On Similarity-based Approximate Reasoning in Interval-valued Fuzzy Environments

Zhi-Qiang Feng [a,b] and Cun-Gen Liu[a]
[a]School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiao Tong University, Shanghai 200240, China
E-mail: fzqstju@yahoo.cn (Z.Q. Feng)

[b]Jiangsu Naval Engineering Centre, Nantong 226010, China
E-mail: cgliu@sjtu.edu.cn (C.G. Liu)

Keywords: Approximate reasoning, interval-valued fuzzy sets, similarity measures, translation for rules

Received: March 30, 2011

This paper proposes a novel approach to similarity-based approximate reasoning in an interval-valued fuzzy environment. In a rule-based system, an 'if ... then ...' rule can be translated into an interval-valued fuzzy relation by suitable implication operations. The similarity grade between a case and the antecedent of a rule is computed and used to modify the relation. A consequent is derived from the well-known projection operation over the modified relation. The inference mechanism is appropriate because the techniques of the conventional Compositional Rule of Inference are incorporated into the existing similarity-based inference. Two examples of shipbuilding processing are utilized to illustrate and validate the effectiveness of the proposed schema.

Povzetek: Članek obravnava metode razmišljanje v mehki logiki, temelječe na podobnosti.

1 Introduction

As the theoretical foundation of fuzzy control, fuzzy inference has achieved successful applications in various fields. The basic Fuzzy Modus Ponens (FMP) often investigated by many researchers can be represented as:

Rule:	If X is A	then	Y is B
Case:	X is A'		
Conclusion:			Y is B'

Here X and Y are two linguistic variables, which can be also regarded as two different universes; A, A' and B, B' are fuzzy subsets of universes X and Y, respectively. Zadeh introduced the concept of Compositional Rule of Inference (CRI) [8]. By constructing a fuzzy relation R between A and B, we can derive the conclusion B' from the compositional operation of A' and R. Many fuzzy systems are based on Zadeh's compositional rule of inference [9]. In spite of their successes in various systems, researchers have pointed out certain drawbacks [4,5] in the mechanism, which motivates the introduction of Similarity-based Approximate Reasoning (SAR) mechanism as proposed in [4-7]. Compared with Zadeh's CRI, it does not require the construction of a fuzzy relation between input and output fuzzy data, and it is conceptually clearer than CRI.

According to the mechanism of SAR methodology, in rule-based system reasoning is based on the computation of similarity grade between the fact and the antecedent of a rule, and the inference result is obtained by directly modifying to the consequent part with the similarity measure. Thus, the inherent relation between the antecedent and the consequent is largely ignored. For the FMP problem, suppose that A is the antecedent part of 'If X is A then Y is B', and A' is an input fact. In light of SAR, we first compute the similarity measure $S(A',A)$ of A' and A, then the result B' is deduced by a modification function f such that $B'(y)=f(S(A',A),B(y))$. Evidently, a same result B' will be concluded by SAR method when A and A' are interchanged. Thus, this result seems somewhat unconvincing because the inference is not always influenced by every change in the input case and the antecedent part.

Combining the conventional CRI and the existing SAR methods, in this paper we extend the works of [4,5] to develop a novel approach to approximate reasoning. First, since interval-valued fuzzy set is considered more flexible than general fuzzy set from the viewpoint of handling imprecise and fuzzy data, we deal with approximation inference within the framework of interval-valued fuzzy sets. Next, to interpret a conditional statement (rule) residing in a rule-base system, an interval-valued fuzzy relation between antecedent and consequent can be constructed by suitable implicators. Furthermore, based on a measure of similarity, the constructed relation is modified to yield a new relation called the induced relation, and the conclusion can be obtained by the well-known projection operation over the induced relation. In the end, we illustrate the effectiveness of the proposed scheme by an example of processing systems of shipbuilding.

The remainder of this article is organized as follows. Section 2 includes a brief introduction of some basic notions and state of the art on approximate reasoning techniques. Section 3 discusses the problems regarding similarity index and an approach to approximate reasoning. Two examples are provided in Section 4 to illustrate the effectiveness of the proposed methods. The final section contains the concluding remarks.

2 Preliminaries and State of the Art

Let X be a universe of discourse. In Fuzzy Sets (FSs) theory, each object $x \in X$ is assigned a single real value, called the grade of membership, between zero and one. In [1-3], Gorzalczany and Turksen proposed the notion of Interval-valued Fuzzy Sets (IVFSs), which allow using interval-based membership instead of using point-based membership as in FSs.

An interval-valued fuzzy set A on X is characterized by a pair of mappings $\underline{A}: X \to [0,1]$ and $\overline{A}: X \to [0,1]$ such that $0 \le \underline{A}(x) \le \overline{A}(x) \le 1$, where $\underline{A}(x)$ and $\overline{A}(x)$ denote a lower and an upper bounds of membership function of A, respectively. An interval-valued fuzzy set A of X can be denoted as $A = \{(x,[\underline{A}(x),\overline{A}(x)]): x \in X\}$.

In other words, the membership degree of x with respect to A is bounded to a subinterval $[\underline{A}(x),\overline{A}(x)]$ of unit interval, which indicates the possible existence of a data value. All the interval-valued fuzzy sets of X is written as $IVFSs(X)$, and A is said to be normal if and only if there exists $x_0 \in X$ such that $\underline{A}(x_0) = \overline{A}(x_0) = 1$. For every $A \in IVFSs(X)$, the lower bound, the upper bound and the kernel function of A can be represented by

$$\underline{A} = \{\underline{A}(x): x \in X\}$$

$$\overline{A} = \{\overline{A}(x): x \in X\}$$

$$\kappa(A) = \{\underline{A}(x) + \overline{A}(x): x \in X\}$$

respectively. Let $A, B \in IVFSs(X)$, the union, intersection and complement operations of interval-valued fuzzy sets are defined as follows:

$$A \cap B = \{(x,[\min\{\underline{A}(x),\underline{A}_B(x)\},\min\{\overline{A}(x),\overline{B}(x)\}]): x \in X\}$$

$$A \cup B = \{(x,[\max\{\underline{A}(x),\underline{A}_B(x)\},\max\{\overline{A}(x),\overline{B}(x)\}]): x \in X\}$$

$$A^c = \{(x,[1-\overline{A}(x),1-\underline{A}(x)]): x \in X\}$$

To improve the flexibility of fuzzy set in handling fuzzy information, Atanassov (1986) proposed the Intuitionistic Fuzzy Sets (IFSs) [14], and Gau et al. (1993) presented the Vague Sets (VSs) [15]. As the IVFSs, IFSs and VSs were proved actually isomorphic and equivalent [16-18], we will put them into a framework of IVFSs in this paper.

