On singularities of submanifolds of higher dimensional Euclidean spaces.

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Summary. - *Generalizations of principle axes are found for surfaces in E^4. The singularities generalize umbilics. The generic indicies are computed For these computations the Thom Transversality Theorem as applied by Feldman to geometry is used. Hower we « reduce the group » rendering the calculations more tractible. Also we show that a torus or sphere cannot be immersed in E^4 with everywhere nonzero curvature of the normal bundle.*

Introduction.

The principal aim of this paper is to study the local geometry of submanifolds of higher dimension and codimension. We study the second order invariants and develop a theory of the second fundamental form. Our local constructions lead to global theorems on the existence of singularities, and these are the main results of this paper. The simplest case, surfaces in E^4, admits a very complete treatment so we deal with it separately.

It was observed a long time ago that the second fundamental form is a vector-valued quadratic form. Even for a surface in E^4 such an object is algebraically rather complicated. However, WILSON and MOORE [17] have shown that for a surface in E^n the second fundamental form can be classfied by a configuration consisting of a point and an ellipse lying in the normal space. They show that this configuration determines the second order scalar invariants and leads to a theory of principal axes. These axes generalize the usual principal axes of surfaces in ordinary space. From this theory one obtains global theorems. For example, if the surface has nonzero EULER characteristic then there must be a point where the mean curvature vector vanishes or an inflection point.

In higher dimensions there is also a classifying configuration, although it has apparently escaped attention. It consists of a point and a VERONESE manifold, or the projection of one.

The second order invariants are completely determined by this configuration, though we have not worked them out explicitly. By studying the VERONESE manifold, and here the classical algebraic geometry proved an inspiration, we again obtain a theory of principal axes. This is the content

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of Chapter III. Constructions are prescribed which «in general» give principal axes. Points where this construction fails are then regarded as singularities of the field of axes. This leads to global theorems which state that if a manifold does not admit a field of axes then it must have a singularity of a given type. These theorems are found in Chapter IV. It is interesting to note that several of the constructions made are independent of the codimension, provided only that it is high enough. These singularities exhibit a certain stability, i.e. they remain generic even if the Euclidean space in which the submanifold lies is imbedded in one of higher dimension.

We have also computed the generic dimension of the locus of the several types of singularities. These computations are based on the work of E. FELDMAN [6, 7]; however, in order to render the calculations tractible we were forced to develop a more refined version of his theory. This matter occupies Chapter II. We were thus able to avoid the complications of the jet bundle and work directly with our configuration.

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CHAPTER I.

Surfaces in E^4.

1. - Notation.

Consider an immersion $X: M \to E^4$ of a compact, oriented, 2-dimensional manifold M in E^4. We assume, unless it is stated explicitly to the contrary, that all maps and manifolds are C^∞. $Xe_1e_2e_3e_4$ will denote an orthonormal frame at p, chosen so that e_1 and e_2 are tangent vectors to M at p with the frame e_1e_2 agreeing with the orientation of the tangent space, and chosen so that e_3 and e_4 are normal to the surface at p with the frame $e_1e_2e_3e_4$ agreeing with a fixed chosen orientation of E^4. We shall use the following seven bundles: TM, the tangent bundle of M; NM, the normal bundle of M; $UTM = \{Xe_1\} =$ the unit tangent bundle of M; $UNM = \{Xe_3\} =$ the unit normal bundle of M; $F_\tau = \{Xe_1e_2\} =$ the bundle of tangent frames; $F_\nu = \{Xe_3e_4\} =$ the bundle of normal frames; and $F = \{Xe_1e_2e_3e_4\} =$ the bundle of all frames. UTM and F_τ can be identified as can UNM and F_ν. These identifications will often be made. Note that F_ν and F_τ are circle bundles and that F is a bundle with the torus as fibre.

