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Research Article

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On singular quasilinear elliptic equations with data measures

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Abstract: The aim of this work is to study a quasilinear elliptic equation with singular nonlinearity and data measure. Existence and non-existence results are obtained under necessary or sufficient conditions on the data, where the main ingredient is the isoperimetric inequality. Finally, uniqueness results for weak solutions are given.

Keywords: weak solution, quasilinear equation, data measures, isoperimetric inequality, singular nonlinearity

MSC: 35J60, 35J62, 35J75, 35R06

1 Introduction

In this work, we restrict our attention to the study of a class of quasilinear elliptic problem with a singular nonlinearity and data measure namely

$$(P_\lambda)\begin{cases} -\Delta u = \dfrac{a(x)}{u^\gamma} + b(x)|\nabla u|^p + \lambda f & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ for $N \geq 2$, with smooth boundary $\partial\Omega$ and $f \in M_b^+(\Omega)$ is a given finite nonnegative Radon measure. We assume that a and b are nonnegative functions, $\gamma > 0$, $\lambda > 0$, $p \geq 1$ and $|.|$ designates the euclidean norm in $\mathbb{R}^N$. We stress that the problem is singular as one asks to the solution to be zero on the boundary.

The study of nonlinear elliptic problems with singular nonlinearities is motivated by its various applications in many fields. For example, we can mention fluid mechanics, newtonian fluids, and glaciology [14]. They are also applicable to model problems arising from boundary layer phenomena for viscous fluids and chemical heterogeneous catalysts. Furthermore, they can be regarded as mathematical models of electrostatic MEMS devices or Micro-Electro Mechanical systems [20].

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In order to trace the objectives of our work, we will start by recalling some previous studies where three types of problems were treated: quasilinear equations with regular data, semilinear problems with singular nonlinearities and coupling of both problems in the regular case.
_ Case where f is regular:

- Case where $b \equiv 0$, the problem is simply written in the form

$$\begin{cases} -\Delta u = \dfrac{a(x)}{u^\gamma} + \lambda f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The homogeneous case (i.e. $\lambda = 0$) was considered in the pioneer works of [15, 23] and references therein. The authors showed using the method of sub- and supersolutions, that if $a(x)$ is a bounded smooth function, then (1.1) has a classical solution.
The case where $a(x)$ is only a function in $L^1(\Omega)$ was treated in [13] where the authors obtained some existence and regularity results for Problem (1.1) depending on the value of γ. In fact, they showed that if $\gamma \le 1$, Problem (1.1) has a weak solution $u \in H_0^1(\Omega)$. Otherwise if $\gamma > 1$, there exists a solution $u \in H_{loc}^1(\Omega)$ such that $u^{\frac{\gamma+1}{2}} \in H_0^1(\Omega)$.
The nonhomogeneous case (i.e. $\lambda > 0$) has also been treated in [17], where the authors proved the existence of bounded solutions to (1.1) in the case where a and f belong to $L^q(\Omega)$ for $q > \frac{N}{2}$.

- Case where $a \equiv 0$, the problem writes

$$\begin{cases} -\Delta u = b(x)|\nabla u|^p + \lambda f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

This problem was considered in [8] in the case where $1 < p \le 2$, $b \in L^\infty(\Omega)$ and f is regular enough. The authors showed that if (1.2) has a subsolution $\underline{u}$ and a supersolution $\overline{u}$ in $W^{2,q}$ $(q > N)$ with $\underline{u} \le \overline{u}$ in Ω, then there exists a solution u to (1.2) such that $\underline{u} \le u \le \overline{u}$.
This problem was also studied in [24], where the authors showed that if $f \in W^{1,\infty}(\Omega)$ and (1.2) has a nonnegative supersolution in $W^{2,q}(\Omega)$ for $(q > N)$, then it has a solution no matter the value of p $(1 \le p < \infty)$. An important step in resolving such problems is to obtain an estimate on the norm of the gradient of the solution in $L^\infty(\Omega)$. The method used to get this estimate was originally introduced by Bernstein and later developed and systematized in [21, 22, 30, 31].

_ Case where f is only integrable or a Radon measure:

- Case where $b \equiv 0$ was treated in [26], where two different cases $\gamma \le 1$ and $\gamma > 1$ were distinguished. For $\gamma \le 1$, using an approximation argument, the authors obtained the existence of a weak solution $u \in W_0^{1,q}(\Omega)$ of (1.1) for $1 \le q < \frac{N}{N-1}$. For $\gamma > 1$, existence and uniqueness of the solution were obtained only in $W_{loc}^{1,q}(\Omega)$ for every $1 \le q < \frac{N}{N-1}$ such that $T_k(u)^{\frac{\gamma+1}{2}} \in H_0^1(\Omega)$ (where $T_k(u)$ represents the truncated function of u). The use of the truncations of u was necessary since the presence of the measure f does not allow to conclude that $u^{\frac{\gamma+1}{2}}$ itself belongs to $H_0^1(\Omega)$.

- Case where $a \equiv 0$ was studied in [6]. Since f is a nonnegative integrable function or, more generally a given finite nonnegative measure on Ω, it is not regular enough. Hence the usual techniques that lead to the $W^{1,\infty}$-solutions can not be applied. This difficulty was the main motivation behind the work [6], where the authors distinguished three cases such that, for different p values in (1.2), existence and nonexistence results are established. Firstly, considering a linear growth on the gradient, they proved the existence of a solution for (1.2) using the isoperimetric inequality. Secondly, they showed that if

$p > 1$, the existence of a solution is obtained if λ is sufficiently small and the measure f does not charge the sets of $W^{1,p'}$ – capacity zero $\left(\frac{1}{p}+\frac{1}{p'}=1\right)$. Finally, if $p = 2$, assuming the existence of a supersolution in $W_0^{1,2}(\Omega)$, the authors obtained the existence of a solution for Problem (1.2).

In [19], the authors studied the existence of weak solutions for the following generalized elliptic Riccati equation

$$\begin{cases} -\Delta u = |\nabla u|^p + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

on a bounded domain $\Omega \in \mathbb{R}^N$ for $N \geq 3$ with smooth boundary $\partial\Omega$, where $p \geq 1$ and μ is a nonnegative function or a finite positive Borel measure $\mu \in M_+(\Omega)$. By involving geometric capacity estimates or pointwise behavior of Riesz potentials, together with sharp estimates of solutions and their gradients; the authors established some necessary and sufficient conditions for the existence of global solutions to (1.3).

_ Case where $\lambda \equiv 0$, $b(x) \equiv 1$ and $1 < p \leq 2$ was treated in [1], in which the model problem was given by

$$\begin{cases} -\Delta u = \dfrac{a}{u^\gamma} + |\nabla u|^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The authors proved that if $p = 2$, (1.4) admits a distributional solution for all $a \in L^1(\Omega)$. The case where $1 < p < 2$ was treated differently depending on the function a. Indeed, if $a(x) \in L^\infty(\Omega)$, then the existence was obtained for every $\gamma > 0$. However, for the general case $a(x) \in L^1(\Omega)$, the existence of a solution to (1.4) was proved under the condition $\gamma > \gamma_0$, where the exact value of the constant γ_0 was given.