It is well-known that Fuzzy Set theory has been extensively applied to the field of approximate reasoning. So far there have been several approaches to approximate reasoning based on fuzzy set or interval-valued fuzzy set, in which the most influential methods are CRI and SAR algorithms. In [19], Li et al. addressed an implication operator based on IVFSs, which is suitable for CRI method. Cornelis et al. investigated a serial of implications in IVFSs and presented an extensional schema of CRI [20]. In [21], based on the CRI method, an extensional model derived from the Mizumoto's model is provided in an interval-valued fuzzy environment. Based on expansion principle, Feng et al. also presented several operators that are applicable to CRI [22].

Although the CRI algorithm had achieved notable success in various fields such as fuzzy control, expert system and decision-making support, some defects of this method were found in terms of inference mechanism, which leads to the yield of another important approach of approximate reasoning—Similarity-based Approximate Reasoning (SAR). For the FMP problem, based on the change of membership grade of the consequent part, Turksen et al. proposed two types of modification procedures—expansion type inference and reduction type inference [4], and B' may be computed by any one of the following form:

$$B'(y) = \min\{1, B(y)/S(A',A)\} \quad \text{(Expansion form)}$$

$$B'(y) = S(A',A) \cdot B(y) \quad \text{(Reduction form)}$$

where $S(A',A)(x)$ is the fuzzy similarity degree between a fact A' and the antecedent part A.

In [23], a new similarity measure of IVFSs was presented and inference result was obtained by Turksen's reduction form. Trough constructing a modified function based on a similarity measure of IVFSs, Tian et al. gave an approach to approximate reasoning [24]. Shi et al. addressed a bidirectional approximate reasoning scheme based on the distances of IVFSs [25], which can be actually regarded as an equivalent form of SAR method. Applying the SAR method, Guan et al. addressed a specific design scheme of fuzzy controller based on IVFSs [26].

In [10], through introducing a definition of fuzzy similarity measure, the authors provided an inference solution for FMP as follows:

$$B'(y) = \vee_{x \in X} \left(S(A',A)(x) \wedge B(y) \right)$$

In [27], the authors proposed the concept of similarity direction between two interval-valued fuzzy sets. Through calculating the similarity grade as well as the similarity direction of interval-valued fuzzy sets, the inference result B' is computed as

$$\underline{B}'(y) = \begin{cases} \underline{B}^s(y) & s \ge 0.5,\ d \ge 0 \\ \underline{B}^{1/s}(y) & s \ge 0.5,\ d < 0 \\ \underline{B}^s(y) & s < 0.5 \end{cases}, \quad \overline{B}'(y) = \begin{cases} \overline{B}^s(y) & s \ge 0.5,\ d \ge 0 \\ \overline{B}^{1/s}(y) & s \ge 0.5,\ d < 0 \\ \overline{B}^s(y) & s < 0.5 \end{cases}$$

where d is the evaluation function of similarity direction between A' and A, and s is the similarity grade of A' and A.

Meng presented a generalized model for fuzzy character spread reasoning [11], which was actually an approach of similarity-based weighted fuzzy inference applicable to multi-dimensional fuzzy reasoning. Through calculating the weighted similarity degree s_i between an input case and the i^{th} rule, the result is expressed by

$$B'(y) = \mathbf{S} \circ B_i(y)$$

Here $\mathbf{S} = (s_1, s_2, \cdots, s_n)$ denotes a weighted similarity vector that holds

$s_i = \sum_{j=1}^{m} w_{ij} \cdot S(A'_j, A_{ij})$

for $i = 1,2,\cdots,n$. The operator '$\circ$' denotes a generalized compositional operation. For example, using $(\Sigma,\cdot)$ operation we then obtain

$B'(y) = \sum_{i=1}^{n} (s_i \cdot B_i(y))$

In [28], a similarity-based approximate reasoning method was given based on IVFSs and fitness techniques. Using data fitness method, the similarity degree k between the interval-valued fuzzy sets is obtained by

$$k_1 = \frac{\ln \underline{A}(x) \cdot \ln \underline{A}'(x)}{\ln^2(\underline{A}(x))},\ k_2 = \frac{\ln \overline{A}(x) \cdot \ln \overline{A}'(x)}{\ln^2(\overline{A}(x))},\ k = \frac{k_1 + k_2}{2}.$$

And then the result is computed by

$\underline{B}'(y) = \underline{B}^k(y)$, $\overline{B}'(y) = \overline{B}^k(y)$.

3 Similarity-based Approximate Reasoning

3.1 Similarity measures of IVFSs

In [13], Zwick et al. surveyed several similarity measures of fuzzy sets and compared their performance in an experiment. In [12], Ke et al. presented a similarity function S to measure the degree of similarity based on fuzzy vectors. Let $E, F \in FSs(X)$, then the similarity grade $S(E,F)$ between E and F can be represented by

Definition1. [12]

$$S(E,F) = \frac{\sum_{x\in X} E(x) \cdot F(x)}{\max\left\{\sum_{x\in X} E^2(x), \sum_{x\in X} F^2(x)\right\}}$$

Here, the Sum-product operations represent the product of fuzzy vectors *E* and *F*, from which we may also derive another definition of similarity index, as follows.

Definition2.

$$S'(E,F) = \frac{\sup_{x\in X} \min\{E(x), F(x)\}}{\max\{\sup_{x\in X} E(x), \sup_{x\in X} F(x)\}}$$

Here, the operations Sum-product in Definition 1 are modified to Sup-min in Definition 2, respectively.

The measure proposed in Definition 2 is based on the computation of overall supremum and therefore, practically difficult to use.

