On singular lagrangians and Dirac's method

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Recibido el 2 de septiembre de 2011; aceptado el 9 de diciembre de 2011

We show some instances of singular lagrangians from the classical mechanics of particles and apply Dirac's method for building the canonical equations. We find then the reason for the singularity, and therefore, we get the Hamilton equations with the familiar procedure, that is without the need of Dirac's procedure. Known cases of singular lagrangians in special relativity are also presented, and their non-singular alternatives.

Keywords: Singular lagrangian; classical mechanics; special relativity.

Se presentan algunos lagrangianos singulares del ámbito de la mecánica clásica de partículas, y se les aplica el método de Dirac para construir las ecuaciones canónicas. Se halla la razón de la singularidad, y, con ello, se obtienen las ecuaciones de Hamilton por el camino acostumbrado, esto es, sin necesidad del método de Dirac. Se presentan también casos conocidos de lagrangianos singulares en la relatividad especial y sus alternativas no singulares.

Descriptores: Lagrangiano singular; mecánica clásica; relatividad especial.

PACS: 45.20.Jj; 11.10.Ef

1. Introduction

The transition from the lagrangian to the hamiltonian formalism is carried out by expressing the generalized velocities $\dot{q}_i$ (i=$1,\ldots,n$) in terms of the momenta $p_j(q,\dot{q},t)$=$\partial L/\partial\dot{q}_j$, and eliminating them in the function $H=\sum p\,\dot{q}-L$. This is possible if the mathematical condition

$$\left\|\frac{\partial p_i}{\partial\dot{q}_j}\right\| = \left\|\frac{\partial^2 L}{\partial\dot{q}_i\partial\dot{q}_j}\right\| \neq 0, \tag{1}$$

is satisfied. This signifies that they build a set of independent variables. But if the determinant vanishes there exists one or more relations between the p's:

$$\phi_k(p,q,t)=0, \qquad k=1,\ldots,\alpha, \tag{2}$$

where $n-\alpha$ is the rank of the matrix $(\partial^2 L/\partial\dot{q}_i\partial\dot{q}_j)$. Thus not all p's are independent. In such situation one says that the lagrangian is degenerated or singular, and the Hamilton equations of motion cannot be obtained by the familiar procedure. In an attempt to generalize the hamiltonian dynamics, Dirac [1-3] developed a method for building the canonical equations starting from the complete hamiltonian

$$H=H_0+\sum v_k\phi_k, \tag{3}$$

where $H_0=\sum\dot{q}_l p_l-L$ depends on the coordinates and the independent p's, and v_k are new independent variables. This comes from taking a virtual variation of H_0 ([7]):

$$\delta H_0{=}\sum\left(\dot{q}_i\delta p_i{-}\frac{\partial L}{\partial q_i}\,\delta q_i\right){=}\sum\left(\dot{q}_i\delta p_i{-}\dot{p}_i\,\delta q_i\right), \tag{3a}$$

or

$$\sum\left(\dot{q}_i-\frac{\partial H_0}{\partial p_i}\right)\delta p_i-\left(\dot{p}_i+\frac{\partial H_0}{\partial q_i}\right)\delta q_i=0, \tag{3b}$$

for all $\delta p_i,\delta q_i$, consistent with the restrictions:

$$\sum_{i=1}^{n}\left(\frac{\partial\phi_k}{\partial q_i}\,\delta q_i+\frac{\partial\phi_k}{\partial p_i}\,\delta p_i\right)=0, \quad k=1,\ldots,\alpha, \tag{3c}$$

that is, α δ's of all $\delta p_i,\delta q_i$ depend on the remaining ones. Eliminating them from Eq. (3b) by the well-known multiplier's procedure, one has

$$\dot{q}_i=\frac{\partial H_0}{\partial p_i}+\sum v_k\,\frac{\partial\phi_k}{\partial p_i},$$
$$\dot{p}_i=-\frac{\partial H_0}{\partial q_i}-\sum v_k\,\frac{\partial\phi_k}{\partial q_i}, \qquad i=1,\ldots,n, \tag{3d}$$

or briefly

$$\dot{q}_i=\frac{\partial H}{\partial p_i}, \quad \dot{p}_i=-\frac{\partial H}{\partial q_i}, \quad i=1,\ldots,n, \tag{3e}$$

with

$$H=H_0+\sum v_k\,\phi_k.$$

Dirac then imposes to the *primary restrictions* ϕ_k the consistency conditions $\dot{\phi}_k=0$, from which one can obtain additional restrictions. Some of these can be identities ($0=0$), others of the form $f_m(q,p)=0$ (like functions (2)), and others of type $g_l(q,p)+v_l h_l(p,q)=0$, that can be used to fix some of the unknown variables v_k. The second possibility is treated in a similar way as conditions $\phi_k=0$.