As usual, define the forms $\omega_i = dX \cdot e_i$ and $\omega_{ij} = de_i \cdot e_j$. The indicies run from 1 to 4. These forms are defined on the frame bundle, F. Since

$e_i \cdot e_j = \delta_{ij}$ we see by differentiation that $\omega_{ij} = -\omega_{ji}$. Also because $dX \cdot e_3 = dX \cdot e_4 = 0$ we have $\omega_3 = \omega_4 = 0$. By use of the POINCARÉ formula $dd = 0$, one obtains the equations of structure of MAURER–CARTAIN:

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j ,$$

$$\omega_i = \sum_k \omega_{ik} \wedge \omega_{kj} ,$$

where d is the exterior derivative. Since $\omega_3 = \omega_4 = 0$ we have

$$0 = d\omega_3 = \omega_{31} \wedge \omega_1 + \omega_{32} \wedge \omega_2 ,$$

$$0 = d\omega_4 = \omega_{41} \wedge \omega_1 + \omega_{42} \wedge \omega_2 .$$

ω_1 and ω_2 are independent forms; in fact, $\omega_1 \wedge \omega_2$ is defined on M and is the area element. Thus it follows by a lemma of CARTAN that

$$\begin{aligned} \omega_{13} &= a\omega_1 + b\omega_2 , \\ \omega_{23} &= b\omega_1 + c\omega_2 , \\ \omega_{14} &= c\omega + f\omega_2 , \\ \omega_{24} &= f\omega_1 + g\omega_2 . \end{aligned} \tag{1}$$

The vector-valued quadratic form

$$(d^2X \cdot e_3)e_3 + (d^2X \cdot e_4)e_4$$

is the second fundamental form of the surface. By use of the above, the second fundamental form may be written

$$(a\omega_1^2 + 2b\omega_1\omega_2 + c\omega_2^2)e_3 + (e\omega_1^2 + 2f\omega_1\omega_2 + g\omega_2^2)e_4 .$$

To see this, note that

$$d^2X \cdot e_3 = -dX \cdot de_3 = -(\omega_1 e_1 + \omega_2 e_2) \cdot de_3 =$$

$$= \omega_1\omega_{31} + \omega_2\omega_{32} = a\omega_1^2 + 2b\omega_1\omega_2 + c\omega_2^2 .$$

Similarly for $d^2X \cdot e_4$.

Let us note that ω_{12} and ω_{34} are the connection forms in the bundles F_τ and F_ν respectively and that $d\omega_{12}$ and $d\omega_{34}$ are the curvature forms in those respective bundles.

The GAUSS curvature is an intrinsic invariant of any Riemannian surface, depending only on the metric and not on the imbedding. The forms ω_1, ω_2 and ω_{12}, which may be regarded as forms on F_τ, depend only on the metric $\omega_1^2 + \omega_2^2$ and the GAUSS curvature may be found by using the formula

$$d\omega_{12} = - K\omega_1 \wedge \omega_2 .$$

The curvature form of F_ν, $d\omega_{34}$, is a well-defined 2-form on the base and as such is a multiple of the area element. The ratio is therefore a scalar invariant. We define the invariant N by the formula

$$d\omega_{34} = - N\omega_1 \wedge \omega_2 .$$

The similarity of K and N are to be noted.

2. - Local Invariants.

The local invariants of surfaces in E^4 have been rather thoroughly studied [5, 9, 17, 18, 19]. It was found, [17], that the invariants are the invariants of a simple configuration; namely, a point and an ellipse in the normal plane. To describe the configuration it is helpful to think of the second fundamental form as giving a map, at each point p, from the tangent circle to the normal space. We call the map η and refer to it as the normal curvature vector.

$$\eta : \quad S^1 \to E^2,$$

where S^1 is the tangent circle at p and E^2 is the normal space at p. η is defined as follows. To each point, e, of the unit tangent sphere at p let $\gamma(s)$ be a curve, parameterized by arc length, through the point p and chosen so that the tangent vector to γ at p is e. Then η is defined by letting $\eta(e)$ be the projection of $d^2\gamma \,|\, ds^2(p)$ on the normal space at p. This definition is independent of the choice of γ because we may write (as in the case of a surface in E^3)

$$\eta(e) = (a \cos^2 \theta + 2b \cos \theta \sin \theta + c \sin^2 \theta) e_3 + \\ + (e \cos^2 \theta + 2f \cos \theta \sin \theta + g \sin^2 \theta) e_4 ,$$

where $e = \cos \theta e_1 + \sin \theta e_2$. ($e_1 e_2$ is a fixed tangent frame).