Example1. Let $X = \{1,2,3,4,5\}$ be the universe of discourse. Consider the following fuzzy sets on X:

$E \triangleq \text{"small"} = 1/1 + 0.8/2 + 0.5/3 + 0.2/4$

$F \triangleq \text{"highly small"} = 1/1 + 0.41/3 + 0.06/3$

According to Definition 2, the calculation result implies that E is identical to F (i.e., $S'(E,F) = 1$), even if E is highly dissimilar to F by our intuitions. This is why we prefer the measure given by Definition 1.

To provide a definition for similarity measure $\tilde{S}(A,B)$ of two interval-valued fuzzy sets A and B, a number of factors must be considered. A primary consideration is that, whatever way we choose to define such an index, it should satisfy the following properties: for every $A,B,C \in IVFSs(X)$,

P1) $\tilde{S}(A,B) \in [0,1]$;

P2) $\tilde{S}(A,B) = \tilde{S}(B,A)$;

P3) $\tilde{S}(A,B) = 1$ if and only if $A = B$;

P4) If $\tilde{S}(A,B) = 0$, and A,B are not simultaneously empty, then $\min\{\underline{A}(x), \underline{B}(x)\} = \min\{\overline{A}(x), \overline{B}(x)\} = 0$ for all $u \in U$;

P5) If $A \subseteq B \subseteq C$ then $\tilde{S}(A,C) \le \min\{\tilde{S}(A,B), \tilde{S}(B,C)\}$.

P4) suggests that *A* and *B* are completely dissimilar only when $A \cap B = \varnothing$. If $A \cap B \neq \varnothing$, then they have some similarity when *A* and *B* have some membership degree in common.

Based on Definition 1, in the following, we develop an expression of similarity function $\tilde{S}$ to measure the degree of similarity between interval-valued fuzzy sets. Let $\underline{A}$, $\overline{A}$ and $\kappa(A)$ be subscript, superscript and kernel function of $A \in IVFSs(X)$.

Definition3. Let $A,B \in IVFSs(X)$. The degree of similarity $\tilde{S}(A,B)$ between *A* and *B* can be measured as follows:

$\alpha = S(\underline{A}, \underline{B})$, $\beta = S(\overline{A}, \overline{B})$, $\gamma = S(\kappa(A), \kappa(B))$

$\tilde{S}(A,B) = (\alpha + \beta + 2\gamma)/4$.

It is easy to verify that the similarity measures given by Definition 3 satisfy axioms P1), P2), P3), P4) and P5). Thus, the similarity measures of the lower bound, the upper bound and the kernel values of two interval-valued fuzzy sets, are incorporated into such an index of IVFSs, where the weighted coefficients of which are given by $\omega_1 = 0.25$, $\omega_2 = 0.25$ and $\omega_3 = 0.5$, respectively.

3.2 Proposed schema for approximation inference

The conventional CRI does not consider the concept of similarity measure in deriving a consequence. The existing SAR methods modify directly the consequence part of a rule, based on a measure of similarity and therefore, the consequence becomes independent of the conditional statement. Here, we intend to integrate the above techniques for an adequate theory of similarity-based approximate reasoning.

According to CRI, a conditional statement (rule) 'If *A* then *B*' can be translated into an interval-valued fuzzy relation, denoted as $R(A,B)$. To construct the relation, some suitable operation operators are used. For example, the relation $R(A,B)$ constructed by the extensional *KD*-implicator can be represented as

$\underline{R}(A,B)(x,y) = \min\{1 - \overline{A}(x), \underline{B}(y)\}$

$\overline{R}(A,B)(x,y) = \min\{1 - \underline{A}(x), \overline{B}(y)\}$

Given a case input A', an interval-valued fuzzy relation between A' and B, denoted as $R(A',B)$, can be obtained by intersection operation of A' and $R(A,B)$. Thus, an inference result B' is computed by the well-known supremum projection operation on $R(A',B)$, i.e.,

$\underline{B}'(y) = \sup_{x\in X} \underline{R}(A',B)(x,y) = \sup_{x\in X} \min\{\underline{A}'(x), \underline{R}(A,B)(x,y)\}$

$\overline{B}'(y) = \sup_{x\in X} \overline{R}(A',B)(x,y) = \sup_{x\in X} \min\{\overline{A}'(x), \overline{R}(A,B)(x,y)\}$

Since the CRI method fails to incorporate the matching computation into the inference procedures, the accuracy of reasoning is not always satisfactory in some application occasions

The primary mechanism of SAR is to deduce result by modifying the consequent part of a rule with similarity measure [4,5]. Applying this principle, in a rule-based system we may first calculate the similarity grade $\tilde{S}(A',A)$ of the fact A' and the antecedent part A. And then, an interval-valued fuzzy relation $R(A',B)$ between A' and B, named as the induced relation, is obtained by modifying the relation $R(A,B)$ with similarity measure $\tilde{S}(A',A)$. Finally, the result B' can be deduced by the projection operation over the induced relation $R(A',B)$.

Given a conditional statement, the following cases should be taken into account to obtain an induced relation using the similarity measure.

*Case*1. If A' equals to A, then $R(A',B)$ equals to $R(A,B)$. This is to say we should not make any modification to $R(A,B)$ when $\tilde{S}(A',A)=1$.
*Case*2. If A' is completely dissimilar to A, then we can conclude nothing from the given conditional statement 'If A then B', i.e., B' is empty. Since $\underline{B}'(y)=\sup_{x\in X}\underline{R}(A',B)(x,y)$, $\overline{B}'(y)=\sup_{x\in X}\overline{R}(A',B)(x,y)$ we then have $\underline{R}(A',B)(x,y)=\overline{R}(A',B)(x,y)=0$. i.e., $R(A',B)$ is empty when $\tilde{S}(A',A)=0$.
*Case*3. As $\tilde{S}(A',A)$ changes from 0 to 1, $R(A',B)$ changes from $\varnothing$ to $R(A,B)$. That means $R(A',B)$ is transformed from the most unknown state into a specific state.

From the cases mentioned-above, a quantitative relationship between the induced relation and similarity measure may be given as following:

Q1. If $S(A',A)=1$, then $\underline{R}(A',B)(x,y)=\underline{R}(A,B)(x,y)$, and $\overline{R}(A',B)(x,y)=\overline{R}(A,B)(x,y)$;
Q2. If $S(A',A)=0$, then $\underline{R}(A',B)(x,y)=\overline{R}(A',B)(x,y)=0$;
Q3. As $S(A',A)$ increase from 0 to 1, $\underline{R}(A',B)(x,y)$ and $\overline{R}(A',B)(x,y)$ increase from 0 to $\underline{R}(A,B)(x,y)$ and $\overline{R}(A,B)(x,y)$, respectively.