2. Cases of singular lagrangians

It is remarkable that most classical mechanics textbooks do not treat the topic on singular lagrangians, or if they do they lack on a discussion of some specific cases (not even 'artificial examples'), although the aim of generally building the appropriate canonical equations have had a considerable development, since the beginning of the way initiated by Dirac [1]. But self in specialized papers one seldom finds examples of systems with such behavior.

Here, we write down a set of particular lagrangians of a special type:

$$L=\frac{1}{2}m\big(\dot q_1^2+\dot q_2^2+l^2\dot q_3{}^2+2\,l\,\dot q_1\,\dot q_3\cos q_3 + 2\,l\,\dot q_2\,\dot q_3\sin q_3\big)+V(q_1,q_2,q_3), \tag{4}$$

where l and m are constants, the Mittelstaedt's lagrangian [4]

$$L=\frac{1}{2\,m}\,(\dot q_1+\dot q_2)^2+\frac{1}{2\,\mu}\,\dot q_3^2+V(q_1,q_2,q_3), \tag{5}$$

that of Cawley ([5])

$$L=\dot q_1\,\dot q_2+V(q_1,q_2,q_3) \qquad \left(V=\frac{1}{2}\,q_2\,q_3^2\right), \tag{6}$$

the lagrangian of Deriglazov [6]

$$L=q_2^2\,\dot q_1^2+q_1^2\,\dot q_2^2+2\,q_1\,q_2\,\dot q_1\,\dot q_2+V(q_1,q_2) \tag{7}$$

$(V=q_1^2+q_2^2)$. They all have in common that the potential energy V depends only on the coordinates of the system, and that they are singular. Certainly, it is not difficult to see that there exists *one* relation between the p's in each instance:

$$\phi_1=p_3-l\,p_1\cos q_3-l\,p_2\sin q_3=0, \tag{4a}$$

$$\phi_1=p_2-p_1=0, \tag{5a}$$

$$\phi_1=p_3=0, \tag{6a}$$

$$\phi_1=q_1\,p_1-q_2\,p_2=0. \tag{7a}$$

Thus in these cases one cannot arrive at the canonical equations of motion using the well-known procedure, and we are forced to use Dirac's method (see however, Sec. 4).

3. Dirac's method for lagrangians (5) and (7)

Actually, we will only give the details for lagrangian (5) because the results for the other are found in the reference [6].

We start obtaining the momenta of the system, using Eq. (5):

$$p_1=\frac{1}{m}\,(\dot q_1+\dot q_2),\quad p_2=\frac{1}{m}\,(\dot q_1+\dot q_2),\quad p_3=\frac{1}{\mu}\,\dot q_3, \tag{8}$$

so, p_2 depends on p_1 and only p_1 (or p_2) and p_3 are independent. The primary restriction is then Eq. (5a) $p_2-p_1=0$. For getting H_0 (Eq. (3)), we eliminate the velocities from $\sum\dot q_i\,p_i-L$ in favor of the independent p's, resulting

$$H_0=\frac{m}{2}\,p_1^2+\frac{\mu}{2}\,p_3^2-V. \tag{9}$$

The complete hamiltonian is then

$$H=\frac{m}{2}\,p_1^2+\frac{\mu}{2}\,p_3^2-V+v\,(p_2-p_1). \tag{10}$$

The consistency condition $\dot\phi_1=\dot p_2-\dot p_1=0$ leads to the secondary restriction

$$\phi_2=\frac{\partial V}{\partial q_1}-\frac{\partial V}{\partial q_2}=0. \tag{11}$$

This is a relation between q_1, q_2 and q_3 which we briefly write as

$$\phi_2=q_2-F(q_1,q_3)=0. \tag{12}$$

We then build the consistency condition $\dot\phi_2=0$, or

$$\dot\phi_2=[\phi_2,H]=0,$$

from which we find

$$v\,(1+F_{,1})-m\,p_1\,F_{,1}-\mu\,p_3\,F_{,3}=0 \qquad \left(F_{,i}\equiv\frac{\partial F}{\partial q_i}\right). \tag{13}$$

$[\phi_2,H]$ is the Poisson bracket of ϕ_2 and H. Eq. (13) allows fixing variable v:

$$v=\frac{m\,p_1\,F_{,1}+\mu\,p_3\,F_{,3}}{1+F_{,1}}. \tag{14}$$

With the additional relations (12) and (14), we can now write the canonical equations of motion:

$$\begin{aligned}&\dot q_1=mp_1-v, && \dot q_2=v, && \dot q_3=\mu p_3,\\ &\dot p_1=V_{,1}, && \dot p_2=V_{,2}, && \dot p_3=V_{,3},\end{aligned} \tag{15}$$

with

$$q_2=F(q_1,q_3),\qquad v=\frac{m\,p_1\,F_{,1}+\mu\,p_3\,F_{,3}}{1+F_{,1}}.$$

Thus, the independent equations of motion are

$$\begin{aligned}&\dot q_1=\frac{m\,p_1-\mu\,p_3\,F_{,3}}{1+F_{,1}}, && \dot q_3=\mu p_3,\\ &\dot p_1=(V_{,1})_{q_2=F}\,, && \dot p_3=(V_{,3})_{q_2=F}\,.\end{aligned} \tag{16}$$

