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ON SIGNORINI PROBLEM FOR VON KÁRMÁN EQUATIONS

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1. INTRODUCTION

In this article we deal with the Signorini boundary value problem for the system of von Kármán equations. The existence theorem (Theorem 4.1) is proved in the case of rather general boundary conditions. All restrictions concerning the shape of the plate and the boundary functions (i.e. (4.2) and (4.8)) seem to be quite natural from the physical points of view. There is only one unpleasant restrictive condition, namely, that the boundary $\partial\Omega$ of Ω is infinitely smooth. (Ω is the simply connected bounded domain corresponding to the middle plane of the undeflected plate.) This can be weakened immediately to $\Omega \in C^3$. Probably, apart from some additional technical difficulties, Theorem 4.1 is available also for an angular domain whose boundary is piecewise of C^3.

Theorem 4.1 is obtained by means of Theorem 5.3 (see also [9]) which is a generalization of the result of J. L. Lions and Q. Stampacchia [6]. We make the application of this abstract result possible by transforming previously our variational problem in Section 6 to a suitable form using the idea of G. H. Knightly (see [3], [2]).

The present paper is a continuation of the article [2] by J. Nečas and the author. J. Naumann in [7] studies the unilateral problems by a different method.

2. CLASSICAL FORMULATION OF THE BOUNDARY VALUE PROBLEM

Let Ω be a simply connected bounded domain in E_2 with infinitely smooth boundary $\partial\Omega$ (see [2], Definition 5) divided into three pairwise disjoint subsets $\Gamma_1, \Gamma_2, \Gamma_3, \bigcup_{i=1}^{3} \Gamma_i = \partial\Omega$. We suppose that either $\Gamma_i = \emptyset$ or Γ_i^0 is a union of finitely many sets λ_j where each λ_j is homeomorphic with an open interval. (Γ_i^0 is the interior of Γ_i with respect to $\partial\Omega$.)

As usual we write w_x instead of $\partial w/\partial x$, w_{xy} instead of $\partial^2 w/\partial x\, \partial y$ etc. The vector function $n = (n_x, n_y)$ maps each point of $\partial\Omega$ onto the unit vector of the outer normal to $\partial\Omega$ at this point and we define the normal derivative w_n and the tangential derivative w_τ as

$$w_n = w_x n_x + w_y n_y , \quad w_\tau = w_x(-n_y) + w_y n_x . \tag{2.1}$$

Denote further

$$\Delta^2 w = \Delta(\Delta w) = w_{xxxx} + 2w_{xxyy} + w_{yyyy} \tag{2.2}$$

and

$$[w, f] = w_{xx} f_{yy} + w_{yy} f_{xx} - 2w_{xy} f_{xy} . \tag{2.3}$$

Finally, let the boundary operators M and T be defined as follows:

$$Mw = \nu \Delta w + (1 - \nu)(w_{xx} n_x^2 + 2w_{xy} n_x n_y + w_{yy} n_y^2) , \tag{2.4}$$

$$Tw = (-\Delta w)_n + (1 - \nu)(w_{xx} n_x n_y - w_{xy}(n_x^2 - n_y^2) - w_{yy} n_x n_y)_\tau \tag{2.5}$$

where ν is Poisson's constant $(0 \leqq \nu < \frac{1}{2})$.

Being interested above all in the variational solutions of boundary value problems we do not give the classical formulation in full detail. We mention it here to make the situation clearer.

Three types of problems will be introduced $-$ R, $S_{\rm I}$ and $S_{\rm II}$. Each problem will be specified on the one hand by the division of $\partial\Omega$ into Γ_i $(i = 1, 2, 3)$ and on the other hand by the following given functions: $q : \Omega \to E_1$ (which represents the density of the perpendicular load), $\Phi_0, \Phi_1 : \partial\Omega \to E_1$ (boundary value of the Airy stress function and its normal derivative) and $k_2, m_2 : \Gamma_2 \to E_1, k_{31}, k_{32}, m_3, r_3 : \Gamma_3 \to E_1$. (Functions k are the coefficients in the boundary conditions concerning elastically supported and elastically clamped part of $\partial\Omega$ while m and r are the given bending moments and shearing forces on the corresponding parts of the boundary.)

2.1. **Definition.** *A pair of functions $w, \Phi \in C^4(\overline{\Omega})$ is said to be a classical solution of the boundary value problem R if*

$$\Delta^2 w = [\Phi, w] + q \quad on \quad \Omega , \tag{2.6}$$

$$\Delta^2 \Phi = -[w, w] \quad on \quad \Omega , \tag{2.7}$$

$$\Phi = \Phi_0 \quad and \quad \Phi_n = \Phi_1 \quad on \quad \partial\Omega , \tag{2.8}$$

$$w = w_n = 0 \quad on \quad \Gamma_1 , \tag{2.9}$$

$$w = 0 , \quad Mw + k_2 w_n = m_2 \quad on \quad \Gamma_2 , \tag{2.10}$$

$$Mw + k_{31} w_n = m_3 \quad on \quad \Gamma_3 , \tag{2.11}$$

$$Tw + k_{32} w = r_3 \quad on \quad \Gamma_3 . \tag{2.12}$$

Remark. This problem was treated in the paper [2]. Remember here at least the meaning of the special case of boundary conditions: Γ_1 is the clamped part of the boundary. If $k_2 = m_2 = 0$ on Γ_2 then Γ_2 is simply supported. If $k_{31} = m_3 = k_{32} = r_3 = 0$ on Γ_3 then Γ_3 is the free part of the boundary $\partial\Omega$.

2.2. **Definition.** *A pair of functions* $w, \Phi \in C^4(\bar{\Omega})$ *is said to be a classical solution of the problem* S_I *if it satisfies the equations* (2.6)–(2.11) *and if*

$$(2.13) \qquad w \geqq 0 \quad and \quad Tw + k_{32}w \geqq r_3 \quad and$$
$$w(Tw + k_{32}w - r_3) = 0 \quad on \quad \Gamma_3 .$$

Remark. In the special case of $k_{32} = r_3 = 0$ the condition (2.13) describes the following situation: The edge Γ_3 of the plate lies on a rigid base so that it can be deflected only upwards ($w \geqq 0$). The possible shearing force is the reaction of the base which acts in the positive sense ($Tw \geqq 0$). If $w(P) > 0$ at a point $P \in \Gamma_3$, no reaction of the base acts there so that $Tw(P) = 0$ and the product $w(P) \,.\, Tw(P) = 0$. If $w(P) = 0$ then $Tw(P)$ can be >0 but the product $w(P) \,.\, Tw(P)$ is zero again.