Let $\tilde{S}(A',A)=s,\underline{R}(A,B)(x,y)=\underline{r},\overline{R}(A,B)(x,y)=\overline{r},\underline{R}(A',B)(x,y)=\underline{r}'$, $\overline{R}(A',B)(x,y)=\overline{r}'$. By Q1 and Q2,

$$\underline{r}'=\begin{cases}0, & s=0\\ \underline{r}, & s=1\end{cases},\quad \overline{r}'=\begin{cases}0, & s=0\\ \overline{r}, & s=1\end{cases}$$

and by Q3, we get

$$\underline{r}'=\underline{r}'(s)=T(s,\underline{r}),\ \overline{r}'=\overline{r}'(s)=T(s,\overline{r})$$

where T is a continuous t-norm. Thus, a modification schema for producing the induced relation $R(A',B)$, named as Q schema, may be represented by

$$\underline{R}(A',B)(x,y)=T(s,\underline{R}(A,B)(x,y))$$
$$\overline{R}(A',B)(x,y)=T(s,\overline{R}(A,B)(x,y))$$

Once the induced relation is derived from Q schema, the inference result B' is then obtained by the supremum projection, i.e.

$$\begin{aligned}\underline{B}'(y)&=\sup_{x\in X}\underline{R}(A',B)(x,y)=\sup_{x\in X}T(s,\underline{R}(A,B)(x,y))\\ \overline{B}'(y)&=\sup_{x\in X}\overline{R}(A',B)(x,y)=\sup_{x\in X}T(s,\overline{R}(A,B)(x,y))\end{aligned}\tag{1}$$

Apparently, in terms of inference mechanism, there exists a distinction between the conventional CRI and the proposed method. A logical interpretation for the CRI method is: from ' X is A' and (X,Y) is $R(A,B)$ ' infer ' Y is B' '. For the proposed method, the inference mechanism can be interpreted as: from ' A' is similar to A and (X,Y) is $R(A,B)$ ' infer ' Y is B' ', where the connective 'and' is associated with t-norm operation.

The proposed algorithm on performing similarity-based approximate reasoning is summarized as follows.

*Step*1. Translate a rule and compute $R(A,B)$ using some suitable operators (Translation);
*Step*2. Compute $\tilde{S}(A',A)$ using some suitable definition, possibly, Definition 3 (Matching);
*Step*3. Modify $R(A,B)$ with $\tilde{S}(A',A)$ to obtain the induce conditional relation $R(A',B)$ using a scheme Q (Modification);
*Step*4. Use supremum projection operation on $R(A',B)$ to obtain B' (Projection).

In Step1, to translate a rule 'If A and B' we should calculate an interval-valued fuzzy relation $R(A,B)$ between A and B. According to the Zadeh's CRI method, there are about eighteen operators applicable to construct $R(A,B)$, which can often be classified two main classes. The first class is called the extensional 'and' operators, such as the Mamdani operator, the Larson operators and the bounded product, etc. The second is called the extensional 'implication' operators, including the well-known S-implicator and the R-implicator. The following proposition will provide a clue on how to select the suitable operators for the construction of $R(A,B)$.

Proposition1. Suppose A is normal and does not completely cover the domain. Let $s=1$.
1) If φ is both increasing, then $B'=B$;
2) If φ is left decreasing and right increasing, then $B'=Y$.

Proof.

1) Since φ is both increasing, then

$$\underline{R}(A,B)(x,y)=\varphi(\underline{A}(x),\underline{B}(y))$$
$$\overline{R}(A,B)(x,y)=\varphi(\overline{A}(x),\overline{B}(y))$$

By Formula (1), we get

$$\underline{B}'(y)=\sup_{x\in X}T(s,\underline{R}(A,B)(x,y))=\sup_{x\in X}T(s,\varphi(\underline{A}(x),\underline{B}(y)))$$
$$=T(s,\sup_{x\in X}\varphi(\underline{A}(x),\underline{B}(y)))=T(s,\varphi(\sup_{x\in X}\underline{A}(x),\underline{B}(y)))$$

As A is normal, then

$$\underline{B}'(y)=T(s,\varphi(1,\underline{B}(y)))=T(s,\underline{B}(y)).$$

Similarly, $\overline{B}'(y)=T(s,\overline{B}(y))$, we then obtain $B'=B$ by $s=1$.

2) As φ is left decreasing and right increasing, then

$$\underline{R}(A,B)(x,y)=\varphi(\overline{A}(x),\underline{B}(y))$$
$$\overline{R}(A,B)(x,y)=\varphi(\underline{A}(x),\overline{B}(y))$$

By Formula (1), we get

$$\underline{B}'(y)=\sup_{x\in X}T(s,\underline{R}(A,B)(x,y))=\sup_{x\in X}T(s,\varphi(\overline{A}(x),\underline{B}(y)))$$
$$=T\left(s,\sup_{x\in X}\varphi(\overline{A}(x),\underline{B}(y))\right)=T\left(s,\varphi\left(\inf_{x\in X}\overline{A}(x),\underline{B}(y)\right)\right)$$

Since A does not completely cover the domain, i.e. $\inf_{x\in X}\underline{A}(x)=\inf_{x\in X}\overline{A}(x)=0$, then

$$\underline{B}'(y)=T(s,\varphi(0,\underline{B}(y)))=T(s,1)=s$$

Similarly, $\overline{B}'(y)=s$. By $s=1$, we then obtain $B'(y)=1$ for every $y\in Y$. Hence, $B'=Y$. □

Remark1. We can conclude from Proposition1 as following:

1) If an implication selected for constructing interval-valued fuzzy relation is both increasing, then the inference result satisfies reductive property when antecedent part is normal. Furthermore, since $\underline{B}'(y)=T(s,\underline{B}(y))$ and $\overline{B}'(y)=T(s,\overline{B}(y))$, we then obtain

$\underline{B}'(y)=s\cdot\underline{B}(y),\ \overline{B}'(y)=s\cdot\overline{B}(y)$

when t-norm takes the algebraic product. This is exactly identical to the Turksen's reduction form as mentioned in Section 2.