Eqs. (16) can easily be written in newtonian form:

$$1+F_{,1}\ddot q_1+F_{,3}\,\ddot q_3+F_{,11}\,\dot q_1^2 + 2\,F_{,13}\,\dot q_1\,\dot q_3+F_{,33}\,\dot q_3^2=m\,\left(V_{,1}\right)_{q_2=F}\,, \tag{17}$$

$$\ddot q_3=\mu\,(V_{,3})_{q_2=F}\,. \tag{18}$$

On the other hand, for Deriglazov's lagrangian (7) it is found that

$$H=\frac{p_1^2}{4\,q_2^2}-V(q_1,q_2)+v\,(q_1\,p_1-q_2\,p_2), \tag{19}$$

and

$$\phi_2=q_1\,V_{,1}-q_2\,V_{,2}=0,\quad\text{or}\quad \phi_2=q_2-F(q_1)=0,$$

$$v=-\frac{p_1}{2F^2\,(F+q_1\,F_{,1})}F_{,1}.$$

Therefore, the independent canonical equations are

$$\dot{q}_1 = \frac{p_1}{2F^2 + 2\, q_1\, F\, F_{,1}},$$
$$\dot{p}_1 = \frac{p_1^2}{2F^2\,(F + q_1\, F_{,1})}\, F_{,1} + (V_{,1})_{q_2=F(q_1)}\,, \qquad (20)$$

and from here one also gets the Newton's equation of motion (Deriglazov uses $V(x,y) = x^2 + y^2$ and $F(x) = \pm x$)

$$2F\,(F + q_1 F_{,1})\,\ddot{q}_1 + 2F(2F_{,1} + q_1 F_{,11})\dot{q}_1^2 - (V_{,1})_{q_2=F(q_1)} = 0, \quad (21)$$

4. An alternative procedure to arrive to the equations of motion

The particular cases here considered, imply the relations $\phi_1 = 0$ (4a) to (7a) between the momenta. Without regarding the hamiltonian formalism, we can deduce the consequence of such relations.

For lagrangian (4) the p's are given by

$$p_1 = m\,\dot{q}_1 + m\,l\,\dot{q}_3\cos q_3,$$
$$p_2 = m\,\dot{q}_2 + m\,l\,\dot{q}_3\sin q_3, \qquad (22)$$
$$p_3 = m\,l^2\dot{q}_3 + m\,l\,(\dot{q}_1\cos q_3 + \dot{q}_2\sin q_3),$$

from which, we know, follows Eq. (4a). If we now take the time derivative of Eq. (4a), substitute there p_1 and p_2 from Eqs. (22) and take into account Lagrange's equations, we write

$$V_{,1}\,l\cos q_3 + V_{,2}\,l\sin q_3 - V_{,3} = 0. \qquad (23a)$$

In a similar way, the implication of $\phi_1 = 0$ for the remaining cases is

$$V_{,1} = V_{,2} \quad \text{or} \quad q_2 = F(q_1, q_3), \qquad (23b)$$
$$V_{,2} = 0, \qquad (23c)$$
$$q_1\, V_{,1} = q_2\, V_{,2} \quad \text{or} \quad q_2 = F(q_1), \qquad (23d)$$

Equations (23a)-(23d) are relations between the coordinates of each system, thus one coordinate cannot be independent. In these cases, the reason for the lagrangian to be singular is that the coordinates are not independent (see Appendix), and so the canonical equations cannot be obtained by the familiar procedure, in which it is necessary that the coordinates be generalized (independent). Therefore, eliminating one of the coordinates from the corresponding lagrangian, it would be possible to build straightforwardly the Hamilton's equations. Let us do it for lagrangians (5) and (7).

Substituting Eq. (23b) into (5) we get

$$L = \frac{1}{2\,m}\,(1 + F_{,1})^2\,\dot{q}_1^2 + \frac{1}{2}\left(\frac{1}{m}F_{,3} + \frac{1}{\mu}\right)\dot{q}_3^2 + \frac{1}{m}\,(1 + F_{,1})\,F_{,3}\,\dot{q}_1\dot{q}_3 + V', \qquad (24)$$

with

$$V'(q_1, q_3) = V\big(q_1, F(q_1, q_3), q_3\big).$$

Likewise, the substitution of Eq. (23d) into (7) leads to

$$L = (q_1\, F_{,1} + F)^2\,\dot{q}_1^2 + V'(q_1),$$
$$V'(q_1) = V(q_1, F(q_1)),$$
$$H = \frac{{p_1}^2}{4\,(F + q_1\,F_{,1})^2} - V'(q_1). \qquad (25)$$