2.3. **Definition.** *A pair of functions* $w, \Phi \in C^4(\bar{\Omega})$ *is said to be a classical solution of the problem* S_{II} *if it satisfies the equations* (2.6)–(2.10), (2.12) *and the condition*

$$(2.14) \qquad w_n \geqq 0 \quad and \quad Mw + k_{31}w_n \geqq m_3 \quad and$$
$$w_n(Mw + k_{31}w_n - m_3) = 0 \quad on \quad \Gamma_3 .$$

3. VARIATIONAL FORMULATION OF THE BOUNDARY VALUE PROBLEMS

First we introduce the necessary notation:

$$(3.1) \qquad (u, v)_{W_0^{2,2}} = \int_\Omega (u_{xx}v_{xx} + 2u_{xy}v_{xy} + u_{yy}v_{yy}) \,\mathrm{d}x \,\mathrm{d}y ,$$

$$(3.2) \qquad (u, v)_V = (u, v)_{W_0^{2,2}} + v \int_\Omega [u, v] \,\mathrm{d}x \,\mathrm{d}y ,$$

$$(3.3) \qquad a(u, v) = \int_{\Gamma_2} k_2 u_n v_n \,\mathrm{d}S + \int_{\Gamma_3} (k_{31}u_n v_n + k_{32}uv) \,\mathrm{d}S ,$$

$$(3.4) \qquad A(u, v) = (u, v)_V + a(u, v) ,$$

$$(3.5) \qquad B(v; u, \varphi) = \int_\Omega (v_{xy}u_x\varphi_y + v_{xy}u_y\varphi_x - v_{yy}u_x\varphi_x - v_{xx}u_y\varphi_y) \,\mathrm{d}x \,\mathrm{d}y .$$

Notice that for $\varphi \in W_0^{2,2}$

$$B(v; u, \varphi) = B(\varphi; v, u) = B(u; \varphi, v) . \tag{3.6}$$

Using the Hölder inequality we obtain

$$|B(w; v, \varphi)| \leqq c\|w\|_{W_0^{2,2}} \|v\|_{W^{1,4}} \|\varphi\|_{W^{1,4}} . \tag{3.7}$$

Define further the subspace V of $W^{2,2}(\Omega)$ as $\overline{\mathscr{V}}$ where

$$\mathscr{V} = \{v \in C^\infty(\overline{\Omega});\ v = v_n = 0 \text{ on } \Gamma_1,\ v = 0 \text{ on } \Gamma_2\} \tag{3.8}$$

(the closure in $W^{2,2}(\Omega)$) and two cones in this subspace

$$\tilde{K}_{\mathrm{I}} = \{v \in V;\ v \geqq 0 \text{ on } \Gamma_3\} , \quad \tilde{K}_{\mathrm{II}} = \{v \in V;\ v_n \geqq 0 \text{ on } \Gamma_3\} . \tag{3.9}$$

3.1. **Definition.** *A pair of functions $w \in V$, $\Phi \in W^{2,2}(\Omega)$ is said to be a variational solution of the problem R if Φ satisfies the condition (2.8) in the sense of traces, if further*

$$A(w, v - w) \geqq B(w; \Phi, v - w) + \int_\Omega q(v - w)\,\mathrm{d}x\,\mathrm{d}y + \tag{3.10}$$
$$+ \int_{\Gamma_2} m_2(v_n - w_n)\,\mathrm{d}S + \int_{\Gamma_3} [m_3(v_n - w_n) + r_3(v - w)]\,\mathrm{d}S$$

holds for each $v \in V$ and if

$$\forall \psi \in W_0^{2,2}(\Omega) : (\Phi, \psi)_{W_0^{2,2}} = -B(w; w, \psi) . \tag{3.11}$$

3.2. **Definition.** *A pair of functions $w \in \tilde{K}_{\mathrm{I}}$, $\Phi \in W^{2,2}(\Omega)$, ($w \in \tilde{K}_{\mathrm{II}}$, $\Phi \in W^{2,2}(\Omega)$, resp.), is said to be a variational solution of the problem S_{I} (S_{II}) if Φ satisfies the condition (2.8) in the sense of traces, if further (3.11) holds and if the inequality (3.10) is satisfied for all $v \in \tilde{K}_{\mathrm{I}}$ ($v \in \tilde{K}_{\mathrm{II}}$).*

Remark. The relation between the variational and classical solution is explained in Section 8.

4. THE MAIN RESULT

Define

$$Y_V = \{v \in V;\ A(v, v) = 0\} . \tag{4.1}$$

It follows from the definition of A by means of the formula (3.4) that Y_V is a linear subset of the set of all polynomials of the first order restricted to the domain Ω.

4.1. **Theorem.** *Suppose that Ω is the domain described in Section 2 and that*

(4.2) *either $\Gamma_2 = 0$ or Γ_2 is a union of finitely many segments of the straight line.*

Let the prescribed functions k_2, k_{31}, k_{32} and q, m_2, m_3, r_3 satisfy the following conditions:

(4.3) $k_2 \in L_p(\Gamma_2)$ $(p > 1)$ *and* $k_2 \geqq 0$ *almost everywhere on* Γ_2 ,

(4.4) $k_{31} \in L_p(\Gamma_3)$ $(p > 1)$ *and* $k_{31} \geqq 0$ *almost everywhere on* Γ_3 ,

(4.5) $k_{32} \in L_1(\Gamma_3)$ *and* $k_{32} \geqq 0$ *almost everywhere on* Γ_3 ,