We conclude this section by recalling some works on the parabolic version of our problem. Recently, the authors in [12] considered the following singular nonlinear parabolic equation

$$\begin{cases} u_t - \Delta u = \dfrac{a}{u^\gamma} + \mu u^r & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = f & \text{in } \Omega, \\ u > 0 & \text{in } \Omega \times (0,T), \end{cases} \tag{1.5}$$

where $\gamma > 0$, $\mu \geq 0$, $r > 0$ and $f \in M_b^+(\Omega)$.
If $r > 1$, the existence of a solution to (1.5) was established for suitable small data a and f. Otherwise, if $0 < r < 1$, there exists a solution for every data.

Closely related to Problem (1.5) is the following one given by (1.6), which has been considered for the p-laplacian operator in [27]

$$\begin{cases} u_t - \Delta_p u = \dfrac{a}{u^\gamma} + f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.6}$$

where $p > 2 - \dfrac{1}{N+1}$ and $\gamma > 0$.
The authors in [27] proved that if $a \in L^1(\Omega \times (0,T))$, $u_0 \in L^1(\Omega)$ and $f \in M_b^+(\Omega \times (0,T))$, then there exists a nonnegative distributional solution $u \in L^1(0,T; W_{loc}^{1,1}(\Omega))$.

Other results concerning the well-posedness of the following triply nonlinear degenerate elliptic parabolic equation were obtained in [10],

$$b(u)_t - div(A(u, \nabla\phi(u)) + \psi(u) = f, \quad u|_{t=0} = u_0. \tag{1.7}$$

Existence, uniqueness and continuous dependence on data u_0 and f when $[b+\psi](\mathbb{R}) = \mathbb{R}$ and $\phi \circ [b+\psi]^{-1}$ is continuous, were established.

The main new aspect of this paper is the fact that λ and the functions a and b are not identically zero. Our aim in this work is to prove the existence of a suitable weak solution to (P_λ). Here, as well as in the proof of other similar results, the first step is to precise in which sense we want to solve our problem. On one hand, a solution to (P_λ) has to be understood in the weak distributional meaning. On the other hand, we have to take into account the singular nonlinearity at zero. For this purpose, we adopt the following definitions:

Definition 1.1. Let $u \in W^{1,1}_{loc}(\Omega)$. We say that $u \le 0$ on $\partial\Omega$ if $(u-\epsilon)^+ \in W^{1,1}_0(\Omega)$ for every $\epsilon > 0$. Furthermore, $u = 0$ on $\partial\Omega$ if u is nonnegative in Ω and $u \le 0$ on $\partial\Omega$.

Definition 1.2. If $\gamma > 0$, then a weak solution to Problem (P_λ) is a function

$$\begin{cases} u \in W^{1,1}_{loc}(\Omega) \text{ and } u = 0 \text{ in } \partial\Omega \text{ in the sense of Definition 1.1,} \\ \forall \omega \subset\subset \Omega, \exists c_\omega,\ u \ge c_\omega > 0 \text{ in } \omega, \\ \int_\Omega \nabla u \nabla \varphi = \int_\Omega \frac{a(x)}{u^\gamma}\varphi + \int_\Omega b(x)|\nabla u|^p \varphi + \lambda \int_\Omega f\varphi,\ \forall \varphi \in C^1_c(\Omega). \end{cases} \tag{1.8}$$

The rest of our paper is organized as follows. Section 2 is devoted to necessary conditions on the data to get existence of weak solutions in (P_λ). In section 3, we investigate the existence of a solution for Problem (P_λ), when $p = 1$. Three different cases will be treated separately depending on the value of γ : the non-singular sublinear problem for any $\gamma > 0$, the singular sublinear problem for $\gamma < 1$ and the strongly singular problem for $\gamma \ge 1$. Finally, in section 4, we show the uniqueness of a solution of (P_λ) when it exists, for every $1 \le p < \frac{N}{N+1}$ and $\gamma > 0$.

Now, in what follows, we give necessary conditions for existence. For this purpose, we prove that for sufficiently large value of λ, the equation (P_λ) has no weak solution.

2 Necessary conditions for existence

2.1 Size condition

Theorem 2.1. Let $p > 1, \gamma > 0$ and $\lambda > 0$. We suppose that $a \in L^1(\Omega)^+$ and there exists a ball B_0 in Ω such that, $b(x) \ge C_0 > 0$ a.e $x \in B_0$ and $\int_{B_0} f > 0$. Then there exists $0 < \lambda^* < \infty$ such that (P_λ) does not have any solution for $\lambda > \lambda^*$.

Furthermore, when (P_λ) has a solution, then

$$\forall \varphi \in C^\infty_0(B_0) \qquad \lambda \int_{B_0} \varphi f \le C_p \int_{B_0} \frac{|\nabla\varphi|^{p'}}{\varphi^{p'-1}}, \tag{2.1}$$

where $C_p = \dfrac{p-1}{p^{\frac{p}{p-1}} C_0^{\frac{1}{p-1}}}$.

Proof. See Theorem 2.1 [Alaa-Pierre, [Theorem 2.1, [6]]] for a similar detailed proof. □

Remark 2.2. The condition (2.1) is at the same time a size and regularity condition on f. It is similar to the results obtained for quasilinear elliptic equations and multidimensional Riccati equations. In other words,
- a regularity condition is required on f as soon as $p > 1$;
- moreover, a size condition is also required if $p > 2$.

For various discussions on the meaning of (2.1) and its relationship with nonlinear capacities, we refer the reader to [6] and [19].

Proposition 2.3. [Alaa-Pierre, [Proposition 2.2, [6]]]
Under the hypothesis of Theorem 2.1, if Problem (P_λ) has a solution for some $\lambda > 0$, then the measure f does not charge the sets of $W^{1,p'}$-capacity zero.

Remark 2.4. We recall that a compact set $K \subset \Omega$ is of $W^{1,p'}$-capacity zero if there exists a sequence of C_0^∞-functions φ_n greater than 1 on K and converging to 0 in $W^{1,p'}$. The above statement in Proposition 2.3 implies that

$$\Big(K\ compact,\ W^{1,p'} - capacity(K) = 0\Big) \Longrightarrow \int_K f = 0.$$

Obviously, this is not true for any measure f as soon as $N > p'$ or $p > \frac{N}{N-1}$. See [11] for more details.

Proof. of Proposition 2.3
See [6] for a similar detailed proof. □

In the following section, we restrict our attention to the existence of a weak solution to (P_λ) for $p = 1$. To this aim, we proceed by an approximation argument. The main step is to get a priori estimates on the approximate solution sequences, for any value of $\gamma > 0$.

3 Existence Results for any finite nonnegative Radon measure

In this section, we present existence results of which the proofs are based on the isoperimetric inequality [25], for linear growth on the gradient ($p = 1$), and for any finite measure $f \in M_b^+(\Omega)$. Three different problems will be treated separately in each subsection: the non-singular sublinear problem for any $\gamma > 0$, the singular sublinear problem for $\gamma < 1$ and the strongly singular problem for $\gamma \geq 1$.