Recall that the mean curvature vector, which we shall call $\mathcal{H}$, is just

$$\frac{1}{2}(a + c) e_3 + \frac{1}{2}(e + g) e_4 .$$

It is an invariant vector. Using the trigonometric identities for double angles we may write

$$\eta(\theta) = \left(\frac{1}{2}(a - c)\cos 2\theta + b \sin 2\theta\right) e_3 + \\ + \left(\frac{1}{2}(e - g)\cos 2\theta + f \sin 2\theta\right) e_4 + \mathcal{H}.$$

As a matrix this takes the form

$$(\eta - \mathcal{H})(\theta) = \begin{bmatrix} \frac{1}{2}(a - b) & b \\ \frac{1}{2}(e - g) & f \end{bmatrix} \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}.$$

Consequently, since the image of a circle under an affine transformation is an ellipse we see that *the normal curvature vector moves on an ellipse in the normal plane about the mean curvature vector.* (cf. Figure 1). This ellipse is called the curvature ellipse.

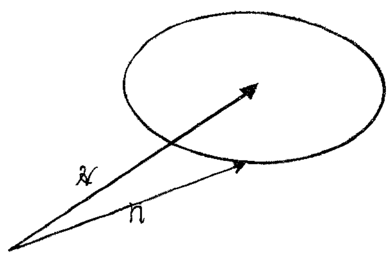

Figure 1

The second fundamental form is defined on the frame bundle F, because the functions a, b, c, e, f, g depend on the choice of tangent and normal frame. However, using the curvature ellipse, we may easily determine the scalar invariants. Since rotations in the tangent space map the unit tangent circle onto itself, the curvature ellipse as a point set in the normal plane is independent of rotations in the tangent space. Furthermore, any invariant quantity of the curvature ellipse, invariant under rotations of the normal plane about the origin, is invariant under rotations in the tangent and normal space and is therefore a scalar invariant.

For example, the vector from the origin to the center of the ellipse is, and invariant vector and is, as we have seen, the mean curvature vector. Its length $\mathcal{H}^2$ is thus a scalar invariant.

The area of the ellipse is a scalar invariant. To find it we use the fact that under an affine transformation the area is multiplied by the determinant of the transformation. Therefore, the area of the ellipse is

$$\left|\frac{1}{2}(a-c)f-\frac{1}{2}(e-g)b\right|\pi=\frac{1}{2}\pi\,|N|.$$

If we let φ be the argument of $\eta(\theta)-\mathcal{H}$ then we may check that $d\varphi/d\theta$ has the same sign as N so that we may regard $\frac{1}{2}\pi N$ as the oriented area of the ellipse, where the orientation is determined by the direction in which η traverses the ellipse.

Consider for a moment the general situation of an ellipse given as the affine image of a circle, say

$$Y(\theta)=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}.$$

If vectors are mapped into the semimajor and semiminor axes of the ellipse then they must necessarily have been at right angles. To see this suppose $Y(\theta_1)$ is along the major axis and $Y(\theta_2)$ is along the minor axis. Then $Y(\theta_2)=\pm\, dY/d\theta\,|_{\theta=\theta_1}=Y(\theta_1\pm 90^\circ)$ so that $\theta_2=\theta_1\pm 90^\circ$.

Now in our case let us choose e_1e_2 so that $\eta-\mathcal{H}$ at $\theta=0$ is the semimajor axis vector $\mathfrak{A}$, and $\theta=45^\circ$ is the semiminor axis vector $\mathfrak{B}$.

$$\mathfrak{A}=\frac{1}{2}(a-c)e_3+\frac{1}{2}(e-g)e_4;$$

$$\mathfrak{B}=be_3+fe_4.$$

If the ellipse is a circle choose any frame e_1e_2 and choose $\mathfrak{A}$ so that $\mathfrak{A}=\eta(0)-\mathcal{H}$. Using the above formulas it is a simple matter to check that $\mathcal{H}^2-K=\mathfrak{A}^2+\mathfrak{B}^2$. Since the area of an ellipse is $\pi\,|\mathfrak{A}|\;|\mathfrak{B}|$ we have

$$|N|=2\,|\mathfrak{A}|\;|\mathfrak{B}|$$

$$\mathcal{H}^2-K=\mathfrak{A}^2+\mathfrak{B}^2.$$

These two equations show that $|N|$ and $\mathcal{H}^2-K$ serve completely to determine the shape of the ellipse.