(4.6) $$q \in L_p(\Omega)\,, \quad (p > 1)\,,$$

(4.7) $$m_2 \in L_p(\Gamma_2) \quad \text{and} \quad m_3 \in L_p(\Gamma_3)\,, \quad (p > 1)\,, \quad r_3 \in L_1(\Gamma_3)\,.$$

Let the functions Φ_0 and Φ_1 defined on $\partial\Omega$ almost everywhere have the following properties:

(4.8) $$\Phi_0 = \Phi_1 = 0 \quad on \quad \Gamma_3\,,$$

(4.9) *there exists a function $F \in C^2(\bar{\Omega})$ for which*

$$F = \Phi_0 \quad and \quad F_n = \Phi_1 \quad on \quad \partial\Omega\,.$$

Then the following assertions hold:

(i) *If $Y_V = \{0\}$ then there exists a variational solution of the problem R.*

(ii) *If $\tilde{K}_{\text{I}} \cap Y_V = \{0\}$ then there exists a variational solution of the problem S_{I}. If $\tilde{K}_{\text{I}} \cap Y_V \neq \{0\}$ and if simultaneously each $z \in \tilde{K}_{\text{I}} \cap Y_V \setminus \{0\}$ satisfies the inequality*

(4.10) $$\int_\Omega qz \,\mathrm{d}x\,\mathrm{d}y + \int_{\Gamma_2} m_2 z_n \,\mathrm{d}S + \int_{\Gamma_3} [m_3 z_n + r_3 z] \,\mathrm{d}S < 0$$

then there exists a variational solution of the problem S_{I}.

(iii) *If $\tilde{K}_{\text{II}} \cap Y_V = \{0\}$ then there exists a variational solution of the problem S_{II}. If $\tilde{K}_{\text{II}} \cap Y_V \neq \{0\}$ and if simultaneously each $z \in \tilde{K}_{\text{II}} \cap Y_V \setminus \{0\}$ satisfies the inequality* (4.10) *then there exists a variational solution of the problem S_{II}.*

In the proof of Theorem 4.1 we employ Knightly's idea of transforming the problems in question to nonlinear operator inequalities (Section 6) to which we apply in Section 7 the abstract Theorem 5.3.

5. ABSTRACT EXISTENCE THEOREM

Let H be a real Hilbert space with its norm $\|\cdot\|_H$ and let p_1 be a seminorm in H. Let $\langle f, v\rangle$ denote the pairing between H' and H.

5.1. **Definition.** *An operator* $\mathcal{T}: H \to H'$ *is said to be semicoercive on* H *with respect to the seminorm* p_1 *if there exists a function*

$$G: \langle 0, +\infty) \to \langle 0, +\infty) \tag{5.1}$$

for which

$$\lim_{z \to +\infty} G(z) = +\infty \tag{5.2}$$

and a positive constant C *such that*

$$\forall v \in H: \langle \mathcal{T}(v), v \rangle \geqq p_1(v) \,.\, G(p_1(v)) - C \,. \tag{5.3}$$

5.2. **Definition.** *An operator* $\mathcal{T}: H \to H'$ *is said to be pseudomonotone if*

$$\mathcal{T} \text{ is bounded on } H \tag{5.4}$$

and

(5.5) $u^n \rightharpoonup u$ *and* $\limsup \langle \mathcal{T}(u^n), u^n - u \rangle \leqq 0$ *implies* $\liminf \langle \mathcal{T}(u^n), u^n - v \rangle \geqq \geqq \langle \mathcal{T}(u), u - v \rangle$ *for all* $v \in H$.

5.3. **Theorem.** *Let* H *be a real Hilbert space with the norm* $\|\cdot\|_H$, *let* p_1 *be a seminorm in* H *and* p_0 *an other norm in* H *such that* H *is pre-Hilbert with respect to* p_0.

Let further the following assumptions be satisfied:

(5.6) $p_0(\cdot) + p_1(\cdot)$ *is a norm equivalent with* $\|\cdot\|_H$,

(5.7) *the subspace* $Y = \{z \in H;\ p_1(z) = 0\}$ *has a finite dimension,*

(5.8) *there exists* $c_1 > 0$ *such that for all* $v \in H$,

$$\inf_{y \in Y} p_0(v + y) \leqq c_1\, p_1(v)\,,$$

(5.9) K *is a closed convex subset of* H, $0 \in K$,

and

(5.10) $\mathcal{T}: H \to H'$ *is a pseudomonotone operator semicoercive with respect to the seminorm* p_1.

Then the following two assertions hold:

(i) *If* $K \cap Y = \{0\}$ *then for each* $f \in H'$ *there exists an element* $w \in K$ *such that*

$$\langle \mathcal{T}(w), v - w \rangle \geqq \langle f, v - w \rangle\,, \quad \forall v \in K\,. \tag{5.11}$$

(ii) *If* $K \cap Y \neq \{0\}$ *then for each* $f \in H'$ *which can be decomposed into a sum* $f = f_0 + f_1$ *in such a way that*

$$(5.12) \qquad \langle f_0, y \rangle < 0 \quad \textit{for each} \quad y \in K \cap Y \setminus \{0\}$$

and

$$(5.13) \qquad \exists c_2 > 0 \colon \ |\langle f_1, v \rangle| \leqq c_2\, p_1(v) \quad \textit{for each} \quad v \in H$$

there exists $w \in K$ *such that* (5.11) *holds.*

Proof. Denote

$$(5.14) \qquad K_n = K \cap \{v \in H;\ \|v\|_H \leqq n\}\,.$$

Each K_n is a closed convex nonempty subset of H. According to [4], Theorem 8.1, there exists for each $f \in H'$ a solution $u \in K_n$ of the inequality

$$(5.15) \qquad \langle \mathcal{T}(u), v - u \rangle \geqq \langle f, v - u \rangle\,, \quad \forall v \in K_n\,.$$

Lemma A. *Let* $u \in K_n$ *be a solution of* (5.15) *and let* $\|u\|_H < n$. *Then* u *is a solution of* (5.11).