3.1 Existence of solutions to the non-singular sublinear problem

Let us consider the following regularized problem in which we regularize the singular term $\dfrac{a(x)}{u^\gamma}$ by $\dfrac{a(x)}{(u+\varepsilon)^\gamma}$ where $\varepsilon > 0$, to become not singular at the origin. The problem then rewrites

$$(P_\varepsilon)\begin{cases} -\Delta u = \dfrac{a(x)}{(u+\varepsilon)^\gamma} + b(x)|\nabla u| + \lambda f & in\ \Omega, \\ u = 0 & on\ \partial\Omega. \end{cases} \tag{3.1}$$

Theorem 3.1. Let $a \in L^1(\Omega)^+$ and $b \in L^{N+\eta}(\Omega)^+$. Then, for all $\gamma > 0$, $\lambda > 0$ and for all $f \in M_b^+(\Omega)$, Problem (P_ε) has a nonnegative weak solution u in $W_0^{1,q}(\Omega)$ for $1 \leq q < \dfrac{N}{N-1}$.

The main tool in the proof of this theorem is the isoperimetric inequality that we will use under the following form [25].

Lemma 3.2. Let $u \in W_0^{1,1}(\Omega)$. Then

$$\frac{-d}{dt}\int_{[u>t]} |\nabla u| \geq N\omega_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}, \tag{3.2}$$

where ω_N is the Lebesgue measure of the unit ball of $\mathbb{R}^N$, and

$$\mu(t) = meas\{x \in \Omega : |u(x)| > t\}. \tag{3.3}$$

Proof. of Theorem 3.1

Step 1. Existence for the approximating problem. Let us approximate our Problem (P_ε). For this purpose, we define the truncated function T_k as follows

$$T_k(r) = \max(-k, \min(r, k)). \tag{3.4}$$

Now, we truncate the functions a, b and f by considering the three sequences a_n, b_n and f_n which are defined by

$$\text{let } n \in \mathbb{N},\ a_n(x) = \min(a(x), n),\ \ b_n(x) = \min(b(x), n), \tag{3.5}$$

and

$$f_n \in C_0^\infty(\Omega),\ \text{ such that } f_n \geq 0,\ ||f_n||_{L^1(\Omega)} \leq ||f||_{M_b(\Omega)} \text{ and } f_n \to f \text{ in } M_b(\Omega). \tag{3.6}$$

Let us now consider the following approximated problem

$$\begin{cases} u_n \in W_0^{1,\infty}(\Omega), \\ \dfrac{1}{n}u_n - \Delta u_n = \dfrac{a_n(x)}{(u_n+\varepsilon)^\gamma} + \dfrac{b_n(x)|\nabla u_n|}{1+\frac{1}{n}b_n(x)|\nabla u_n|} + \lambda f_n \quad \text{in } \Omega. \end{cases} \tag{3.7}$$

The constant $\overline{M} = \max((2n||a_n||_\infty)^{\frac{1}{\gamma+1}} - \dfrac{1}{n} + 2\lambda n||f_n||_\infty)$ is a supersolution of (3.7) and $\underline{M} = 0$ is a subsolution. Then by applying the classical theory (see p.34 of [7] and the Main theorem of [2]), we obtain the existence of u_n solution of (3.7).

Step 2. Estimates on the approximating solutions. At this level, we will prove the existence of a constant C independent of n such that

$$\int_\Omega |\nabla u_n|^q \leq M, \qquad 1 \leq q < \frac{N}{N-1}. \tag{3.8}$$

First of all, we introduce the following function

$$p_{t,h}(r) = \begin{cases} 0 & \text{if } r \leq t, \\ \dfrac{r-t}{h} & \text{if } t \leq r \leq t+h, \\ 1 & \text{if } r > t+h. \end{cases} \tag{3.9}$$

We multiply (3.7) by $p_{t,h}(u_n)$ and then we integrate on Ω to obtain

$$\int_\Omega \left[\frac{1}{n}u_n - \Delta u_n\right] p_{t,h}(u_n) = \int_\Omega \left[\frac{a_n}{(u_n+\varepsilon)^\gamma} + \frac{b_n|\nabla u_n|}{1+\frac{1}{n}b_n|\nabla u_n|} + \lambda f_n\right] p_{t,h}(u_n). \tag{3.10}$$

We observe that

$$\frac{b_n|\nabla u_n|}{1+\frac{1}{n}b_n|\nabla u_n|} \leq b|\nabla u_n|, \tag{3.11}$$

thus,

$$\begin{aligned} \frac{1}{h}\int_{[t\leq u_n \leq t+h]} |\nabla u_n|^2 \leq & \int_{[t\leq u_n\leq t+h]} \frac{a_n}{(u_n+\varepsilon)^\gamma}\frac{u_n-t}{h} + \int_{[u_n\geq t+h]} \frac{a_n}{(u_n+\varepsilon)^\gamma} \\ & + \int_{[u_n\geq t]} b|\nabla u_n| + \lambda||f_n||_{L^1(\Omega)}. \end{aligned} \tag{3.12}$$

Since $0 \le \frac{u_n - t}{h} \le 1$ on the set $[t \le u_n \le t+h]$ and $||f_n||_{L^1(\Omega)} \le ||f||_{M_b(\Omega)}$, we obtain

$$\frac{1}{h}\int_{[t\le u_n\le t+h]} |\nabla u_n|^2 \le \int_{[u_n\ge t]} \frac{a_n}{(u_n+\varepsilon)^\gamma} + ||b||_{L^{N+\eta}(\Omega)}\left(\int_{[u_n\ge t]} |\nabla u_n|^q\right)^{\frac{1}{q}} + \lambda||f||_{M_b(\Omega)}, \tag{3.13}$$

where $q = (N+\eta)' = \frac{N}{N-1} - \varepsilon(\eta)$, $\varepsilon(\eta) > 0$.
Hence

$$\frac{1}{h}\int_{[t\le u_n\le t+h]} |\nabla u_n|^2 \le \int_{[u_n\ge t]} \frac{a_n}{(t+\varepsilon)^\gamma} + ||b||_{L^{N+\eta}(\Omega)}\left(\int_{[u_n\ge t]} |\nabla u_n|^q\right)^{\frac{1}{q}} + \lambda||f||_{M_b(\Omega)}. \tag{3.14}$$

Finally, we get

$$\frac{1}{h}\int_{[t\le u_n\le t+h]} |\nabla u_n|^2 \le \frac{C_1}{\varepsilon^\gamma} + C_q\left(\int_{[u_n\ge t]} |\nabla u_n|^q\right)^{\frac{1}{q}} + C_\lambda, \tag{3.15}$$

where $C_1 = ||a||_{L^1(\Omega)}$, $C_q = ||b||_{L^{N+\eta}(\Omega)}$ and $C_\lambda = \lambda||f||_{M_b(\Omega)}$. Now, we assume that $N \ge 2$ so that $q < 2$, and we use the following two inequalities

$$\frac{1}{h}\int_{[t\le u_n\le t+h]} |\nabla u_n|^q \le \left(\frac{1}{h}\int_{[t\le u_n\le t+h]} |\nabla u_n|^2\right)^{\frac{q}{2}} \left(\frac{\mu(t)-\mu(t+h)}{h}\right)^{\frac{2-q}{2}} \tag{3.16}$$

and

$$\frac{1}{h}\int_{[t\le u_n\le t+h]} |\nabla u_n| \le \left(\frac{1}{h}\int_{[t\le u_n\le t+h]} |\nabla u_n|^q\right)^{\frac{1}{q}} \left(\frac{\mu(t)-\mu(t+h)}{h}\right)^{\frac{q-1}{q}}. \tag{3.17}$$