Let us write the equation of motion for Deriglazov's lagrangian (25), $dp_1/dt = \partial L/\partial q_1$:

$$2\,(F + q_1\,F_{,1})^2\,\ddot{q}_1 + 2\,(F + q_1\,F_{,1})\,(2\,F_{,1} + q_1\,F_{,11})\dot{q}_1^2 - V'_{,1} = 0. \qquad (26)$$

This equation is exactly the same as Eq. (21). This can be seen from Eq. (23d) that we write at $q_2 = F(q_1)$:

$$q_1\,(V_{,1})_{q_2=F(q_1)} = (q_2\,V_{,2})_{q_2=F(q_1)}\,, \qquad (27)$$

so that

$$(V_{,1})_{q_2=F(q_1)} = \frac{1}{q_1}\,F\,(V_{,2})_{q_2=F(q_1)}\,, \qquad (28)$$

$$V'_{,1} = \frac{1}{q_1}\,(F + q_1\,F_{,1})\,(V_{,2})_{q_2=F(q_1)}\,, \qquad (29)$$

and thus factor F cancels out from Eq. (21), and $F + q_1\,F_{,1}$ from Eq. (26).

Regarding (5) we get, after substituting (23b) into (5),

$$L = \frac{1}{2\,m}A^2\dot{q}_1^2 + \left(\frac{B^2}{2m} + \frac{1}{2\mu}\right)\dot{q}_3^2 + \frac{A\,B}{m}\,\dot{q}_1\dot{q}_3 + V', \quad (30)$$

where we have done the abbreviations

$$A = 1 + F_{,1}, \quad B = F_{,3},$$
$$V'(q_1, q_3) = V\big(q_1, q_2 = F(q_1, q_3), q_3\big). \qquad (31)$$

The two momenta and the generalized velocities are then given by

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \frac{A^2}{m}\,\dot{q}_1 + \frac{A\,B}{m}\,\dot{q}_3,$$
$$p_3 = \frac{A\,B}{m}\,\dot{q}_1 + \left(\frac{B^2}{m} + \frac{1}{\mu}\right)\dot{q}_3, \qquad (32)$$

$$\dot{q}_1 = \frac{m + \mu\,B^2}{A^2}\,p_1 - \frac{\mu\,B}{A}\,p_3, \quad \dot{q}_3 = -\frac{\mu\,B}{A}\,p_1 + \mu\,p_3. \quad (33)$$

Thus the hamiltonian is

$$H = \frac{m}{2\,A^2}\,p_1 + \frac{\mu}{2\,A^2}\,(B\,p_1 - A\,p_3)^2 - V'(q_1, q_3). \quad (34)$$

We can now easily write the canonical equations of motion:

$$\dot q_1 = \frac{m+\mu B^2}{A^2} p_1 - \frac{\mu B}{A} p_3, \qquad \dot q_3 = -\frac{\mu\, B}{A}\, p_1 + \mu\, p_3,$$
$$\dot p_1 = V'_{,1} + \frac{m}{A^3}\, A_{,1}\, p_1 - \frac{\mu}{A^2}\,(B\, p_1 - A\, p_3)\left(B_{,1}\, p_1 - \frac{B}{A} A_{,1}\, p_3\right), \tag{35}$$
$$\dot p_3 = V'_{,3} + \frac{m}{A^3}\, A_{,3}\, p_1 - \frac{\mu}{A^2}\,(B\, p_1 - A\, p_3)\left(B_{,3}\, p_1 - \frac{B}{A} A_{,3}\, p_3\right).$$

From here we come to the equations of motion for q_1 and q_3 by eliminating p_1 and p_3 in the two last equations (35). For this purpose, we derive Eqs. (32) with respect to t and substitute the result in (35). We get, after solving for $\ddot q_1$ and $\ddot q_3$ and taking into account that

$$A_{,3} = B_{,1} = F_{,13}, \tag{36}$$

the equations

$$A\, \ddot q_1 + A_{,1}\, \dot q_1^2 + B_{,3}\, \dot q_3^2 + 2A_{,3}\, \dot q_1 \dot q_3 + \mu\, B\, V'_{,3} - \frac{m+\mu\, B^2}{A} V'_{,1} = 0, \tag{37}$$

$$\ddot q_3 - \mu V'_{,3} + \mu \frac{B}{A} V'_{,1} = 0. \tag{38}$$

By Eq. (23b), regarding that

$$(V_{,1})_{q_2=F} = (V_{,2})_{q_2=F}\,,$$

we can see that Eqs. (37) and (38) are fully equivalent to Eqs. (17) and (18).

For the cases presented here we then see that even though there exists a relation between the momenta, it is not necessary to apply Dirac's method for building the canonical equations. We can continue using the conventional procedure, without the need of invoking any generalization of the dynamics.