Indeed, fix $v \in K$ and denote $w_\lambda = \lambda v + (1 - \lambda)\, u$. We can choose $\bar{\lambda} \in (0, 1)$ such that $w_{\bar{\lambda}} \in K_n$. Substituting into (5.15) we have

$$\langle \mathcal{T}(u), w_{\bar{\lambda}} - u \rangle \geqq \langle f, w_{\bar{\lambda}} - u \rangle$$

which implies immediately

$$\langle \mathcal{T}(u), v - u \rangle \geqq \langle f, v - u \rangle$$

and Lemma A is proved.

According to Lemma A it suffices to prove that

(5.16) in each sequence $\{u^n\}_{n=1}^{\infty}$ ($u^n \in K_n$) of solutions of the inequalities (5.15) there exists u^{n_0} such that $\|u^{n_0}\|_H < n_0$.

Let (5.16) be not valid, i.e.,

(5.17) there exists a sequence $\{\tilde{u}^n\}_{n=1}^{\infty}$ ($\tilde{u}^n \in K_n$) of solutions of the inequalities (5.15) such that $\|\tilde{u}^n\|_H = n$ for all $n \in N$.

We shall prove that under the assumptions of Theorem 5.3, (5.17) leads to a contradiction.

Denote

$$(5.18) \qquad w^n = \frac{\tilde{u}^n}{n}\,, \quad n \in N\,.$$

It is $w^n \in K$ for each $n \in N$. Because of the boundedness of $\{w^n\}_{n=1}^{\infty}$ in H there exists a weakly convergent subsequence $\{w^{n_k}\}_{k=1}^{\infty}$,

$$w^{n_k} \rightharpoonup w . \tag{5.19}$$

K is closed and convex which implies that K is weakly closed and so $w \in K$.

Lemma B. *Let w be defined by* (5.19). *It is* $p_1(w) = 0$ (*i.e.* $w \in Y$).

To prove that we put in (5.15) $\tilde{u}^{n_k}$ for u and 0 for v. We have

$$\langle \mathcal{T}(\tilde{u}^{n_k}), \tilde{u}^{n_k} \rangle \leqq \langle f, \tilde{u}^{n_k} \rangle$$

and using the inequality (5.3) (semicoerciveness),

$$p_1(\tilde{u}^{n_k})\, G(p_1(\tilde{u}^{n_k})) - C \leqq \langle f, \tilde{u}^{n_k} \rangle .$$

Hence it follows after substituting (5.18) that

$$p_1(w^{n_k})\, G(n_k\, p_1(w^{n_k})) - \frac{C}{n_k} \leqq \langle f, w^{n_k} \rangle \leqq \|f\|_{H'} .$$

Taking into account (5.2) we obtain finally

$$\lim_{k \to \infty} p_1(w^{n_k}) = 0 . \tag{5.20}$$

Denote now

$$P\colon H \to Y \text{ (the projection of } H \text{ on } Y \text{ with respect to } p_0). \tag{5.21}$$

From (5.6) it follows that P is a continuous mapping in H (with respect to $\|\cdot\|_H$). It is $w^{n_k} - Pw^{n_k} \rightharpoonup w - Pw$ in H. This together with (5.6), (5.8) and (5.20) yeilds $p_0(w - Pw) + p_1(w) \leqq M\|w - Pw\|_H \leqq M \liminf \|w^{n_k} - Pw^{n_k}\|_H \leqq MM_1 \liminf . \,.\, [p_0(w^{n_k} - Pw^{n_k}) + p_1(w^{n_k})] \leqq MM_1(c_1 + 1) \lim p_1(w^{n_k}) = 0$ and so $p_1(w) = 0$.

Lemma C. *Let $\{w^{n_k}\}$ be defined by* (5.19) *and let P be the projection defined by* (5.21). *Then* $p_0(Pw) > 0$.

Suppose that $p_0(Pw) = 0$. As the operator P is continuous in H and its range Y has a finite dimension it is totally continuous. So there exists a subsequence $\{z^n\}_{n=1}^{\infty}$ of $\{w^{n_k}\}_{k=1}^{\infty}$ such that

$$\lim_{n \to \infty} p_0(Pz^n) = p_0(Pw) = 0 . \tag{5.22}$$

On the other hand,

$$1 = \|z^n\|_H \leqq M_1(p_0(z^n) + p_1(z^n)) \leqq$$

$$\leqq M_1(p_0(z^n - Pz^n) + p_1(z^n) + p_0(Pz^n)) \leqq M_1((c_1 + 1)\, p_1(z^n) + p_0(Pz^n)) .$$

Using (5.22) and (5.20) we obtain a contradiction and Lemma C is proved.

So the assumption (5.17) yields the existence of an element $w \in K$ which belongs to Y (Lemma B) and is not zero (Lemma C). But this is impossible if $K \cap Y = \{0\}$. Thus in this case (5.16) takes place and the assertion (i) of Theorem 5.3 is proved.

Lemma D. *Let $K \cap Y \neq \{0\}$ and let $f \in H'$, $f = f_0 + f_1$ where $f_0 \in H'$ and $f_1 \in H'$ with f_0, f_1 satisfying (5.12), (5.13). Then (5.16) holds.*

Suppose on the contrary that (5.17) holds and let $\{w^{n_k}\}_{k=1}^{\infty}$ be defined by (5.18) and (5.19). Substituting in (5.15) $\tilde{u}^{n_k}$ for u and 0 for v and using the condition (5.3) of semicoerciveness we get

$$(5.23) \qquad \langle f, w^{n_k} \rangle \geqq -\frac{C}{n_k}, \quad k \in \boldsymbol{N}.$$

On the other hand,

$$(5.24) \qquad \langle f, w^{n_k} \rangle = \langle f_0, w^{n_k} - Pw^{n_k} \rangle + \langle f_0, Pw^{n_k} \rangle + \langle f_1, w^{n_k} \rangle .$$

Estimating $\langle f_0, w^{n_k} - Pw^{n_k} \rangle$ by $\|f_0\|_{H'} \|w^{n_k} - Pw^{n_k}\|_H$ and using (5.6), (5.8) and (5.20) we get

$$(5.25) \qquad \lim_{k \to \infty} \langle f_0, w^{n_k} - Pw^{n_k} \rangle = 0 .$$

From (5.13) it follows immediately that

$$(5.26) \qquad \lim_{k \to \infty} \langle f_1, w^{n_k} \rangle = 0 .$$

Using finally the fact that $Pw^{n_k} \rightharpoonup Pw$ together with Lemma C ($Pw \neq 0$) and Lemma B ($Pw = w \in K \cap Y$) we conclude

$$(5.27) \qquad \lim_{k \to \infty} \langle f_0, Pw^{n_k} \rangle = \langle f_0, Pw \rangle < 0 .$$

Passing to the limit in the equality (5.24) we obtain from (5.25)–(5.27) that $\langle f, w \rangle < 0$ while (5.23) yields that $\langle f, w \rangle \geqq 0$. So the assumption (5.17) leads to a contradiction and the proof of Lemma D is complete.

