_k$ is not graphical. What does this say about $\sigma_1$? In order to gain some insight into the answer to this question, we present the following necessary condition for a standard sequence to be graphical.

**Theorem 8.** *Let $\sigma \colon d_1, d_2, \ldots, d_p$ be a standard sequence. If $\sigma$ is graphical, then for every pair $a, b$ of nonnegative integers with $1 \leqslant a + b \leqslant p$ such that $d_a \geqslant 0$ when $a > 0$ and $d_{p-b+1} \leqslant 0$ when $b > 0$, it follows that*

$$\sum_{i=1}^{a} d_i - \sum_{i=p-b+1}^{p} d_i \leqslant (p-1)(a+b) - 2ab.$$

Proof. Since $\sigma$ is graphical, there exists a signed graph $S$ with vertices $v_1$, $v_2$, ..., $v_p$ such that $\operatorname{sdeg} v_i = d_i$ for $i = 1, 2, \ldots, p$. Let $a$ and $b$ be nonnegative integers with $1 \leqslant a+b \leqslant p$ such that $d_a \geqslant 0$ when $a > 0$ and $d_{p-b+1} \leqslant 0$ when $b > 0$.

Let $A = \{v_1, v_2, \ldots, v_a\}$ and $B = \{v_{p-b+1}, v_{p-b+2}, \ldots, v_p\}$. Furthermore, let $C$ denote the set consisting of the remaining $p-a-b$ vertices of $S$. If each vertex $v_i$ is incident with $d_i^+$ positive edges and $d_i^-$ negative edges, then

$$\sum_{i=1}^{a} d_i - \sum_{i=p-b+1}^{p} d_i = \sum_{i=1}^{a} (d_i^+ - d_i^-) - \sum_{i=p-b+1}^{p} (d_i^+ - d_i^-).$$

Note that $\sum_{i=1}^{a} d_i$ is the sum of the degrees of the vertices in $A$, while $\sum_{i=p-b+1}^{p} d_i$ is the sum of the degrees of the vertices in $B$. Now if $e$ is a positive edge joining a vertex of $A$ to a vertex of $B$, then $e$ contributes 1 to $\sum_{i=1}^{a} (d_i^+ - d_i^-)$ and also contributes 1 to $\sum_{i=p-b+1}^{p} (d_i^+ - d_i^-)$. Hence the contribution of $e$ to the expression $\sum_{i=1}^{a} d_i - \sum_{i=p-b+1}^{p} d_i$ is 0. A similar situation results if $e$ is a negative edge. Thus, when computing $\sum_{i=1}^{a} d_i - \sum_{i=p-b+1}^{p} d_i$, the edges joining vertices of $A$ to vertices of $B$ may be ignored.

Any other positive edge incident with the vertices $v_i$ $(1 \leqslant i \leqslant a)$ either joins two vertices of $A$ or joins a vertex of $A$ to a vertex of $C$. Similarly, any other negative edge incident with the vertices of $v_i$ $(p-b+1 \leqslant i \leqslant p)$ joins two vertices of $B$ or joins a vertex of $B$ and a vertex of $C$. Thus

$$\begin{aligned}\sum_{i=1}^{a} d_i - \sum_{i=p-b+1}^{p} d_i &\leqslant a(a-1) + a(p-a-b) + b(b-1) + b(p-a-b)\\ &= (p-1)(a+b) - 2ab.\end{aligned}$$

□

This theorem gives us a way to check if a given sequence is *not* graphical, namely, we have the following corollary, which is simply a statement of the contrapositive of Theorem 8.

**Corollary 8a.** *Let $\sigma\colon d_1, d_2, \ldots, d_p$ be a standard sequence. If there exist nonnegative integers $a$ and $b$ with $1 \leqslant a+b \leqslant p$, where $d_a \geqslant 0$ if $a > 0$ and $d_{p-b+1} \leqslant 0$ if $b > 0$ such that*

$$\sum_{i=1}^{a} d_i - \sum_{i=p-b+1}^{p} d_i > (p-1)(a+b) - 2ab,$$

*then $\sigma$ is not graphical.*

As in example, consider the sequence $\sigma$: $7,7,7,7,6,3,3,0$. Now choose $a = 4$ and $b = 1$, and observe that

$$\sum_{i=1}^{a} d_i - \sum_{i=p-b+1}^{p} d_i = 7+7+7+7-0 = 28$$

while $(p-1)(a+b) - 2ab = 7 \cdot 5 - 8 = 27$. Since $28 > 27$, it follows by Corollary 8a that $\sigma$ is not graphical.

We now return to the question at hand, namely, if we are given a standard sequence $\sigma_1$ and its corresponding $H$-sequence $\sigma_1, \sigma_2, \ldots, \sigma_k$, where the first term of $\sigma_k$ equals the number of terms in $\sigma_k$, then what can we say about $\sigma_1$? As our next result states, in this case, $\sigma_{k-1}$ is not graphical.

**Theorem 9.** *Let $\sigma_1$ be a standard sequence, and let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be the $H$-sequence corresponding to $\sigma_1$. If the first term of $\sigma_k$ equals the number of terms in $\sigma_k$, then $\sigma_{k-1}$ is not graphical.*

Proof. Suppose that $\sigma_k$ contains $r$ terms. Then the first term of $\sigma_k$ is $r$. Since $\sigma_k$ was obtained by applying procedure $H$ to $\sigma_{k-1}$ and then standardization, it follows that $\sigma_{k-1}$ contains $r+1$ terms and contains either (1) the terms $r$ and $-(r-1)$, or (2) the terms $r-1$, $-(r-1)$, and $-(r-1)$. We consider these two cases.