Next, we take the q^{th} power of (3.17) and we multiply it by the square of (3.16) to find

$$\left(\frac{1}{h}\int_{[t\le u_n\le t+h]} |\nabla u_n|\right)^q \left(\frac{1}{h}\int_{[t\le u_n\le t+h]} |\nabla u_n|^q\right) \le \left(\frac{1}{h}\int_{[t\le u_n\le t+h]} |\nabla u_n|^2\right)^q \left(\frac{\mu(t)-\mu(t+h)}{h}\right). \tag{3.18}$$

Now, we plug the inequality (3.15) into the previous inequality, and we let h tend to zero, to obtain a differential inequality satisfied by $\sigma_n(t) = \int_{[u_n\ge t]} |\nabla u_n|^q$ and defined in the following sense

$$\left(-\frac{d}{dt}\int_{[u_n\ge t]} |\nabla u_n|\right)^q \left(-\sigma_n'(t)\right) \le \left(\frac{C_1}{\varepsilon^\gamma} + C_q\left(\sigma_n(t)\right)^{\frac{1}{q}} + C_\lambda\right)^q. \tag{3.19}$$

On the other hand, according to the isoperimetric inequality (3.2), we get

$$N^q\omega_n^{\frac{q}{N}}\mu_n(t)^{q(1-\frac{1}{N})}(-\sigma_n'(t)) \le \left(\frac{C_1}{\varepsilon^\gamma} + C_q\left(\sigma_n(t)\right)^{\frac{1}{q}} + C_\lambda\right)^q \left(-\mu_n'(t)\right). \tag{3.20}$$

Using Young's inequality on the right hand side term leads to

$$-\sigma_n'(t) \le N^{-q}\omega_n^{\frac{-q}{N}}\left(\frac{D_1}{\varepsilon^{\gamma q}} + D_q\,\sigma_n(t) + D_\lambda\right)\mu_n(t)^{q(\frac{1}{N}-1)}(-\mu_n'(t))), \tag{3.21}$$

where $D_1 = {C_1}^q$, $D_q = {C_q}^q$ and $D_\lambda = {C_\lambda}^q$. This implies that

$$-\sigma_n'(t) \le \left(\frac{\hat{D_1}}{\varepsilon^{\gamma q}} + \hat{D_q}\sigma_n(t) + \hat{D_\lambda}\right)\mu_n(t)^{q(\frac{1}{N}-1)}(-\mu_n'(t))), \tag{3.22}$$

where $\hat{D_1} = N^{-q}\omega_n^{\frac{-q}{N}} D_1$, $\hat{D_q} = N^{-q}\omega_n^{\frac{-q}{N}} D_q$ and $\hat{D_\lambda} = N^{-q}\omega_n^{\frac{-q}{N}} D_\lambda$.
This can be rewritten as

$$-\frac{d}{dt}\left(e^{-k\mu_n(t)^\alpha}\sigma_n(t)\right) \le \frac{d}{dt}\left(e^{-k\mu_n(t)^\alpha}\right)\frac{1}{\hat{D_q}}\left(\frac{\hat{D_1}}{\varepsilon^{\gamma q}} + \hat{D_\lambda}\right), \tag{3.23}$$

where $\alpha = 1 - q\,\frac{N-1}{N}$ and $k\alpha = \hat{D_q}$.
Then by integrating from $t = 0$ to $t = ||u_n||_\infty$, and knowing $\sigma_n(||u_n||_\infty) = 0$ and $\mu_n(||u_n||_\infty) = 0$, we obtain

$$e^{-k\mu_n(0)^\alpha}\sigma_n(0) \le \frac{1}{\hat{D_q}}\left(\frac{\hat{D_1}}{\varepsilon^{\gamma q}} + \hat{D_\lambda}\right). \tag{3.24}$$

Since $\mu_n(0) \le |\Omega|$, we get

$$\int_\Omega |\nabla u_n|^q \le C, \tag{3.25}$$

where $C = e^{k|\Omega|^\alpha}\frac{1}{\hat{D_q}}\left(\frac{\hat{D_1}}{\varepsilon^{\gamma q}} + \hat{D_\lambda}\right)$.

Step 3. Passage to the limit. We have

$$\left|\left|\frac{a_n}{(u_n+\varepsilon)^\gamma}\right|\right|_{L^1(\Omega)} \le \frac{||a||_{L^1(\Omega)}}{\varepsilon^\gamma}, \tag{3.26}$$

and then from (3.7) and (3.25), we deduce

$$||\Delta u_n||_{L^1(\Omega)} \le C \quad and \quad ||u_n||_{W_0^{1,q}(\Omega)} \le C. \tag{3.27}$$

This yields to the compactness of u_n in $W_0^{1,q}(\Omega)$ for $1 \le q < \dfrac{N}{N-1}$. Then there exists a function u such that (up to not relabeled sub-sequences), the sequence u_n converges to u strongly in $W_0^{1,q}(\Omega)$, and $(u_n, \nabla u_n)$ converges to $(u, \nabla u)$ a.e in Ω.
Moreover, by compact embedding, we obtain that u_n converges strongly to u in $L^1(\Omega)$.
Thus, taking φ in $C_c^1(\Omega)$, we have that

$$\left|\frac{a_n\varphi}{(u_n+\varepsilon)^\gamma}\right| \le \frac{||\varphi||_{L^\infty(\Omega)}}{\varepsilon^\gamma}a. \tag{3.28}$$

Applying Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_\Omega \frac{a_n\varphi}{(u_n+\varepsilon)^\gamma} = \int_\Omega \frac{a\,\varphi}{u^\gamma}. \tag{3.29}$$

Finally, since $b \in L^{N+\eta}(\Omega)$, then $b|\nabla u_n|$ converges strongly to $b|\nabla u|$ in $L^1(\Omega)$. This concludes the proof since it is straightforward to pass to the limit in the last term containing f_n. □

3.2 Existence of solutions to the singular sublinear problem and for every nonnegative Radon measure

Theorem 3.3. Let $0 < \gamma < 1$, $a \in L^\infty(\Omega)^+$ and $b \in L^{N+\eta}(\Omega)^+$. Then for all $\lambda > 0$ and all $f \in M_b^+(\Omega)$, Problem (P_λ) has a solution u in $W_0^{1,q}(\Omega)$ for every $1 \le q < \dfrac{N}{N-1}$.

The proof of Theorem 3.3 strictly follows the main steps of the previous proof of Theorem 3.1. We will then sketch it by enlightening the main differences. Estimates will mainly be based on the isoperimetric inequality, and so they will be formally very similar to the previous proof. The main challenge in this case will be to control the singular term $\frac{1}{{u_n}^{\gamma}}$, for which we will show that u_n is bounded from below on the compact subsets of Ω.