Lemma D and Lemma A imply immediately the validity of the assertion (ii) of Theorem 5.3.

6. APPLICATION OF KNIGHTLY'S IDEA

From now we still suppose the assumptions (4.2)–(4.9) to be satisfied. Let $F \in C^2(\bar{\Omega})$ be any function for which $F = \Phi_0$ and $F_n = \Phi_1$ on $\partial\Omega$. Define

$$(6.1) \qquad g = \Phi - \zeta F$$

where ζ is an auxiliary function from $C^\infty(\bar{\Omega})$ for which

$$(6.2) \qquad \zeta = 1 \quad \text{on} \quad \partial\Omega \quad \text{and} \quad \zeta_n = 0 \quad \text{on} \quad \partial\Omega .$$

Substitute now from (6.1) into (3.10) and (3.11) realizing that the equation (3.11) is then equivalent to the inequality

(6.3) $$\forall \psi \in W_0^{2,2}(\Omega):$$

$$(g, \psi - g)_{W_0^{2,2}} + (\zeta F, \psi - g)_{W_0^{2,2}} + B(w; w, \psi - g) \geqq 0 .$$

Adding the inequality from (6.3) to the result of the substitution into (3.10) we obtain the inequality

(6.4) $$A(w, v - w) + (g, \psi - g)_{W_0^{2,2}} - B(w; g, v - w) + B(w; w, \psi - g) +$$

$$+ (\zeta F, \psi - g)_{W_0^{2,2}} - B(w; \zeta F, v - w) \geqq$$

$$\geqq \int_\Omega q(v - w) \, dx \, dy + \int_{\Gamma_2} m_2(v_n - w_n) \, dS + \int_{\Gamma_3} [m_3(v_n - w_n) + r_3(v - w)] \, dS .$$

6.1. **Definition.** *A couple of functions* $w \in V$ *and* $g \in W_0^{2,2}(\Omega)$ ($w \in \tilde{K}_\mathrm{I}$ *and* $g \in W_0^{2,2}(\Omega)$, $w \in \tilde{K}_{\mathrm{II}}$ *and* $g \in W_0^{2,2}(\Omega)$, *respectively*) *is said to be a solution of the problem* R_ζ, ($S_{\mathrm{I}\zeta}$, $S_{\mathrm{II}\zeta}$) *if the inequality* (6.4) *is satisfied for each couple* $v \in V$ *and* $\psi \in W_0^{2,2}(\Omega)$ ($v \in \tilde{K}_\mathrm{I}$ *and* $\psi \in W_0^{2,2}(\Omega)$, $v \in \tilde{K}_{\mathrm{II}}$ *and* $\psi \in W_0^{2,2}(\Omega)$).

Remark. It follows from the procedure described above that the existence of a variational solution to the problem R, (S_I, S_{II}, respectively) will be proved if we prove the existence of a solution of the problem R_ζ ($S_{\mathrm{I}\zeta}$, $S_{\mathrm{II}\zeta}$) for a fixed function ζ. The formulation of all three problems is based on the same inequality (6.4) but each problem has its specific convex set of solutions and test functions.

Now we rewrite the problems R_ζ, $S_{\mathrm{I}\zeta}$ and $S_{\mathrm{II}\zeta}$ in a form which permits to apply to them Theorem 5.3.

Denote by

(6.5) $$H = V \times W_0^{2,2}(\Omega)$$

the linear space whose elements

(6.6) $$U = [w, g] , \quad w \in V , \quad g \in W_0^{2,2}(\Omega)$$

are normed in the usual way as

(6.7) $$\|U\|_H = \|w\|_{W^{2,2}} + \|g\|_{W_0^{2,2}} .$$

(Remember here that $\|g\|_{W_0^{2,2}} = (g, g)_{W_0^{2,2}}^{1/2}$ and $\|w\|_{W^{2,2}} = \|w\|_{L_2} + \|w\|_{W_0^{2,2}}$.)

The space H with the norm $\|\cdot\|_H$ defined by (6.7) is a real Hilbert space. Denote further

(6.8) $$K_\mathrm{I} = \tilde{K}_\mathrm{I} \times W_0^{2,2}(\Omega) , \quad K_{\mathrm{II}} = \tilde{K}_{\mathrm{II}} \times W_0^{2,2}(\Omega)$$

and define the functional $Q: H \to E_1$ and the operator $\mathcal{T}_\zeta: H \to H'$ as

$$Q(U) = \int_\Omega qw \, \mathrm{d}x \, \mathrm{d}y + \int_{\Gamma_2} m_2 w_n \, \mathrm{d}S + \int_{\Gamma_3} [m_3 w_n + r_3 w] \, \mathrm{d}S \tag{6.9}$$

and

$$\langle \mathcal{T}_\zeta(U), Z \rangle = A(w, v) + (g, \psi)_{W_0^{2,2}} - B(w; g, v) + B(w; w, \psi) - \\ - B(w; \zeta F, v) + (\zeta F, \psi)_{W_0^{2,2}}, \tag{6.10}$$

where $Z = |v, \psi|$.

6.2. **Definition.** *A solution of the problem R_ζ, ($S_{\mathrm{I}\zeta}$, $S_{\mathrm{II}\zeta}$, respectively) is such an element of H ($K_{\mathrm{I}}, K_{\mathrm{II}}$) for which the inequality*

$$\langle \mathcal{T}_\zeta(U), Z - U \rangle \geqq \langle Q, Z - U \rangle \tag{6.11}$$

holds for each $Z \in H$, ($Z \in K_{\mathrm{I}}$, $Z \in K_{\mathrm{II}}$).