Case 1: *The sequence $\sigma_{k-1}$ contains $r$ and $-(r-1)$.* Then choose $a = 1$ and $b = 1$ in Corollary 8a. Now observe that

$$r - \big(-(r-1)\big) = 2r - 1 > 2r - 2.$$

Hence it follows that $\sigma_{k-1}$ is not graphical.

Case 2: *The sequence $\sigma_{k-1}$ contains the terms $r-1$, $-(r-1)$, and $-(r-1)$.* Then choose $a = 1$ and $b = 2$. Then

$$r - 1 + 2(r-1) = 3r - 3 > 3r - 4.$$

Thus, again by Corollary 8a, $\sigma_{k-1}$ is not graphical. □

The following result extends the previous one to say that in addition to $\sigma_{k-1}$, the two sequences $\sigma_{k-2}$ and $\sigma_{k-3}$ are also not graphical when the first term of $\sigma_k$ equals the number of terms in $\sigma_k$. The tedious proof is similar to that of Theorem 9 and hence is omitted.

**Theorem 10.** *Let $\sigma_1$ be a standard sequence with corresponding $H$-sequence $\sigma_1$, $\sigma_2, \ldots, \sigma_k$. If the first term of $\sigma_k$ equals the number of terms in $\sigma_k$, then $\sigma_{k-1}$, $\sigma_{k-2}$, and $\sigma_{k-3}$ are not graphical.*

Now let us consider some examples. First, we have already seen that the sequence $\sigma_1 : 7,7,7,7,6,3,3,0$ is not graphical. Observe that the $H$-sequence corresponding to $\sigma_1$ is $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$, $\sigma_5$, where

$$\begin{aligned}
&\sigma_1 : 7,7,7,7,6,3,3,0,\\
&\sigma_2 : 6,6,6,5,2,2,-1,\\
&\sigma_3 : 5,5,4,1,1,-2,\\
&\sigma_4 : 4,3,0,0,-3,\\
&\sigma_5 : 4,1,1,-2.
\end{aligned}$$

Since $\sigma_5$ contains four terms, the first of which is 4, it follows by Theorem 10 that $\sigma_2$, $\sigma_3$, and $\sigma_4$ are not graphical. It is important to note, however, that we cannot use Theorem 10 to show that $\sigma_1$ is not graphical. Recall that, in fact, we used Corollary 8a to verify this.

As a second example, consider the sequence $\sigma_1 : 5,5,5,5,4,2$ and its corresponding $H$-sequence $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$, $\sigma_5$. So we have

$$\begin{aligned}
&\sigma_1 : 5,5,5,5,4,2,\\
&\sigma_2 : 4,4,4,3,1,\\
&\sigma_3 : 3,3,2,0,\\
&\sigma_4 : 2,1,-1,\\
&\sigma_5 : 2,0.
\end{aligned}$$

Again, by Theorem 10, we know that $\sigma_2$, $\sigma_3$, and $\sigma_4$ are not graphical.We would like to determine whether $\sigma_1$ is graphical. Obviously, we cannot use Theorem 10 to show that $\sigma_1$ is not graphical. In the previous example, we used Corollary 8a. It turns out that the same approach will not work here, that is, for every pair $a$, $b$ of nonnegative integers with $1 \leqslant a+b \leqslant 6$ such that $d_a \geqslant 0$ if $a>0$ and $d_{p-b+1} \leqslant 0$ if $b>0$, it follows that

$$\sum_{i=1}^{a} d_i - \sum_{i=p-b+1}^{p} d_i \leqslant (p-1)(a+b) - 2ab.$$

Of course, from this we will neither be able to conclude that $\sigma_1$ is graphical nor not graphical. Notice that $\sigma_1$ has no zeros or negative terms. Thus the only choice for $b$ is 0. Therefore, the inequality above simplifies to

$$\sum_{i=1}^{a} d_i \leqslant (p-1)a,$$

which is true for each choice of $a$. So the sequence $\sigma_1$ cannot be shown to be not graphical by Theorem 8.

We now prove that, in fact, $\sigma_1$ is not graphical, for suppose, instead, that $\sigma_1$ is graphical. Then there exists a signed graph $S$ with degree sequence $\sigma_1$. So four vertices of $S$ have degree 5, that is, four vertices are joined by positive edges to every other vertex of $S$. These edges account for 14 of the edges of $S$. Since $S$ can have at most 15 edges, it is clear that the two remaining vertices of $S$ do not have the required degrees. So $\sigma_1$ is not graphical.

**Conjecture 1.** Let $\sigma_1$ be a given standard sequence and let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be its corresponding $H$-sequence. Then $\sigma_1$ is graphical if and only if $\sigma_k$ is graphical.

The truth of this conjecture can be established from the following conjecture.

**Conjecture 2.** If $\sigma\colon d_1, d_2, \ldots, d_p$ is a standard graphical sequence, then there exists a signed graph containing a vertex of degree $d_1$ incident only with positive edges.

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