Proof. **Step 1. Existence for the approximating problem.**
Let us now consider the following approximated problem

$$\begin{cases} u_n \in W_0^{1,q}(\Omega), \ 1 \le q < \dfrac{N}{N-1}, \\ -\Delta u_n = \dfrac{a(x)}{\left(u_n + \dfrac{1}{n}\right)^{\gamma}} + b(x)|\nabla u_n| + \lambda f \quad in\ \Omega. \end{cases} \tag{3.30}$$

The existence of a solution for (3.30) is ensured by Theorem 3.1 by letting $\varepsilon = \frac{1}{n}$ in Problem (P_ε).

Step 2. Local uniform bound from below. Here, we show that u_n is bounded from below on the compact subsets of Ω. In particular, we check that the sequence u_n is such that for every $\omega \subset\subset \Omega$, there exists a constant $c_\omega > 0$ such that

$$u_n(x) \ge c_\omega \text{ in } \omega, \text{ for every } n \in \mathbb{N}. \tag{3.31}$$

In fact, we have

$$-\Delta u_n \ge \lambda f. \tag{3.32}$$

Hence, using the uniform Hopf principle as formulated in [3] and [16], there exists a constant C only depending on Ω such that

$$Gf \ge C(\Omega) \left(\int_\Omega f\phi_1 \right) \phi_1, \tag{3.33}$$

where ϕ_1 denotes the first eigenfunction of $-\Delta$ with Dirichlet homogeneous boundary conditions, and G denotes the inverse in $L^1(\Omega)$ of the operator $-\Delta$ under homogeneous Dirichlet conditions.
Therefore we have

$$u_n \ge \lambda\, Gf \ge \lambda\, C(\Omega) \left(\int_\Omega f\phi_1 \right) \phi_1. \tag{3.34}$$

Thus, for all compact subset ω of Ω, there exists a constant c_ω (not depending on n) such that $u_n \ge c_\omega$, in which c_ω can be taken as $c_\omega = \lambda \tilde{C}(\Omega) \min\{\phi_1(x), x \in \omega\}$, where $\tilde{C}(\Omega) = C(\Omega) \left(\int_\Omega f\phi_1 \right)$.

Step 3. Estimates on the approximating solutions. Let us take $\varphi = p_{t,h}(u_n)$ as a test function in the weak formulation (3.30), where $p_{t,h}$ is given by (3.9). Applying (3.34), we have $\frac{1}{{u_n}^{\gamma}} \le \frac{C}{{\phi_1}^{\gamma}}$, in which $C = \left(\lambda\, \tilde{C}(\Omega)\right)^{\gamma}$. Using the fact that $u_n + \frac{1}{n} \ge u_n$, we obtain

$$\int_\Omega \nabla u_n \nabla p_{t,h}(u_n) \le C \int_\Omega \frac{a(x) p_{t,h}(u_n)}{{\phi_1}^{\gamma}} + \int_\Omega b(x)|\nabla u_n| p_{t,h}(u_n) + \lambda \int_\Omega f p_{t,h}(u_n),$$

which implies that

$$\begin{aligned} \frac{1}{h} \int_{[t \le u_n \le t+h]} |\nabla u_n|^2 \le C \int_{[t \le u_n \le t+h]} \frac{a(x)}{{\phi_1}^{\gamma}} \frac{(u_n - t)}{h} + C \int_{[u_n \ge t+h]} \frac{a(x)}{{\phi_1}^{\gamma}} \\ + \int_{[u_n \ge t]} b(x)|\nabla u_n| + \lambda ||f||_{M_b(\Omega)}. \end{aligned}$$

Since $0 \le \frac{u_n - t}{h} \le 1$ on the set $[t \le u_n \le t+h]$, we obtain

$$\frac{1}{h}\int_{[t\le u_n\le t+h]} |\nabla u_n|^2 \le C\int_\Omega \frac{a(x)}{{\phi_1}^\gamma} + \int_{[u_n\ge t]} b(x)|\nabla u_n| + \lambda||f||_{M_b(\Omega)}.$$

Therefore

$$\frac{1}{h}\int_{[t\le u_n\le t+h]} |\nabla u_n|^2 \le C\int_\Omega \frac{a(x)}{{\phi_1}^\gamma} + ||b||_{L^{N+\eta}(\Omega)}\left(\int_{[u_n\ge t]} |\nabla u_n|^q\right)^{\frac{1}{q}} + \lambda||f||_{M_b(\Omega)},$$

in which $q = (N+\eta)' = \frac{N}{N-1} - \varepsilon(\eta)$, $\varepsilon(\eta) > 0$.
Thus

$$\frac{1}{h}\int_{[t\le u_n\le t+h]} |\nabla u_n|^2 \le C_1\int_\Omega \frac{1}{{\phi_1}^\gamma} + C_q\left(\int_{[u_n\ge t]} |\nabla u_n|^q\right)^{\frac{1}{q}} + C_\lambda,$$

where $C_1 = C||a||_{L^\infty(\Omega)}$, $C_q = ||b||_{L^{N+\eta}(\Omega)}$ and $C_\lambda = \lambda||f||_{M_b(\Omega)}$.
Analogously to the proof of the previous theorem, we finally get

$$\int_\Omega |\nabla u_n|^q \le C. \tag{3.35}$$

Step 4. Passage to the limit Similarly to the passage to the limit in (3.7), we may assume that u_n converges strongly to u in $L^1(\Omega)$ and a.e. in Ω.
Thus, taking φ in $C_c^1(\Omega)$, we have that

$$\left|\frac{a_n\varphi}{\left(u_n + \frac{1}{n}\right)^\gamma}\right| \le \frac{||\varphi||_{L^\infty(\Omega)}}{{\phi_1}^\gamma} a. \tag{3.36}$$

Finally, applying Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_\Omega \frac{a_n\varphi}{\left(u_n + \frac{1}{n}\right)^\gamma} = \int_\Omega \frac{a\,\varphi}{u^\gamma}. \tag{3.37}$$

By a straightforward re-adaptation of the previous theorem, u is a solution to (P_λ). □

3.3 The strongly singular case: $\gamma \ge 1$

In this case, only local estimates on the approximated solution u_n can be obtained. Our aim here is mainly to give global estimates on $T_k^{\frac{\gamma+1}{2}}(u)$ in $H_0^1(\Omega)$ in order to provide at least a weak sense to u on the boundary of Ω.

Theorem 3.4. Let $\gamma \ge 1$, $a \in L^1(\Omega)^+$ and $b \in L^{N+\eta}(\Omega)^+$. Then for all $f \in M_b^+(\Omega)$ and $\lambda > 0$, (P_λ) has a solution u in $W_{loc}^{1,q}(\Omega)$ for every $1 \le q < \dfrac{N}{N-1}$. Furthermore, $T_k(u)^{\frac{\gamma+1}{2}} \in H_0^1(\Omega)$.