2) If an implication selected for constructing interval-valued fuzzy relation is hybrid monotonic, then the inference result equals to the whole set when antecedent part does not completely cover the domain. In this case, the result becomes the most unspecific case because $B'=Y$ means 'B' is anything' from the viewpoint of semantics.

Proposition2. Suppose A is normal, and φ is both increasing, then $B'\subseteq B$.

Proof. Since

$$\begin{aligned}\underline{B}'(y)&=\sup_{x\in X}T\left(s,\underline{R}(A,B)(x,y)\right)=\sup_{x\in X}T\left(s,\varphi\left(\underline{A}(x),\underline{B}(y)\right)\right)\\&=T\left(s,\sup_{x\in X}\varphi\left(\underline{A}(x),\underline{B}(y)\right)\right)=T\left(s,\varphi\left(\sup_{x\in X}\underline{A}(x),\underline{B}(y)\right)\right)\\&\le\varphi\left(\sup_{x\in X}\underline{A}(x),\underline{B}(y)\right)=\varphi\left(1,\underline{B}(y)\right)=\underline{B}(y)\end{aligned}$$

$$\begin{aligned}\overline{B}'(y)&=\sup_{x\in X}T\left(s,\overline{R}(A,B)(x,y)\right)=\sup_{x\in X}T\left(s,\varphi\left(\overline{A}(x),\overline{B}(y)\right)\right)\\&=T\left(s,\sup_{x\in X}\varphi\left(\overline{A}(x),\overline{B}(y)\right)\right)=T\left(s,\varphi\left(\sup_{x\in X}\overline{A}(x),\overline{B}(y)\right)\right)\\&\le\varphi\left(\sup_{x\in X}\overline{A}(x),\overline{B}(y)\right)=\varphi\left(1,\overline{B}(y)\right)=\overline{B}(y)\end{aligned}$$

for every $y\in Y$. Hence, we get $B'\subseteq B$. □

Example2. Suppose there is a conditional statement 'If A then B' such that

$A=\{(x_1,[0.7,0.8]),(x_2,[0.4,0.5]),(x_3,[0.2,0.3]),(x_4,0)\}$

$B=\{(y_1,1),(y_2,[0.5,0.6]),(y_3,0)\}$

Given a case input $A'=\{(x_1,1),(x_2,[0.7,0.8]),(x_3,[0.3,0.3]),(x_4,0)\}$, compute the inference result B' using the Turksen's reduction form and the proposed method, respectively.

From Definition 3, the similarity grade between the fact A' and the antecedent part A can be calculated as $\tilde{S}(A',A)=0.70$. Appling the Turksen's SAR method, we have

$B_{\mathrm{I}}'=\tilde{S}(A',A)\cdot B=\{(y_1,0.7),(y_2,[0.35,042]),(y_3,0)\}$.

According to the proposed algorithm in this paper, we first compute an interval-valued fuzzy relation of A and B using the Mamdani operator, i.e.

$$R(A,B)=\begin{bmatrix}[0.7,0.8] & [0.5,0.6] & 0\\ [0.4,0.5] & [0.4,0.5] & 0\\ [0.2,0.3] & [0.2,0.3] & 0\\ 0 & 0 & 0\end{bmatrix}$$

and then we have an induced relation $R(A',B)$ by the schema Q, i.e.

$$R(A',B)=\tilde{S}(A',A)\cdot R(A,B)=\begin{bmatrix}[0.49,0.56] & [0.35,0.42] & 0\\ [0.28,0.35] & [0.28,0.35] & 0\\ [0.14,0.21] & [0.14,0.21] & 0\\ 0 & 0 & 0\end{bmatrix}$$

where the t-norm is the algebraic product. Finally, we have the result B_{II}' by supremum projection, i.e.

$$\begin{aligned}B_{\mathrm{II}}'&=\sup\nolimits_{x\in X}R(A',B)(x,y)\\&=\{([0.49,0.56],y_1),([0.35,0.42],y_2),(0,y_3)\}\end{aligned}$$

Suppose that the fact A' and the antecedent part A are interchanged. An identical result B_{I}' can be deduced from the Turksen's SAR method, whereas a different result B_{II}' can be derived from the proposed method, i.e.

$B_{\mathrm{II}}'=\{(0.7,y_1),([0.35,042],y_2),(0,y_3)\}$.

Remark2. It can be seen from Example 2 that every change in the concept, as it appears in the conditional premise and in the fact, is incorporated into the induced interval-valued fuzzy relation. Hence, through the projection operation on the induced relation, the inference result is influenced by the changes in the fact and the antecedent of a rule.

4 Case Study

In shipbuilding technologies, some operational systems are so complex that it is very difficult for us to describe them with precise mathematical models. For these systems, the operational determinations can be acquired by means of the experiences of operators accumulated in practices. As the experiential knowledge is often fuzzy, which is suitable to describe by IVFSs, in the sequel we provide two technological cases modelled by IVFSs to illustrate applications of similarity-based method proposed in this article.

4.1 Layout of heating lines on plate

The processing practices indicate that length and density of heating lines exert a great impact on the forming of sheet metals. Let *X*, *Y* and *Z* be the linguistic variables representing curvature of a bending plate, the length of heating lines, and the space of heating lines, respectively. And the linguistic values are composed of several interval-valued fuzzy sets $A_i\in IVFSs(X)$, $B_j\in IVFSs(Y)$ and $C_k\in IVFSs(Z)$, as shown in Table 1, where $X=Y=Z=\{1,2,3,4,5\}$. And the operational rules derived from experiences of operators are summarized in Table 2

IVFSs \ Universe	1	2	……	5
A_1 (Large)	0	[0.1,0.2]	……	1
A_2 (Medium)	[0.2,0.3]	[0.7,0.8]	……	[0.1,0.2]
A_3 (Small)	1	[0.4,0.4]	……	0
B_1 (Long)	0	[0,0.1]	……	1
B_2 (Medium)	[0.2,0.4]	[0.5,0.5]	……	[0.2,0.3]
B_3 (Short)	1	[0.6,0.8]	……	0
C_1 (Large)	0	[0.1,0.1]	……	1
C_2 (Medium)	[0.1,0.3]	[0.4,0.5]	…..	[0.3,0.4]
C_3 (Small)	1	[0.7,0.8]	……	0

Table 1: Description for linguistic values based on IVFSs.