Remark. The equivalence of Definitions 6.1 and 6.2 is evident. In accordance with our programme we are looking for the function ζ for which the operator $\mathcal{T}_\zeta$ is pseudomonotone and semicoercive with respect to a suitable seminorm.

6.3. **Lemma.** *The functional Q defined by (6.9) is a continuous linear functional on H.*

Proof. The assertion follows from (4.6) and (4.7) by means of Sobolev imbedding theorems and theorems of traces (see e.g. [8]).

6.4. **Lemma.** *Let $\zeta \in C^\infty(\bar{\Omega})$ be a function with the property (6.2). Then the operator $\mathcal{T}_\zeta$ defined by (6.10) is a bounded operator from H to H'.*

Proof. The assertion follows from the estimate (3.7), Sobolev imbedding theorems and theorems of traces.

7. PROOF OF THEOREM 4.1

In the space H defined by the relations (6.5)–(6.7) set for $U = |w, g|$

$$p_0(U) = \|w\|_{L_2} + \|g\|_{W_0^{2,2}}, \tag{7.1}$$

$$p_1(U) = [A(w, w)]^{1/2} + \|g\|_{W_0^{2,2}}. \tag{7.2}$$

H is pre-Hilbert with respect to the norm p_0. From (4.3)–(4.5) it follows that p_1 is a seminorm in H. From the definition (3.4) of $A(u. v)$ we obtain easily that $p_0(\cdot) + p_1(\cdot)$ is equivalent with the norm $\|\cdot\|_H$. So the assumption (5.6) of Theorem 5.3 is satisfied.

Denoting $Y = \{U \in H;\ p_1(U) = 0\}$ we have immediately

$$Y = Y_V \times \{0\} \tag{7.3}$$

where Y_V is defined by (4.1). Since Y_V is a linear subset of the set of all polynomials of the first order restricted to the domain Ω, it is finite dimensional and the assumption (5.7) of Theorem 5.3 is satisfied.

To prove (5.8) we have to show

$$\underset{c>0}{\exists}\ \underset{u \in v}{\forall}\ \inf_{z \in Y_v} \|u + z\|_{L_2} \leq c[A(u, u)]^{1/2}\,. \tag{7.4}$$

Let $X \perp Y_V$ be the orthogonal decomposition of the space V with respect to the scalar product in $L_2(\Omega)$. Suppose that (7.4) does not take place. Then there exists a sequence $\{u^n\}_{n=1}^{\infty}$ in V such that

$$\inf_{z \in Y_V} \|u^n + z\|_{L_2} = 1 \quad \text{and} \quad A(u^n, u^n) \leq \frac{1}{n}\,, \quad n = 1, 2, \ldots\,. \tag{7.5}$$

Let w^n be an orthogonal projection of u^n onto X. According to (7.5) it is

$$\|w^n\|_{L_2} = 1\,, \quad n \in N \tag{7.6}$$

and $A(w^n, w^n) \to 0$ so that $\|w^n\|_{W_0{}^{2,2}} \to 0$. Thus the sequence $\{w^n\}_{n=1}^{\infty}$ is bounded in $W^{2,2}(\Omega)$. Thanks to the compact imbedding $W^{2,2}(\Omega) \subset L_2(\Omega)$ there exists a subsequence $\{w^{n_k}\}_{k=1}^{\infty}$ convergent in $L_2(\Omega)$ to an element w. Because of $D^\alpha w^{n_k} \to 0$ in $L_2(\Omega)$ for each α, $|\alpha| = 2$, it is $w^{n_k} \to w$ in $W^{2,2}(\Omega)$. As $w \in V$ and $A(w, w) = 0$ we obtain that $w \in Y_V$, which implies

$$\int_\Omega w^{n_k} w \,\mathrm{d}x \,\mathrm{d}y = 0\,, \quad k \in N\,.$$

Thus $w = 0$ which contradicts (7.6).

Each of the sets H, K_{I} and K_{II} is a convex closed subset of H containing zero so that in all three cases the assumption (5.9) is satisfied.

To prove the semicoerciveness of $\mathscr{T}_\zeta(U)$ with respect to p_1 for an auxiliary function ζ we use the following lemma the proof of which will be sketched at the end of this section.

7.1. **Lemma.** *There exists a function $\tilde{\zeta} \in C^\infty(\bar{\Omega})$ satisfying (6.2) for which*

$$|B(w; \tilde{\zeta}F, w)| \leq \tfrac{1}{8}\, p_1^2(U) \quad \textit{for all} \quad U = [w, g] \in H\,. \tag{7.7}$$

Let us estimate now $\langle \mathscr{T}_\zeta(U), U \rangle$. We obtain by means of (3.6), (7.2) and (7.7) for each $U = [w, g]$

$$(7.8) \qquad \langle \mathscr{T}_\zeta(U), U \rangle = A(w, w) + (g, g)_{W_0^{2,2}} - B(w; g, w) + B(w; w, g) - \\ - B(w; \tilde{\zeta}F, w) + (\tilde{\zeta}F, g)_{W_0^{2,2}} \geqq \tfrac{1}{2} p_1^2(U) - |B(w; \tilde{\zeta}F, w)| - \\ - |(\tilde{\zeta}F, g)_{W_0^{2,2}}| \geqq \tfrac{3}{8} p_1^2(U) - |(\tilde{\zeta}F, g)_{W_0^{2,2}}| .$$

Estimating

$$|(\tilde{\zeta}F, g)_{W_0^{2,2}}| \leqq C\|g\|_{W_0^{2,2}} \leqq \bar{C} + \tfrac{1}{8}\|g\|^2_{W_0^{2,2}} \leqq \bar{C} + \tfrac{1}{8} p_1^2(U)$$

and substituting it into (7.8) we obtain finally that

$$(7.9) \qquad \langle \mathscr{T}_\zeta(U), U \rangle \geqq \tfrac{1}{4} p_1^2(U) - \bar{C} .$$