Proof. Analogously to **Step 1** and **Step 2** in the proof of Theorem 3.3, we obtain the existence of a solution u_n for the approximated Problem (3.30) for $\gamma \ge 1$, such that for all $\omega \subset\subset \Omega$, there exists a constant $c_\omega > 0$ such that

$$u_n(x) \ge c_\omega \text{ in } \omega. \tag{3.38}$$

In what follows, we show that $T_k(u)^{\frac{\gamma+1}{2}} \in H_0^1(\Omega)$.
To this aim, let H be a function in $C^1(\mathbb{R})$ defined by

$$H(s) = \begin{cases} 0 \text{ } if \text{ } |s| \ge 1, \\ s \text{ } if \text{ } |s| \le \frac{1}{2}. \end{cases} \tag{3.39}$$

We introduce the test function

$$\phi = H\Big(\frac{u_n}{k}\Big)^{\gamma} e^{\beta u_n}. \tag{3.40}$$

Next, we multiply (3.30) (with $\gamma \geq 1$) by ϕ and we integrate on Ω to obtain

$$\begin{aligned} &\frac{\gamma}{k}\int_\Omega |\nabla u_n|^2 H'\left(\frac{u_n}{k}\right) H\left(\frac{u_n}{k}\right)^{\gamma-1} e^{\beta u_n} + \beta\int_\Omega |\nabla u_n|^2 H\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n} \\ &= \int_\Omega \frac{a_n}{(u_n+\frac{1}{n})^\gamma} H\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n} + \int_\Omega b_n|\nabla u_n| H\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n} \\ &+ \int_\Omega \lambda f H\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n}. \end{aligned} \tag{3.41}$$

Hence

$$\begin{aligned} &c(k)\int_\Omega \left|\nabla H\left(\frac{u_n}{k}\right)^{\frac{\gamma+1}{2}}\right|^2 + \beta\int_\Omega |\nabla u_n|^2 H\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n} \\ &\leq \int_{[u_n\leq k]} \frac{a_n}{(u_n+\frac{1}{n})^\gamma} H\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n} + \int_{[u_n>k]} \frac{a_n}{(u_n+\frac{1}{n})^\gamma} H\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n} \\ &+ \int_\Omega b_n|\nabla u_n| H\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n} + \int_\Omega \lambda f H\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n}. \end{aligned} \tag{3.42}$$

By using the definition of H given by (3.39), the second term of the right-hand side of the inequality (3.42) vanishes and for the first term, we have

$$\int_{[u_n\leq k]} \frac{a_n}{(u_n+\frac{1}{n})^\gamma} H\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n} \leq e^{\beta k}\int_\Omega \frac{a_n}{(u_n+\frac{1}{n})^\gamma}\left(\frac{u_n}{k}\right)^{\gamma}. \tag{3.43}$$

Since $\dfrac{u_n^\gamma}{(u_n+\frac{1}{n})^\gamma} \leq 1$, then

$$\int_{[u_n\leq k]} \frac{a_n}{(u_n+\frac{1}{n})^\gamma} H\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n} \leq C_k ||a||_{L^1(\Omega)}. \tag{3.44}$$

Using Young's inequality in the third term of (3.42), we obtain

$$\begin{aligned} &c(k)\int_\Omega \left|\nabla H\left(\frac{u_n}{k}\right)^{\frac{\gamma+1}{2}}\right|^2 + \beta\int_\Omega |\nabla u_n|^2 H\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n} \\ &\leq C_k||a||_{L^1} + \varepsilon\int_\Omega |\nabla u_n|^2 H\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n} + C(\varepsilon,k)\int_\Omega b^2 \\ &+ \int_\Omega \lambda f H\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n}. \end{aligned} \tag{3.45}$$

Concerning the last term in (3.45), we have

$$\lambda\int_\Omega fH\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n} = \lambda\int_{[u_n\leq k]} fH\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n} + \lambda\int_{[u_n>k]} fH\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n}. \tag{3.46}$$

Since $H\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n} \leq \left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n} \leq e^{\beta k}$ on the set $[u_n \leq k]$, we get

$$\lambda\int_\Omega fH\left(\frac{u_n}{k}\right)^{\gamma} e^{\beta u_n} \leq C_k{}'||f||_{M_b}. \tag{3.47}$$

Thus

$$c(k)\int_\Omega \left|\nabla H\left(\frac{u_n}{k}\right)^{\frac{\gamma+1}{2}}\right|^2 + (\beta-\varepsilon)\int_\Omega |\nabla u_n|^2 H\left(\frac{u_n}{k}\right)^\gamma e^{\beta u_n} \le C_k||a||_{L^1} + C(\varepsilon,k)\int_\Omega b^2 + C_k{}'||f||_{M_b}. \tag{3.48}$$

Choosing β such that $\beta-\varepsilon>0$ leads to

$$c(k)\int_\Omega \left|\nabla H\left(\frac{u_n}{k}\right)^{\frac{\gamma+1}{2}}\right|^2 \le C_k||a||_{L^1} + C(\varepsilon,k)\int_\Omega b^2 + C_k{}'||f||_{M_b}. \tag{3.49}$$

Finally, we get

$$\int_\Omega \left|\nabla H\left(\frac{u_n}{k}\right)^{\frac{\gamma+1}{2}}\right|^2 \le \widehat{C}(k). \tag{3.50}$$

Now, we observe that

$$\int_{[u_n\le\frac{k}{2}]} \left|\nabla H\left(\frac{u_n}{k}\right)^{\frac{\gamma+1}{2}}\right|^2 \le \int_\Omega \left|\nabla H\left(\frac{u_n}{k}\right)^{\frac{\gamma+1}{2}}\right|^2 \le \widehat{C}(k). \tag{3.51}$$

From the definition of H, we obtain that

$$\frac{1}{k^{\gamma+1}}\int_{[u_n\le\frac{k}{2}]} |\nabla u_n^{\frac{\gamma+1}{2}}|^2 \le \widehat{C}(k). \tag{3.52}$$

Consequently

$$\int_\Omega |\nabla T_{\frac{k}{2}}(u_n)^{\frac{\gamma+1}{2}}|^2 \le C(k). \tag{3.53}$$

Even if we replace $\frac{k}{2}$ by k, we obtain the desired result.
Next, we show the boundedness of u_n in $W^{1,q}_{loc}(\Omega)$ into two steps.
For fixed $k>0$, we will make use of the two truncations functions $T_k(r)$ given by (3.4) and $G_k(r)$ defined as

$$G_k(r) = (|r|-k)^+\, sign(r).$$

Step 1: $G_1(u_n)$ is bounded in $W^{1,q}_0(\Omega)$ for all $1\le q<\frac{N}{N-1}$.
In other words, we have to prove that there exists a constant $\bar{C}_k$ depending only on k such that

$$\int_{[u_n\ge 1]} |\nabla u_n|^q \le \bar{C}_k. \tag{3.54}$$

Analogously to the case $\gamma<1$, we take $\phi = p_{t,h}(u_n)$ as a test function in (3.30), and we obtain

$$\int_\Omega -\Delta u_n\, p_{t,h}(u_n) = \int_\Omega \left[\frac{a_n}{(u_n+\frac{1}{n})^\gamma} + b_n|\nabla u_n| + \lambda f_n\right] p_{t,h}(u_n). \tag{3.55}$$

Hence

$$\frac{1}{h}\int_{[t\le u_n\le t+h]} |\nabla u_n|^2 \le \int_{[u_n\ge t]} \frac{a_n}{(u_n+\frac{1}{n})^\gamma} + \int_{[u_n\ge t]} b|\nabla u_n| + \lambda||f||_{M_b(\Omega)}.$$