These lagrangians are of the type[i]

$$L = L_0(Q_m, \dot Q_m) + V(Q, Q_m)$$

which is known to be singular. Of course, this does not change the fact that they are so because one uses more coordinates than there are degrees of freedom. *Lowering* the number of coordinates accordingly, the problems reduce to ordinary ones (see Appendix). Moreover, restrictions (23b) and (23d) are not set 'on the fly', rather they are a consequence of the way we build the lagrangian.

We can summarize the results in other terms. If we interpret the velocity dependent part of (4) to (7) as the kinetic energy of the system[ii],

$$T = \frac{1}{2}\, m \left(\frac{d\mathbf{s}}{dt}\right)^2 = \frac{1}{2}\, m g_{ij}\, \dot q_i\, \dot q_j,$$

where g_{ij} are the components of the metric tensor, and sum over repeated indices is understood, then the volume element in the space of the system can be written as

$$d\tau = \sqrt{\|g_{mn}\|}\, dq_1\, dq_2\, dq_3, \tag{39}$$

where $\|g_{mn}\|$ is the determinant of the metric tensor. But if, as it is here the case, Eq. (1) is violated, then the volume element vanishes, and thus the system is restricted to a space of lower dimension (*e.g.* a surface).

5. Relativistic lagrangians

There are several possibilities to build the free particle relativistic lagrangian that reduce to the classical expression in the limit $c \to \infty$. For instance,

$$L = -m\, c \sqrt{c^2 - \dot q^2}, \tag{40}$$

for the one dimensional motion is

$$L \approx -m\, c^2 + \frac{1}{2}\, m\, \dot q^2$$

when the velocity of the particle is much smaller than c. The corresponding hamiltonian is, therefore

$$H = p\, \dot q - L = c\, \frac{p^2 + m^2 c^2}{\sqrt{p^2 + m^2 c^2}} = c\sqrt{p^2 + m^2 c^2}, \tag{41}$$

where

$$p = \frac{m\, c\, \dot q}{\sqrt{c^2 - \dot q^2}}, \qquad \dot q = \frac{c\, p}{\sqrt{p^2 + m^2 c^2}}.$$

One tries to come in another way to the hamiltonian by using the proper time τ of the particle:

$$c^2 d\tau^2 = c^2\, dt^2 - dq^2, \tag{42}$$

instead of the coordinate time t. q and t are then functions of the parameter τ: $q(\tau)$, $t(\tau)$, so that the lagrangian now is

$$L = -m\, c\, \sqrt{c^2\, t'^{\,2} - q'^{\,2}}, \tag{43}$$

where

$$t' = \frac{dt}{d\tau}, \qquad q' = \frac{dq}{d\tau}.$$

For the lagrangian (43) we can construct two momenta p_0 and p, given by

$$p_0 = \frac{\partial L}{\partial t'} = -\frac{m\, c^3\, t'}{\sqrt{c^2\, t'^{\,2} - q'^{\,2}}},$$
$$p = \frac{\partial L}{\partial q'} = \frac{m\, c\, q'}{\sqrt{c^2\, t'^{\,2} - q'^{\,2}}}. \tag{44}$$

It is not difficult to see that there exists a relation between them:

$$p_0^2 = c^2\, p^2 + m^2\, c^4, \tag{45}$$

so that (43) is singular. This lagrangian is peculiar in a certain sense. For all lagrangians of the form

$$L(t', q') = F(c^2\, t'^{\,2} - q'^{\,2}), \tag{46}$$

where F is an arbitrary function of the invariant $c^2\, t'^{\,2} - q'^{\,2}$, the only function F that violates Eq. (1) is just the square root. Indeed, the determinant (1) for the function (46) is

$$\left\| \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right\| = -4c^2 F' \left(F' + 2\left(c^2\, t'^{\,2} - q'^{\,2} \right) F'' \right), \tag{47}$$

where F' is the derivative of F with respect of its argument, so that the determinant is zero for a function F satisfying the equation

$$F' + 2\left(c^2\, t'^{\,2} - q'^{\,2} \right) F'' = 0, \tag{48}$$

that is

$$F(x) = a\sqrt{x} + b, \tag{49}$$

where a and b are constants.