It remains to prove the pseudomonotonicity of the operator $\mathscr{T}_\zeta$. We can write

$$(7.10) \qquad \langle \mathscr{T}_\zeta(U^n), U^n - V \rangle = \langle \mathscr{T}_\zeta(U^n) - \mathscr{T}_\zeta(U), U^n - U \rangle + \\ + \langle \mathscr{T}_\zeta(U), U^n - U \rangle + \langle \mathscr{T}_\zeta(U^n), U - V \rangle .$$

If we prove that the situation

$$(7.11) \qquad U^n \rightharpoonup U \quad \text{and} \quad \limsup \langle \mathscr{T}_\zeta(U^n), U^n - U \rangle \leqq 0$$

implies that the first two members on the right hand side of (7.10) tend to zero and $\mathscr{T}_\zeta(U^n) \rightharpoonup \mathscr{T}_\zeta(U)$ then the pseudomonotonicity will be established. Obviously, $\lim \langle \mathscr{T}_\zeta(U), U^n - U \rangle = 0$. Further, according to Lemma 7.1,

$$(7.12) \qquad \langle \mathscr{T}_\zeta(U^n) - \mathscr{T}_\zeta(U), U^n - U \rangle = A(w^n - w, w^n - w) + \\ + (g^n - g, g^n - g)_{W_0^{2,2}} - B(w^n - w; \tilde{\zeta}F, w^n - w) + \\ + \{B(w^n; w^n, g^n - g) + B(w; g, w^n - w) - B(w^n; g^n, w^n - w) - \\ - B(w; w, g^n - g)\} \geqq \tfrac{3}{8} p_1^2(U^n - U) + \\ + \{B(w^n; w^n, g^n - g) + B(w; g, w^n - w) - B(w^n; g^n, w^n - w) - B(w; w, g^n - g)\} .$$

For $n \to \infty$ the expression in figure brackets tends to zero. (This follows from the estimate (3.7) and from the compactness of the imbedding $W^{2,2}(\Omega) \subset W^{1,4}(\Omega)$.) Using (7.11) we obtain from (7.12) that

$$\lim \langle \mathscr{T}_\zeta(U^n) - \mathscr{T}_\zeta(U), U^n - U \rangle = 0 .$$

Simultaneously we get from (7.12) that

$$(7.13) \qquad \lim_{n \to \infty} p_1(U^n - U) = 0 .$$

Let now $Z = [v, \psi]$. It is

$$(7.14) \qquad \langle \mathcal{T}_\zeta(U^n) - \mathcal{T}_\zeta(U), Z \rangle = A(w^n - w, v) + (g^n - g, \psi)_{W_0^{2,2}} -$$
$$- B(w^n; g^n, v) + B(w; g, v) + B(w^n; w^n, \psi) - B(w; w, \psi) - B(w^n - w; \tilde{\zeta}F, v) .$$

From (7.13) and (3.7) it follows that

$$(7.15) \qquad \lim_{n \to \infty} \{A(w^n - w, v) + (g^n - g, \psi)_{W_0^{2,2}} - B(w^n - w; \tilde{\zeta}F, v)\} = 0 ,$$

$$(7.16) \qquad \lim_{n \to \infty} \{B(w; g, v) - B(w^n; g^n, v)\} = \lim_{n \to \infty} B(w - w^n; g, v) -$$
$$- \lim_{n \to \infty} B(w^n; g^n - g, v) = 0$$

and

$$(7.17) \qquad \lim_{n \to \infty} \{B(w^n; w^n, \psi) - B(w; w, \psi)\} = \lim_{n \to \infty} B(w^n - w; w^n, \psi) +$$
$$+ \lim_{n \to \infty} B(w; w^n - w, \psi) = 0 .$$

Using (7.15)–(7.17) we obtain from (7.14) that

$$\lim_{n \to \infty} \langle \mathcal{T}_\zeta(U^n) - \mathcal{T}_\zeta(U), Z \rangle = 0 .$$

In this way we have established that under the conditions of Theorem 4.1 all assumptions of Theorem 5.3 are satisfied for the operator $\mathcal{T}_\zeta$ in the space H. The assertion of Theorem 4.1 is nothing else than the assertion of Theorem 5.3 rewritten for the operator $\mathcal{T}_\zeta$.

Sketch of the proof of Lemma 7.1. Under the conditions (4.2), (4.8) and (4.9) we are able to prove the following fact: For each $\gamma > 0$ there exists a function $\tilde{\zeta} \in C^\infty(\overline{\Omega})$ satisfying (6.2) for which

$$(7.18) \qquad |B(w; \tilde{\zeta}F, w)| \leqq \gamma \|w\|^2_{W^{2,2}} \quad \text{for all} \quad w \in V .$$

(For the details see [2], Section 6.) Put $\gamma = 1/16(c^2 + 2)$ where c is the constant from (7.4) and denote by $\tilde{\zeta}$ an auxiliary function for which (7.18) holds. For $w \in V$ and $z \in Y_V$ we have

$$|B(w + z; \tilde{\zeta}F, w + z)| \leqq \frac{1}{16(c^2 + 2)} \|w + z\|^2_{W^{2,2}} \leqq$$
$$\leqq \frac{1}{8(c^2 + 2)} (\|w + z\|^2_{L_2} + \|w\|^2_{W_0^{2,2}})$$

and, using the definition of $A(w, w)$ and the relation (7.4),

$$\inf_{z\in Y_V} |B(w + z; \tilde{\zeta}F, w + z)| \leqq \frac{1}{8(c^2 + 2)} (c^2 + 2) A(w, w) \leqq \tfrac{1}{8} p_1^2(U) . \tag{7.19}$$

a) In the case $Y_V = \{0\}$ the proof is complete.