Rule No.	Antecedent part	Consequent part	
1	X is A_1	Y is B_1	Z is C_3
2	X is A_1	Y is B_2	Z is C_3
3	X is A_2	Y is B_2	Z is C_2
4	X is A_3	Y is B_3	Z is C_1
5	X is A_3	Y is B_3	Z is C_2

Table 2: Operational rules on the layout of heating lines.

Now let us conduct approximate reasoning using the scheme proposed in Section 3, and the inference procedures can be summarized as follows.

*Step*1. Translate the l^{th} rule r_l and compute $R\left(A^{(l)}, B^{(l)} \times C^{(l)}\right)$ using the extensional Mamdani operator, for $l = 1, 2, \cdots, 5$;
*Step*2. Compute the similarity grade s_l between input case A' and the antecedent $A^{(l)}$ by Definition 3;
*Step*3. Combine s_l with $R\left(A^{(l)}, B^{(l)} \times C^{(l)}\right)$ to induce an interval-valued fuzzy relation $R\left(A', B^{(l)} \times C^{(l)}\right)$ using the modification schema Q;
*Step*4. Deduce a conclusion output $B'^{(l)} \times C'^{(l)}$ by the supremum projection over $R\left(A', B^{(l)} \times C^{(l)}\right)$;
*Step*5. Derive the general output $B' \times C'$ from union operation over $B'^{(l)} \times C'^{(l)}$;
*Step*6. Decouple the synthetic output $\kappa\left(B' \times C'\right)$ to obtain $\kappa\left(B'\right)$ and $\kappa\left(C'\right)$ via the projection operations on the universes Z and Y, respectively.
*Step*7. Obtain the determination values by defuzzification operations, using the maximum membership method.

In *Step*1, since $B^{(l)} \times C^{(l)}$ is a synthetic consequent part, which can be interpreted as a binary interval-valued fuzzy relation $R\left(B^{(l)}, C^{(l)}\right)$ such that

$$\underline{R}\left(B^{(l)}, C^{(l)}\right)(y, z) = \min\left(\underline{B}^{(l)}(y), \underline{C}^{(l)}(z)\right)$$
$$\overline{R}\left(B^{(l)}, C^{(l)}\right)(y, z) = \min\left(\overline{B}^{(l)}(y), \overline{C}^{(l)}(z)\right)$$

Thus, a ternary interval-valued fuzzy relation $R\left(A^{(l)}, B^{(l)} \times C^{(l)}\right)$ is constructed by

$$\underline{R}\left(A^{(l)}, B^{(l)} \times C^{(l)}\right)(x, y, z) = \min\left(\underline{A}^{(l)}(x), \underline{R}\left(B^{(l)}, C^{(l)}\right)(y, z)\right)$$
$$\overline{R}\left(A^{(l)}, B^{(l)} \times C^{(l)}\right)(x, y, z) = \min\left(\overline{A}^{(l)}(x), \overline{R}\left(B^{(l)}, C^{(l)}\right)(y, z)\right)$$

In *Step*3, according to the modification schema Q, we then obtain

$$\underline{R}\left(A', B^{(l)} \times C^{(l)}\right)(x, y, z) = s_l \cdot \underline{R}\left(A^{(l)}, B^{(l)} \times C^{(l)}\right)(x, y, z)$$
$$\underline{R}\left(A', B^{(l)} \times C^{(l)}\right)(x, y, z) = s_l \cdot \underline{R}\left(A^{(l)}, B^{(l)} \times C^{(l)}\right)(x, y, z)$$

where t-norm is the algebraic product. From *Step*4, we have

$$\begin{aligned}\left(\underline{B}'^{(l)} \times \underline{C}'^{(l)}\right)(y, z) &= s_l \cdot \min\left(\sup\nolimits_{x \in X} \underline{A}^{(l)}(x), \underline{R}\left(B^{(l)}, C^{(l)}\right)(y, z)\right) \\ \left(\overline{B}'^{(l)} \times \overline{C}'^{(l)}\right)(y, z) &= s_l \cdot \min\left(\sup\nolimits_{x \in X} \overline{A}^{(l)}(x), \overline{R}\left(B^{(l)}, C^{(l)}\right)(y, z)\right)\end{aligned} \quad (2)$$

If $A^{(l)}$ is normal, then Formula (2) can be simplified as

$$\begin{aligned}\left(\underline{B}'^{(l)} \times \underline{C}'^{(l)}\right)(y, z) &= s_l \cdot \underline{R}\left(B^{(l)}, C^{(l)}\right)(y, z) \\ \left(\overline{B}'^{(l)} \times \overline{C}'^{(l)}\right)(y, z) &= s_l \cdot \overline{R}\left(B^{(l)}, C^{(l)}\right)(y, z)\end{aligned} \quad (3)$$

Now let a case input

$$A' \triangleq \text{"more or less large"} = 0/1 + [0.32, 0.45]/2 + [0.55, 0.63]/3 + [0.90, 0.95]/4 + 1/5$$

For the 1st rule, according to Definition 3, the degree of similarity between A' and $A^{(1)}$ is calculated by

$$s_1 = \left(S\left(\underline{A}', \underline{A}^{(1)}\right) + S\left(\overline{A}', \overline{A}^{(1)}\right) + 2S\left(\kappa\left(A'\right), \kappa\left(A^{(1)}\right)\right)\right)/4 = 0.87$$

Since $A^{(1)}$ is normal, we then obtain $B'^{(1)} \times C'^{(1)}$ by Formula (3), i.e.

$$B'^{(1)} \times C'^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ [0, 0.09] & [0, 0.08] & [0, 0.09] & [0, 0.09] & 0 \\ [0.35, 0.35] & [0.35, 0.35] & [0.26, 0.35] & [0.09, 0.17] & 0 \\ [0.61, 0.78] & [0.61, 0.70] & [0.26, 0.35] & [0.09, 0.17] & 0 \\ [0.87, 0.87] & [0.61, 0.70] & [0.26, 0.35] & [0.09, 0.17] & 0 \end{bmatrix}$$