(43) is in this sense the 'worst' choice one can take, much in the same manner as the construction of lagrangian (A3) in the Appendix. If one should have started with the relativistic covariant Newton's second law

$$m\, \frac{d^2 q^i}{d\tau^2} = m\, \frac{dq'^{\,i}}{d\tau} = 0, \tag{50}$$

where $q^0 = c\,t$, $q^1 = q$, and the line element is given by $d\mathbf{s} = (dq^0, dq)$, and the metric tensor g_{ij} has components

$$g_{00} = 1, \qquad g_{11} = -1, \qquad g_{10} = g_{01} = 0, \tag{51}$$

one would have arrived at

$$L = \frac{1}{2}\, m\, (c^2\, t'^{\,2} - q'^{\,2}), \tag{52}$$

that is certainly not singular. With lagrangian (52) one can directly get the hamiltonian by the familiar procedure:

$$H = \frac{p_0^2}{2\, m\, c^2} - \frac{p^2}{2\, m}. \tag{54a}$$

The equations of motion are according to (43)

$$\frac{c^3 m q' \left(q'\, t'' - t'\, q'' \right)}{\left(c^2 t'^{\,2} - q'^{\,2} \right)^{3/2}} = 0, \tag{53}$$

$$\frac{c^3 m t' \left(q'\, t'' - t'\, q'' \right)}{\left(c^2 t'^{\,2} - q'^{\,2} \right)^{3/2}} = 0, \tag{54}$$

and they clearly reduce to only one equation, from which it follows

$$q = c_1\, t + c_2, \tag{55}$$

a relation between q and t. On the contrary, from (52) one get the equations

$$t'' = 0, \qquad q'' = 0, \tag{56}$$

or

$$t = a_1\, \tau + b_1, \qquad q = a_2\, \tau + b_2. \tag{57}$$

Lagrangian (40) describes a relativistic particle if we demand it to be real, so that $v < c$. In the case represented by Eq. (54), one can add the condition $v < c$ for completeness, or demand that the proper time τ (appearing in Eq. (59), for example) must be real.

We are not diminishing the interesting properties of lagrangian (43), like invariance, parametrization independence^{iii}, rather we are only showing here the consequences for the existence of a relation between momenta, and how can one overcome it without the necessity of generalize the classical dynamics.

There is another example of a (relativistic) singular lagrangian, namely ([6], we write it here for a 'one' dimensional motion)

$$L = \frac{1}{2\, q} (\dot{q}_0^2 - \dot{q}_1^2) + \frac{1}{2}\, m^2 q, \tag{58}$$

where $q_0 = q_0(\tau)$, $q_1 = q_1(\tau)$, $q = q(\tau)$ are the unknowns and m is a constant. L is singular because

$$\frac{\partial L}{\partial \dot{q}} = 0, \qquad \text{or} \qquad p = 0 \tag{59}$$

and this is a relation between p's.

On the other hand, the equations of motion are

$$\frac{d}{d\tau}\left(\frac{\dot{q}_0}{q} \right) = 0, \quad \frac{d}{d\tau}\left(\frac{\dot{q}_1}{q} \right) = 0,$$
$$\frac{1}{q^2} (\dot{q}_0^2 - \dot{q}_1^2) - m^2 = 0, \tag{60}$$

from which the third, that is a consequence of Eq. (59), can be solved for $q(\tau)$:

$$q = \pm \frac{1}{m} \sqrt{\dot{q}_0^2 - \dot{q}_1^2}. \tag{61}$$

The first two Eqs. (60) can thus be expressed in the form

$$\frac{d}{d\tau}\left(\frac{\dot{q}_0}{\sqrt{\dot{q}_0^2 - \dot{q}_1^2}} \right) = 0, \qquad \frac{d}{d\tau}\left(\frac{\dot{q}_1}{\sqrt{\dot{q}_0^2 - \dot{q}_1^2}} \right) = 0, \tag{62}$$

and they are equivalent to the equations of motion resulting from (43).

According to Deriglazov, Dirac's method applied to (60), leads to the hamiltonian

$$H = \frac{q}{2} \left(p_0^2 - p_1^2 - m^2 \right) + v\, p, \tag{63}$$

and hence the canonical equations are

$$\dot{q}_i = q\, p_i, \quad \dot{p}_i = 0, \quad \dot{q} = v, \quad \dot{p} = 0, \qquad i = 0, 1, \tag{64}$$

with the conditions (primary and secondary)

$$p = 0, \qquad p_0^2 - p_1^2 - m^2 = 0. \tag{65}$$

The secondary condition is similar to the primary one (45) for lagrangian (43).

The canonical equations of motion (64) contain an undetermined variable v, that equals $\dot{q}$. One can intend to fix it employing the Eqs. (64). From Eqs. (64) one sees that p_0 and p_1 are constant, so that

$$\dot{q}_1 = \frac{p_1}{p_0}\,\dot{q}_0 = A\,\dot{q}_0, \qquad A = \text{constant}, \tag{66}$$

or

$$q_1 = A\,q_0 + B, \tag{67}$$

where B is an arbitrary constant. On the other side, variable q can be written as

$$q^2 = \frac{1-A^2}{m^2}\,\dot{q}_0^2, \tag{68}$$

from which it follows

$$\dot{q} = \pm\sqrt{\frac{1-A^2}{m^2}}\,\ddot{q}_0, \tag{69}$$

and hence

$$v = \pm\sqrt{\frac{1-A^2}{m^2}}\,\ddot{q}_0, \tag{70}$$

Of course, from the canonical equations of motion q_0 and q_1 cannot be determined as functions of τ, so that v, like q, remains undetermined.