Therefore

$$\frac{1}{h}\int_{[t\le u_n\le t+h]} |\nabla u_n|^2 \le \int_\Omega \frac{a_n}{(t+\frac{1}{n})^\gamma} + ||b||_{L^{N+\eta}(\Omega)}\left(\int_{[u_n\ge t]} |\nabla u_n|^q\right)^{\frac{1}{q}} + \lambda||f||_{M_b(\Omega)}, \tag{3.56}$$

where $q = (N+\eta)' = \frac{N}{N-1} - \varepsilon(\eta)$, $\varepsilon(\eta) > 0$. Then, we get

$$\frac{1}{h}\int\limits_{[t\le u_n\le t+h]} |\nabla u_n|^2 \le \frac{||a||_{L^1(\Omega)}}{t^\gamma} + ||b||_{L^{N+\eta}(\Omega)}\left(\int\limits_{[u_n\ge t]} |\nabla u_n|^q\right)^{\frac{1}{q}} + \lambda ||f||_{M_b(\Omega)}. \tag{3.57}$$

We thus get the following inequality

$$\frac{1}{h}\int\limits_{[t\le u_n\le t+h]} |\nabla u_n|^2 \quad\le\quad \frac{C_1}{t^\gamma} + C_q\left(\int\limits_{[u_n\ge t]} |\nabla u_n|^q\right)^{\frac{1}{q}} + C_\lambda, \tag{3.58}$$

where $C_1 = ||a||_{L^1(\Omega)}$, $C_q = ||b||_{L^{N+\eta}(\Omega)}$ and $C_\lambda = \lambda||f||_{M_b(\Omega)}$.
Let us now plug the inequality (3.58) into the previous inequality (3.18). By tending h to zero, we obtain a differential inequality satisfied by the function σ_n which is defined in the following sense $\sigma_n(t) = \int\limits_{[u_n\ge t]} |\nabla u_n|^q$,

$$(-\frac{d}{dt}\int\limits_{[u_n\ge t]} |\nabla u_n|)^q(-\sigma_n'(t)) \le \left(\frac{C_1}{t^\gamma} + C_q\sigma_n(t)^{\frac{1}{q}} + C_\lambda\right)^q(-\mu_n'(t)). \tag{3.59}$$

On the other hand, according to the isoperimetric inequality (3.2), we get

$$N^q\omega_n^{\frac{q}{N}}\mu_n(t)^{q(1-\frac{1}{N})}(-\sigma_n'(t)) \le \left(\frac{C_1}{t^\gamma} + C_q\left(\sigma_n(t)\right)^{\frac{1}{q}} + C_\lambda\right)^q\left(-\mu_n'(t)\right). \tag{3.60}$$

Using Young's inequality on the right-hand side term leads to

$$-\sigma_n'(t) \le N^{-q}\omega_n^{\frac{-q}{N}}\left(\frac{D_1}{t^{\gamma q}} + D_q\ \sigma_n(t) + D_\lambda\right)\mu_n(t)^{q(\frac{1}{N}-1)}(-\mu_n'(t))), \tag{3.61}$$

where $D_1 = {C_1}^q$, $D_q = {C_q}^q$ and $D_\lambda = {C_\lambda}^q$. This implies that

$$-\sigma_n'(t) \le \left(\frac{\hat{D_1}}{t^{\gamma q}} + \hat{D_q}\sigma_n(t) + \hat{D_\lambda}\right)\mu_n(t)^{q(\frac{1}{N}-1)}(-\mu_n'(t))), \tag{3.62}$$

where $\hat{D_1} = N^{-q}\omega_n^{\frac{-q}{N}}D_1$, $\hat{D_q} = N^{-q}\omega_n^{\frac{-q}{N}}D_q$ and $\hat{D_\lambda} = N^{-q}\omega_n^{\frac{-q}{N}}D_\lambda$.
This can be rewritten as

$$-\frac{d}{dt}\left(e^{-k\mu_n(t)^\alpha}\sigma_n(t)\right) \le \frac{d}{dt}\left(e^{-k\mu_n(t)^\alpha}\right)\frac{1}{\hat{D_q}}\left(\frac{\hat{D_1}}{t^{\gamma q}} + \hat{D_\lambda}\right), \tag{3.63}$$

where $\alpha = 1 - q\ \frac{N-1}{N}$ and $k\alpha = \hat{D_q}$.

Integrating between 1 and $||u_n||_\infty$, since $\sigma_n(||u_n||_\infty) = 0$ and $\mu_n(||u_n||_\infty) = 0$, we get

$$\int\limits_{[u_n\ge 1]} |\nabla u_n|^q \le \hat{C}_N[e^{k\mu_n(1)^\alpha} - 1]. \tag{3.64}$$

This concludes the statement of **Step 1**.

Step 2: $T_1(u_n)$ is bounded in $H^1_{loc}(\Omega)$.
We have to investigate the behavior of (u_n) for its small values ($u_n \le 1$). To do so, we need to prove that $\forall\omega \subset\subset \Omega$,

$$\int\limits_{\omega} |\nabla T_1(u_n)|^2 \le C'. \tag{3.65}$$

First, we take $T_1^\gamma(u_n)$ as a test function in (3.30), and we get

$$\gamma \int_\omega |\nabla T_1(u_n)|^2 \, T_1^{\gamma-1}(u_n) = \int_\Omega \left[\frac{a_n}{(u_n + \frac{1}{n})^\gamma} + b_n|\nabla u_n| + \lambda f_n \right] T_1^\gamma(u_n) \le C. \tag{3.66}$$

Furthermore, according to (3.38), we have $u_n \ge c_\omega$ on ω, and we observe that

$$\gamma \, {c_\omega}^{\gamma-1} \int_\omega |\nabla T_1(u_n)|^2 \le \gamma \int_\Omega |\nabla T_1(u_n)|^2 \, T_1^{\gamma-1}(u_n) \le C. \tag{3.67}$$

Now since $u_n = T_1(u_n) + G_1(u_n)$, we deduce that u_n is bounded in $W^{1,q}_{loc}(\Omega)$ for every $1 \le q < \dfrac{N}{N-1}$. By (3.30), we obtain

$$||\Delta u_n||_{L^1_{loc}(\Omega)} \le C(\omega),$$

This yields to the compactness of (u_n) in $W^{1,q}_{loc}(\Omega)$ for $1 \le q < \dfrac{N}{N-1}$ by applying the following Lemma:

Lemma 3.5. [Baras-Pierre, [Lemma A.2, [11]]]
Let $u_n \in W^{1,q}_{loc}(\Omega)$, $1 \le q < \dfrac{N}{N-1}$ such that

$$||u_n||_{W^{1,q}_{loc}} \le C \quad and \quad ||\Delta u_n||_{L^1_{loc}(\Omega)} \le C. \tag{3.68}$$

Then we can extract a subsequence of (u_n) still denoted u_n such that

$$\begin{aligned} u_n &\rightarrow u \text{ in } W^{1,q}_{loc}(\Omega),\\ u_n &\rightarrow u \text{ almost everywhere in } \Omega. \end{aligned}$$

Proof. See Lemma A.2 of [11] for a detailed proof. □

Now, for the passage to the limit in (3.30), let $\omega \subset\subset \Omega$ and $\varphi \in C_0^\infty(\Omega)$, such that $supp\ \varphi = \omega$.
Since $u_n \ge c_\omega$ in ω, we have

$$\left| \frac{a_n}{\left(u_n + \frac{1}{n}\right)^\gamma} \, \varphi \right| \le \frac{||\varphi||_{L^\infty(\Omega)}}{{c_\omega}^\gamma} a \in L^1(\Omega). \tag{3.69}$$

Hence, applying Lebesgue's dominated convergence theorem, we deduce that $\dfrac{a_n}{\left(u_n + \frac{1}{n}\right)^\gamma}$ converges to $\dfrac{a}{u^\gamma}$ in $L^1_{loc}(\Omega)$.