Similarly, we can derive the outputs from other rules in Table 2, and the general output is calculated by

$$B' \times C' = \bigcup_{l=1}^{5}\left(B'^{(l)} \times C'^{(l)}\right) = \begin{bmatrix} [0.17, 0.35] & [0.17, 0.35] & [0.17, 0.35] & [0.11, 0.21] & [0.13, 0.21] \\ [0.44, 0.44] & [0.44, 0.44] & [0.26, 0.35] & [0.26, 0.26] & [0.15, 0.21] \\ [0.87, 0.87] & [0.61, 0.70] & [0.51, 0.51] & [0.26, 0.32] & [0.15, 0.21] \\ [0.61, 0.78] & [0.61, 0.70] & [0.32, 0.36] & [0.26, 0.32] & [0.15, 0.21] \\ [0.87, 0.87] & [0.61, 0.70] & [0.26, 0.35] & [0.11, 0.17] & [0.15, 0.21] \end{bmatrix}$$

From *Step*6, we have

$$\kappa\left(B'\right)(y) = \sup\nolimits_{z \in Z}\left(\kappa\left(B' \times C'\right)\right)(y, z) = \{0.52, 0.88, 1.74, 1.39, 1.74\}$$
$$\kappa\left(C'\right)(z) = \sup\nolimits_{y \in Y}\left(\kappa\left(B' \times C'\right)\right)(y, z) = \{1.74, 1.31, 0.61, 0.28, 0.36\}$$

According to *Step*7, since

$$\sup\nolimits_{y \in Y} \kappa\left(B'\right)(y) = \kappa\left(B'\right)(3) = \kappa\left(B'\right)(5)$$
$$\sup\nolimits_{z \in Z} \kappa\left(C'\right)(z) = \kappa\left(C'\right)(1)$$

Hence, $(y', z') = (3, 1)$ or $(y', z') = (5, 1)$ are selected as determination values, which can be interpreted as the conclusion is 'Y is long or medium' and 'Z is small' when the case input is 'more or less large'.

4.2 Welding deformation prediction on high-tensile steel structure

Welding experiment shows that, welding deformation of high-tensile steel structure not only relates to the leg size of weld seam, but also relates to the thickness of steel structure and welding current. Through a large amount of welding experiments, a rule-set including nine rules is summarized by the experienced welding operators, as shown in Table 3, where the linguistic values are the interval-valued fuzzy sets. For example, let Y be the thickness universe. The membership function of A_{21}, A_{22} and A_{23} is given as Table4, where linguistic values 'Thick', 'Medium' and 'Thin' are represented by A_{21}, A_{22} and A_{23}, respectively.

Rule No.	Antecedent part			Consequent part
1	X is A_{11}	Y is A_{21}	Z is A_{31}	W is D_3
2	X is A_{12}	Y is A_{21}	Z is A_{31}	W is D_3
……	……	……	……	……
9	X is A_{13}	Y is A_{23}	Z is A_{33}	W is D_2

Table 3: Decision rule-set of welding deformation.

Linguistic value \ Y	1	2	3	4	5
Thick	[0,0]	[0,0]	[0.2,0.2]	[0.6,0.7]	[1,1]
Medium	[0,0]	[0.1,0.3]	[0.9,0.9]	[0.3,0.3]	[0,0]
Thin	[1,1]	[0.8,0.8]	[0.2,0.3]	[0,0]	[0,0]

Table 4: Membership function of linguistic value of thickness universe.

Once the knowledge model based on decision rules of welding deformation is obtained, we can set up a model of approximate reasoning, using the proposed method in this article. Through fuzzification of input data, similarity-based reasoning as well as defuzzification of fuzzy data, we then obtain the inference result of welding deformation. The running interface on welding deformation prediction system is shown as Figure 1. To examine the effectiveness of inference model, we arrange a welding experiment including ten test samples, as shown in Table 5.

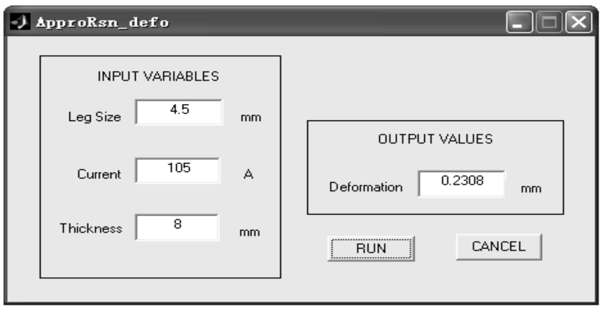

Figure 1: Running interface on welding deformation prediction system.

Table 5: Test data on welding experiment.

Test sample	Leg size /mm	Thickness /mm	Current /A	Deformation value /mm
p_1	5.5	6	130	0.53
p_2	4.5	8	95	0.26
……	……	……	……	……
p_{10}	4.5	5	115	0.46

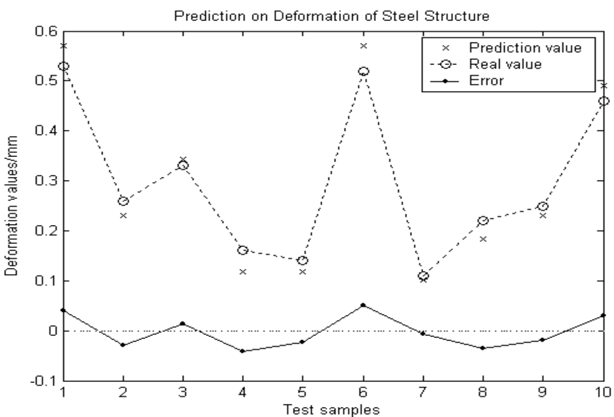

Figure 2: Comparison of prediction values and real values

From error curve of prediction values and real values shown in Figure 2, we can calculate the maximum error $E_{\max} = 0.050$, the mean error $E_m = 0.029$ and the standard error $E_{\mathrm{std}} = 0.0317$, respectively. In terms of prediction accuracy of welding deformation, the result justifies the effectiveness of the proposed method.

5 Conclusion

In this paper, we investigate the similarity measures of interval-valued fuzzy sets. Based on the Turksen's reasoning model, we develop an approach to inference by combining the conventional CRI with similarity-based approximate reasoning. It is shown that a general representation for inference conclusion can be yielded by the procedures including translation, matching, modification, and projection. Besides, as the approximation inference is performed under the framework of IVFSs, the proposed method seems more flexible than is done with the general FSs. In the end, we utilize two examples concerning shipbuilding techniques to illustrate and validate the proposed schema.