For lagrangians (43) and (58), one cannot avoid the use of Dirac's method for constructing the hamiltonian, not even by employing the restrictions as was done in Sec. 4. In the case of (43) the alternative is to take a different lagrangian, for instance that given by Eq. (52).

6. Conclusions

In the classical mechanics of particles, there is no case reported of a singular lagrangian for a real system; all instances that we know are of artificially built systems. Even that of the system described in the Appendix is really not singular. Thus, it seems that the lagrangians of classical mechanics are basically non degenerate.

Singularities appears first when we treat to generalize to cases where there is not a previously given rule for building L, like in the special relativity. There one has the freedom to choose the lagrangian among several possibilities, some of which are regular and others singular. One does not care too much about this because one have a method, Dirac's method, for working out the problem, even at the expense of frequently introducing undetermined variables like the v's.

Perhaps it would be more natural to set the condition on new lagrangians to be regular. One can argue against this that the additional variables v that appear in the theory can reveal symmetries of the system, like gauges. However, if two lagrangians, one regular and the other singular, lead to the same set of equations (for example, field equations), they must share comparable symmetries.

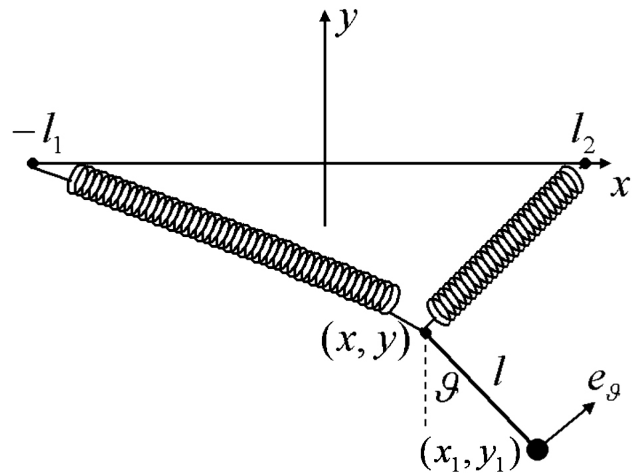

FIGURE 1. Two ends of the springs are fixed at $x = -l_1$ and $x = l_2$. The other ends are joined at a point, whose coordinates are (x, y) at time t. The pendulum is inclined ϑ at this time. The position of the mass is given by (x_1, y_1).

Appendix

A. Singular lagrangian of a physical system

The system in the plane shown in Fig. 1 consists of two massless springs of lengths l_1 and l_2 and constants k_1 and k_2 with ends fixed at $x = -l_1, y = 0$ and $x = l_2, y = 0$, the other ends being joined at the free point (x, y) where a pendulum of length l and mass m hangs.

In setting the newtonian equations of motion for the mass m one takes into account that at the point (x, y)

$$\mathbf{F}_1 + \mathbf{F}_2 + \boldsymbol{\tau} = 0, \tag{A1}$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ are the forces exerted by each spring and $\boldsymbol{\tau}$ is the tension of the string. This implies that the resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$ must have the same inclination ϑ as the string, that is

$$\tan\vartheta = -\frac{F_x}{F_y}, \tag{A2}$$

where $F_x = F_{1x} + F_{2x}$ and $F_y = F_{1y} + F_{2y}$.

The equilibrium condition (A2) can also be deduced directly from the Lagrange's equations of motion, for which the lagrangian is given by

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + l^2\dot{\vartheta}^2 + 2\,l\,\dot{x}\,\dot{\vartheta}\cos\vartheta + 2\,l\,\dot{y}\,\dot{\vartheta}\sin\vartheta\right) - V(x,y) - m\,g\,(l\cos\vartheta + y). \tag{A3}$$

$V(x, y)$ stands for the potential energy of both springs:

$$V(x,y) = \frac{1}{2}\,k_1\,(r_1 - l_1)^2 + \frac{1}{2}\,k_2\,(r_2 - l_2)^2, \tag{A4}$$
$$r_1 = \sqrt{(l_1 + x)^2 + y^2}, \qquad r_2 = \sqrt{(l_2 - x)^2 + y^2}.$$