Finally, we deduce that u is a solution to (P_λ) by a straightforward re-adaptation of the passage to the limit in the previous theorem. □

4 Uniqueness of weak solutions

Theorem 4.1. Let $a \in L^1(\Omega)^+$, $b \in L^{N+\eta}(\Omega)^+$ and $1 \le p < \dfrac{N}{N-1}$. Then for all $\gamma > 0$, $\lambda > 0$ and $f \in M_b^+(\Omega)$, the solution of (P_λ) is unique if it exists.

In order to prove this result, we start by recalling the following technical lemma

Lemma 4.2. Let us consider $j(r) = |r|^p$. The function j is convex and we have

$$\forall r \in \mathbb{R}^n, \ \exists A \in \partial j(r) \text{ such that } \forall \hat{r} \in \mathbb{R}^n, \ j(r) - j(\hat{r}) \ge \langle A, r - \hat{r} \rangle, \tag{4.1}$$

where $\partial j(r)$ is the sub-differential of $j(r)$ defined as follows:
(i) if $p > 1$, $\partial j(r) := \nabla j(r) = p|r|^{p-2} r$,

(ii) if $p = 1$,

$$\partial j(r) = \begin{cases} \dfrac{r}{|r|} & r \neq 0, \\ \{r \in \mathbb{R}^n;\ |r| \leq 1\} & r = 0. \end{cases} \tag{4.2}$$

Consequently, we deduce that for $u \in W^{1,p}(\Omega)$, there exists $A(x) \in \partial j(\nabla u)$ such that

$$\forall \hat{u} \in W^{1,p}(\Omega),\ |\nabla u|^p - |\nabla \hat{u}|^p \geq < A, \nabla(u - \hat{u}) > . \tag{4.3}$$

Furthermore, for $1 < p < \frac{N}{N-1}$, we deduce from statement (i) in Lemma 4.2 that $A \in \left(L^{p'}(\Omega)\right)^N$, in which the conjugate p' verifies $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ and $p' > N$. Hence $A \in \left(L^{N+\eta}(\Omega)\right)^N$ for $\eta > 0$.
For $p = 1$, we may deduce from (4.2), that $||A(x)|| \leq 1$, for all $x \in \Omega$. Hence $A \in \left(L^\infty(\Omega)\right)^N$. Thus, again we have $A \in \left(L^{N+\eta}(\Omega)\right)^N$.
Finally, the uniqueness result that we obtain is a consequence of the following two lemmas:

Lemma 4.3. [Alaa-Pierre,[Lemma 4.6, [6]]]
Let $\vec{a} \in L^{N+\varepsilon}(\Omega, \mathbb{R}^n)$, $\varepsilon > 0$, $\alpha \geq 0$ and ω a solution of

$$\begin{cases} \omega \in W_0^{1,1}(\Omega), \\ \alpha\omega - \Delta\omega \leq \vec{a}.\nabla\omega \quad in\ \mathcal{D}'(\Omega). \end{cases} \tag{4.4}$$

Then $\omega \leq 0$.

Lemma 4.4. Let $A \in \left(L^{N+\eta}(\Omega)\right)^N$ and $\theta \in W^{1,q}_{loc}(\Omega)$ for $1 \leq q < \dfrac{N}{N-1}$, such that $\theta \geq 0$ in Ω and $\theta = 0$ on $\partial\Omega$ in the sense of Definition 1.1. Furthermore, we assume that θ verifies

$$-\Delta\theta \leq A.\nabla\theta \quad in\ \mathcal{D}'(\Omega). \tag{4.5}$$

Then $\theta = 0$ in Ω.

Proof. of Lemma 4.4
We have $(\theta - \varepsilon)^+ \in W_0^{1,q}(\Omega)$ for all $\varepsilon > 0$, and by mean of Kato's inequality up to the boundary (see [28]), we obtain

$$\begin{aligned} -\Delta(\theta - \varepsilon)^+ &\leq -\Delta(\theta - \varepsilon)\,\chi_{[\theta-\varepsilon>0]} \\ &\leq -\Delta\theta\,\chi_{[\theta-\varepsilon>0]} \\ &\leq A.\nabla\theta\,\chi_{[\theta-\varepsilon>0]} \end{aligned}$$

Hence

$$\begin{cases} (\theta - \varepsilon)^+ \in W_0^{1,1}(\Omega), \\ -\Delta(\theta - \varepsilon)^+ \leq A.\nabla(\theta - \varepsilon)^+ \quad in\ \mathcal{D}'(\Omega). \end{cases} \tag{4.6}$$

Now using the previous Lemma 4.3, we obtain

$$\theta - \varepsilon \leq 0,\ i.e.\ \theta \leq \varepsilon,\ \forall \varepsilon > 0, \tag{4.7}$$

and since $\theta \geq 0$ in Ω, then $\theta = 0$ in Ω. □

Proof. of Theorem 4.1 Let u be a supersolution of (P_λ) and $\hat{u}$ a subsolution, and let $w = u - \hat{u}$.

We take the difference between the equations associated to u and $\hat{u}$ respectively, we obtain

$$-\Delta w = \frac{a(x)}{u^\gamma} - \frac{a(x)}{\hat{u}^\gamma} + b(x)\left(|\nabla u|^p - |\nabla \hat{u}|^p\right). \tag{4.8}$$

By the convexity, there exists $A \in L^{N+\eta}(\Omega, \mathbb{R}^n)$ for $\eta > 0$ such that

$$|\nabla u|^p - |\nabla \hat{u}|^p \geq A.\nabla(u - \hat{u}). \tag{4.9}$$

Hence

$$-\Delta w \leq \frac{a(x)}{u^\gamma} - \frac{a(x)}{\hat{u}^\gamma} + \tilde{A}.\nabla(u - \hat{u}), \tag{4.10}$$

Then by Kato's inequality, we obtain

$$-\Delta w^+ \leq \chi_{[u-\hat{u}>0]}\left[\frac{a(x)}{u^\gamma} - \frac{a(x)}{\hat{u}^\gamma} + \tilde{A}.\nabla(u - \hat{u})\right]. \tag{4.11}$$

which implies that

$$-\Delta w^+ \leq \tilde{A}.\nabla w^+. \tag{4.12}$$

Therefore, thanks to Lemma 4.4, we get $w = 0$, which completes the proof. □

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