For the nonlinear and coupling shipbuilding technology, the conventional modelling schema mainly contains physical simulation and Finite Element Analysis (FEA). As to the former, it not only costs a large amount of lab funds but also limits to experimental conditions. As for FEA, although the precise of this method is relatively high, the program is so time-consuming that it can hardly be applied to manufacture practices. Compared with the traditional methods, modelling based on fuzzy data can fully take advantage of the experiences of experts in their field, and accuracy of inference result is also adequate to meet the needs of technological practices. Therefore, we have lots of research opportunities for future applications of similarity-based inference to complex shipbuilding systems, such as layout of heating lines, welding parameters design and welding deformation prediction, etc.

Acknowledgement

This research was partially supported by the national key research project 'Knowledge Based Ship Design and Hyper Integrated Platform'.

References

[1] M.B. Gorzalczany, 'A method of inference in approximate reasoning based on interval-valued

fuzzy sets', *Fuzzy Sets and Systems*, vol. 21, pp. 1-17, 1987.

[2] M.B. Gorzalczany, An interval-valued fuzzy inference method — some basic properties, *Fuzzy Sets and Systems*, vol. 31, pp. 243-251, 1989.

[3] I.B. Turksen, 'Interval valued fuzzy sets based on normal forms', *Fuzzy Sets and Systems*, vol. 20, pp. 191-210, 1986.

[4] I.B. Tursken, Z. Zhao, 'An approximate analogical reasoning based on similarity measures', *IEEE Transactions on Systems Man, and Cybernetics*, vol.18, pp. 1049-1056, 1988.

[5] I.B. Tursken, Z. Zhao, 'An approximate analogical reasoning scheme based on similarity measures and interval valued fuzzy sets', *Fuzzy Sets and Systems*, vol. 34, pp. 323-346, 1990.

[6] S.M. Chen, 'A new approach to handling fuzzy decision making problems', *IEEE Transaction SMC*, vol. 18, pp. 1012-1016, 1988.

[7] D.S. Yeung, E.C.C. Tsang, 'A comparative study on similarity based fuzzy reasoning methods', *IEEE Transaction SMC Part B: Cybernetic*, vol. 27, pp. 216-227, 1997.

[8] L.A. Zadeh, 'Fuzzy logic and approximate reasoning', *Syntheses*, vol. 30, pp. 407-428, 1975.

[9] L.A. Zadeh, 'A theory of approximate reasoning', *Machine Intelligence*, vol. 9, pp. 149-194, 1979.

[10] D.G. Wang et al., 'A fuzzy similarity inference method for fuzzy reasoning', *Computers and Mathematics with Applications*, vol. 56, pp. 2445-2454, 2008.

[11] Y.P. Meng, 'Similarity-based information weighted fuzzy reasoning', *Journal of Beijing Normal University*, vol. 44, pp. 21-24, 2008.

[12] J.S. Ke, G.T. Her, 'A fuzzy information retrieval system model', in: Proc. *1983 National Computer Symposium, Taiwan, Republic of China*, pp. 147-155, 1983.

[13] R. Zwick, E. Carlstein, and D.R. Budescu, 'Measures of similarity among fuzzy concepts: a comparative analysis', *International Journal of Approximate Reasoning*, vol. 1, pp. 221-242, 1987.

[14] K.T. Atanassov, 'Intuitionistic fuzzy sets', *Fuzzy Sets and Systems*, vol. 20, pp. 87-96, 1986.

[15] W.L. Gau, D.J. Buehrer, 'Vague sets', *IEEE Transactions on Systems, Man and Cybernetics*, vol. 23, pp. 610-614, 1993.

[16] Atnassov K, Gargov G., 'Interval valued intuitionistic fuzzy sets', *Fuzzy Sets and Systems*, vol.31, pp.343-349, 1989.

[17] Bustince H, Burillo P, 'Vague sets are intuitionistic fuzzy sets', *Fuzzy Sets and Systems*, vol. 79, pp. 403-405, 1996.

[18] Deschrijver G, Kerre E E., 'On the relationship between some extensions of fuzzy set theory', *Fuzzy Sets and Systems*, vol. 133, pp. 227-235, 2003.

[19] F. Li et al., 'Implication operator based on vague sets', *J. Huazhong Univ. Sci. & Tech.* (Nature Science Edition), vol. 29, pp. 53-55, 2001.

[20] C. Cornelis, G. Deschrijver et al., 'Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: construction, classification, application', *International Journal of Approximate Reasoning*, vol.35, pp. 55-95, 2004.

[21] X.L. X, Y.J. Lei, 'Approximate reasoning method based on some intuitionistic fuzzy relations', *Computer Engineering*, vol. 34, pp. 4-6, 2008.

[22] Z.Q Feng et al., 'Vague propositional logic and approximation inference based on linear transformation', *Journal of Harbin Engineering Universe*, vol. 9, pp. 1222-1227, 2010.

[23] F. Li et al., 'An approximate reasoning approach based on the measures of similarity between vague sets', *J. Huazhong Univ. Sci. & Tech.* (Nature Science Edition), vol. 32, pp. 44-46, 2004.

[24] Y. Tian et al., 'An approximate reasoning approach based on the measures of similarity between intuitionistic fuzzy sets', *Journal of Air Force Engineering University* (Nature Science Edition), vol. 8, pp. 81-83, 2007.

[25] Y.Q. Shi et al., 'Bidirectional approximate reasoning based on distances of vague set', *Journal of Chinese Computer System*, vol. 28, pp. 661-665, 2007.

[26] X.Z. Guan et al., 'Controller design based on the measures of similarity reasoning using vague sets', *Control Engineering of China*, vol. 13, pp. 15-24, 2006.

[27] T.J. Wang et al., 'Bidirectional approximate reasoning based on weighted similarity measures of vague set', *Mini-Micro Systems*, vol. 25, pp. 211-215, 2004.

[28] X.M. Li et al., 'Approximate reasoning method based on intuitionistic fuzzy', *Journal of Air Force Engineering University* (Nature Science Edition), vol. 9, pp. 82-85, 2008.