It is not difficult to show that the momenta

$$p_x = m\,\dot{x} + m\,l\,\dot{\vartheta}\cos\vartheta,$$
$$p_y = m\,\dot{y} + m\,l\,\dot{\vartheta}\sin\vartheta, \tag{A5}$$
$$p_\vartheta = m\,l^2\dot{\vartheta} + m\,l\,(\dot{x}\cos\vartheta + \dot{y}\sin\vartheta),$$

are not independent, but satisfy the relation

$$p_x\,l\cos\vartheta + p_y\,l\sin\vartheta - p_\vartheta = 0. \tag{A6}$$

The consequence of restriction (A6) on the p's can be found by deriving it by t (and taking into account that $\dot{p} = \partial L/\partial q$):

$$m\,g\,l\sin\vartheta = \dot{p}_\vartheta = F_x\,l\cos\vartheta + (F_y + m\,g)\,l\sin\vartheta + l\,\dot{\vartheta}\,(-p_x\sin\vartheta + p_y\cos\vartheta), \tag{A7}$$

where F_x and F_y are the components of the net force of the springs. Substituting now here p_x and p_y as given by Eqs. (A5), one gets

$$F_x\cos\vartheta + (F_y + m\,g)\sin\vartheta = m\,g\sin\vartheta,$$

that is

$$\tan\vartheta = -\frac{F_x}{F_y}, \tag{A8}$$

and it fully agrees with the equilibrium condition (A2). Since

$$F_x = -\frac{\partial V(x,y)}{\partial x}, \qquad F_y = -\frac{\partial V(x,y)}{\partial y},$$

are certain functions of (x, y), expression (A8) is a relation between x, y and ϑ, so that alone two of the three coordinates are independent. Our system has only (obviously) two degrees of freedom. In other words, the existence of a relation of p's in the present case implies the presence of a restriction in the coordinates.

Of course, because the system has only two degrees of freedom, we might as well have used the two obvious coordinates x_1 and y_1 for characterizing the position of the mass point m. The potential energy $V(x, y)$ depends on the point (x, y) and needs to be expressed in terms of (x_1, y_1) by the relations (see Fig. 1)

$$x_1 = x + l\sin\vartheta, \qquad y_1 = y + l\cos\vartheta, \tag{A9}$$

where ϑ is a given function of (x, y) (Eq. A2), what can be done by inverting the relation (A9). Perhaps it is simpler to take x and y as independent variables and x_1 and y_1 depending on them trough Eqs. (A9). Then, lagrangian

$$L_1 = L_1(x_1, y_1, \dot{x}_1, \dot{y}_1), \tag{A10}$$

is transformed into

$$L(x, y, \dot{x}, \dot{y}) = L_1\big(x_1(x, y), y_1(x, y), \dot{x}_1((x, y, \dot{x}, \dot{y})), \dot{y}_1(x, y, \dot{x}, \dot{y})\big). \tag{A11}$$

By the well-known property, the coordinate transformation leaves invariant Lagrange's equations. Thus, variables x and y are well suited for building the equations of motion (lagrangian (A9)) as x_1 and y_1.

On the other side, the function H_0 that does not contain the dependent momentum p_ϑ is, taking into account Eq. (A6),

$$H_0 = \frac{p_x^2 + p_y^2}{2\,m} + V(x, y) + m\,g\,(l\cos\vartheta + y), \tag{A12}$$

and so, the complete hamiltonian is given by (Eq. (3))

$$H = \frac{p_x^2 + p_y^2}{2\,m} + V(x, y) + m\,g\,(l\cos\vartheta + y) + v\,(p_x\,l\cos\vartheta + p_y\,l\sin\vartheta - p_\vartheta). \tag{A13}$$

Thus, specially, the velocities are

$$\dot{x} = \frac{p_x}{m} + lv\cos\theta \quad \dot{y} = \frac{p_y}{m} + lv\sin\theta, \quad \dot{\vartheta} = -v. \tag{A14}$$

The one-dimensional version of the system shown in Fig. 1 is a horizontal spring of length l_1 and constant k_1 in series with another of length l_2 and constant k_2, with a mass m attached at its end. The mass has the coordinate x_2 relative to the joint of the springs, which has the coordinate x_1 with respect to the fixed end of the first spring. Thus, the lagrangian is given by

$$L = \frac{1}{2}\,m\,(\dot{x}_1 + \dot{x}_2)^2 - V(x_1, x_2), \tag{A15}$$

where the potential energy V is expressed as

$$V(x_1, x_2) = \frac{1}{2}\,k_1\,(x_1 - l_1)^2 + \frac{1}{2}\,k_2\,(x_2 - l_2)^2\,. \tag{A16}$$

(A15) is singular (because the additional condition at the union point), with the relation between the momenta

$$\phi = p_2 - p_1 = 0. \tag{A17}$$

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i. For lagrangian (5) $Q_1 = q_1 + q_2, Q_3 = q_3, Q = q_2$, whereas for lagrangian (7) $Q_1 = q_1\,q_2, Q = q_2$.

ii. As seen in the Appendix, for lagrangian (4) this is really the case, for (5), (6) and (7) we cannot assure that, because we do not know the physical system, they refer to.

iii. Generally, parameters are not observable quantities, so that one prefers parameter independent theories. However, there are interesting procedures in mechanics, which contain unobservable variables, for example, Lagrange's treatment of a constrained system through *Lagrange multipliers*, that are not observable, the mechanics of Hertz, the Kaluza-Klein theory; even Dirac's method introduces variables v (Eq. (3)) that are not observable.

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