b) In the case $Y_V \neq \{0\}$ we must prove that $B(w + z; \tilde{\zeta}F, w + z) = B(w; \tilde{\zeta}F, w)$ for $w \in V$, $z \in Y_V$ or, equivalently (remembering that z are polynomials of the first order)

$$B(w; \tilde{\zeta}F, z) = 0 \tag{7.20}$$

for all $w \in C^\infty(\bar{\Omega})$ for which $w = w_n = 0$ on Γ_1 and $w = 0$ on Γ_2 and all polynomials $z(x, y) = ax + by + c$ from $Y_V \setminus \{0\}$. We obtain by Green's formula (using also (6.2), (4.8) and (4.9)) that

$$B(w; \tilde{\zeta}F, z) = \int_{\partial\Omega} (\tilde{\zeta}F)_\tau (-bw_x + aw_y)\, dS = \tag{7.21}$$

$$= \int_{\Gamma_2} (\Phi_0)_\tau (-bw_x + aw_y)\, dS .$$

If now $\Gamma_2 = \emptyset$, the last integral equals zero. Let $\Gamma_2 \neq \emptyset$. As $z \in Y_V \subset V$, $ax + by + c = 0$ on Γ_2. This implies that $(-b, a)$ is a tangential vector so that the expression $-bw_x + aw_y$ equals cw_τ with some real constant c. But $w_\tau = 0$ on Γ_2 as $w = 0$ on Γ_2 and so $B(w, \tilde{\zeta}F, z) = 0$ q.e.d.

8. RELATION BETWEEN CLASSICAL AND VARIATIONAL SOLUTIONS OF THE PROBLEM

The sufficiently smooth variational solution is a classical one. We sketch the proof in the case of the problem S_I. From Green's theorem we obtain (for u, v and w sufficiently regular)

$$(w, v)_V = \int_\Omega \Delta^2 w\, v\, dx\, dy + \int_{\partial\Omega} Tw\, v\, dS + \int_{\partial\Omega} Mw\, v_n\, dS , \tag{8.1}$$

$$(w, v)_{W_0^{2,2}} = \int_\Omega \Delta^2 w\, v\, dx\, dy \quad \text{if} \quad v \in W_0^{2,2}(\Omega) , \tag{8.2}$$

$$B(u; w, v) = \int_\Omega [u, w]\, v\, dx\, dy - \int_{\partial\Omega} (w_x u_{y\tau} - w_y u_{x\tau})\, v\, dS . \tag{8.3}$$

Let w, Φ be a sufficiently smooth variational solution of the problem S_1 (see Definition 3.2); using (3.11), (8.2) and (8.3) we obtain

$$\int_\Omega (\Delta^2\Phi + [w, w])\, \psi \,\mathrm{d}x\,\mathrm{d}y = 0\,, \quad \forall \psi \in W_0^{2,2}(\Omega)\,.$$

This yields

$$\Delta^2\Phi = -[w, w] \quad \text{on} \quad \Omega\,,$$

which is the equation (2.7).

Setting in (3.10) $v = w \pm \varphi$ where φ is a smooth function with a compact support in Ω we obtain from (8.1) and (8.3) that the functions w and Φ satisfy the equation (2.6). Because of $w \in V$, the condition (2.9) is satisfied automatically and $w = 0$ on Γ_2 as well.

Using these facts together with the formulas (8.1) and (8.3) we transform (3.10) into

$$\int_{\Gamma_2} (Mw + k_2 w_n - m_2)(v_n - w_n)\,\mathrm{d}S + \int_{\Gamma_3} (Mw + k_{31} w_n - m_3)(v_n - w_n)\,\mathrm{d}S +$$
$$+ \int_{\Gamma_3} (Tw + k_{32} w - r_3)(v - w)\,\mathrm{d}S + \int_{\partial\Omega} (\Phi_x w_{y\tau} - \Phi_y w_{x\tau})(v - w)\,\mathrm{d}S \geqq 0\,.$$

The integral $\int_{\partial\Omega} (\Phi_x w_{y\tau} - \Phi_y w_{x\tau})(v - w)\,\mathrm{d}S$ in (8.4) equals zero because $v - w = 0$ on $\Gamma_1 \cup \Gamma_2$ and $\Phi_x = \Phi_y = 0$ on Γ_3 (see the assumption (4.8)). If we take now $v = w \pm \varphi$ with $\varphi \in C^\infty(\bar{\Omega})$ and such that $\varphi = 0$ on $\partial\Omega$, $\varphi_n = 0$ on $\Gamma_1 \cup \Gamma_2$ we get $Mw + k_{31} w_n = m_3$ on Γ_3 which is the boundary condition (2.11). Analogously we obtain that satisfies the condition $Mw + k_2 w_n = m_2$ on Γ_2.

It remains to prove that w satisfies the inequalities (2.13). Until now we have proved that the inequality (8.4) has the form

$$(8.5) \qquad \int_{\Gamma_3} (Tw + k_{32} w - r_3)(v - w)\,\mathrm{d}S \geqq 0\,, \quad \forall v \in \tilde{K}_1\,.$$

Firstly, $w \geqq 0$ according to the definition of $\tilde{K}_1$. Secondly, if $(Tw + k_{32} w - r_3)(P) < 0$ at a point $P \in \Gamma_3$ we obtain (by the continuity) that for a suitably chosen function $v \in \tilde{K}_1$ it is

$$\int_{\Gamma_3} (Tw + k_{32} w - r_3)(v - w)\,\mathrm{d}S < 0\,,$$

which contradicts (8.5). So it is $Tw + k_{32} w - r_3 \geqq 0$ on Γ_3. Finally, substituting $v = 0$ and $v = 2w$ we obtain from (8.5) that

$$(8.6) \qquad \int_{\Gamma_3} (Tw + k_{32} w - r_3)\, w \,\mathrm{d}S = 0\,.$$

As we have proved the non-negativeness of the subintegral function, the formula (8.6) implies that $(Tw + k_{32} w - r_3)\, w = 0$ on Γ_3.

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Souhrn

O SIGNORINIOVĚ PROBLÉMU PRO VON KÁRMÁNOVY ROVNICE

OLDŘICH JOHN

Článek volně navazuje na práci [2]. Je v něm dokázána existence variačního řešení zobecněného Signoriniova problému převedením příslušné okrajové úlohy na nerovnici s pseudomonotonním semikoercitivním operátorem. Řešitelnost této nerovnice plyne z abstraktní Věty 5.3, která je zobecněním výsledku J. L. Lionse a Q. Stampacchii z článku [6] na nelineární případ.

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