We expect yet another invariant; namely, a quantity which, expresses how the ellipse is oriented with respect to the line through $\mathcal{H}$. A further invariant is

$$\Delta = \frac{1}{4}\begin{vmatrix} a & 2b & c & 0 \\ e & 2f & g & 0 \\ 0 & a & 2b & 0 \\ 0 & e & 2f & 0 \end{vmatrix}.$$

Before we show that Δ is an invariant it will be helpful to develop two invariant quadratic forms. Write $e = xe_1 + ye_2$ and consider

$$de \cdot e_3 \wedge de \cdot e_4.$$

Now $de = xde_1 + dxe_1 + yde_2 + dye_2$ so that $de \cdot e_3 = x\omega_{13} + y\omega_{23}$ and $de \cdot e_4 = x\omega_{14} + y\omega_{24}$. Thus, using equations (1), we may write

$$de \cdot e_3 \wedge de \cdot e_4 = \mathcal{S}(x, y)\omega_1 \wedge \omega_2$$

where $\mathcal{S}(x, y)$ is a quadratic form in x and y, namely

$$\mathcal{S}(x, y) = (af - be)x^2 + (ag - ce)xy + (bg - cf)y^2.$$

Notice that a rotation in the fibre of F_ν will change $de \cdot e_3 \wedge de \cdot e_4$ only by the determinant of the transformation, but that is 1. Hence the coefficients of $\mathcal{S}(x, y)$ are defined on F_τ. A rotation in F_τ will leave $\omega_1 \wedge \omega_2$ unchanged but it will change (x, y) to (x', y') by the adjoint of the rotation which sends e_1e_2 to $e_1'e_2'$. (Here $e = xe_1 + ye_2 = x'e_1' + y'e_2'$.) Thus $\mathcal{S}(x, y)$ is an invariantly defined quadratic form. Its trace and determinant are scalar functions defined on the manifold.

In a similar fashion consider a normal vector ν. We write $\nu = xe_3 + ye_4$ and consider $d\nu \cdot e_1 \wedge d\nu \cdot e_2$. As before we may define a quadratic form $\mathcal{F}(x, y)$ by

$$d\nu \cdot e_1 \wedge d\nu \cdot e_2 = \mathcal{F}(x, y)\omega_1 \wedge \omega_2.$$

An easy computation shows that

$$\mathcal{F}(x, y) = (ac - b^2)x^2 + (ag + ce - 2bf)xy + (eg - f^2)y^2.$$

In a fashion similar to our consideration of $\mathcal{S}(x, y)$, we may show that the coefficients of $\mathcal{F}(x, y)$ are functions defined on F_ν and that the determinant and trace of $\mathcal{F}(x, y)$ are scalar functions on the manifold.

One may with a simple computation show that

$$\Delta = \det \mathcal{S} = \det \mathcal{F},$$

This then shows that Δ is an invariant. One also notices that $N =$ trace $\mathfrak{S}$ and $K =$ trace $\mathfrak{F}$.

Let us now try to describe Δ in terms of the configuration. In the course of this discussion it will become clear that Δ is independent of $\mathcal{H}^2$, K, N. Δ is the resultant of the two polynomials $ax^2 + 2bxy + cz^2$ and $ex^2 + 2fxy + gy^2$, (x, y) homogeneous coordinates of a point. Thus we see that if the ellipse passes through the origin so that $\eta(\theta) = 0$ for some θ then the two polynomials have a common root, namely $(\cos\theta, \sin\theta)$, so that $\Delta = 0$. In fact, in this case the common root is real. Since roots of a quadratic are either both real or both imaginary, they have a common real root only if all four roots are real. The condition for this is that $b^2 - ac \geq 0$, $f^2 - eg \geq 0$. Hence $K \leq 0$ in order for the ellipse to pass through the origin. The above reasoning may be reversed to show this is sufficient.

Next we ask what the condition $\Delta = 0$ and $K > 0$ may mean. $\Delta = 0$ means the quadratic equations have a common root and $K > 0$ means at least one root is imaginary. Since imaginary roots occur in conjugate pairs, one equation must be a multiple of the other and hence the ellipse is a radial line segment; i.e. the point is an inflection point. Since also at an inflection point $\Delta = 0$ we see that

$\Delta = 0$ *at a point if and only if the point is an inflection point or a point where* $\eta(\theta) = 0$ *for some* θ.

One may also check that if the point is not an inflection point then the origin is inside, on, or outside the ellipse as Δ is respectively positive, zero, or negative.

Suppose that $\Delta > 0$ so that the origin lies outside the ellipse. Then we may draw the tangent lines from the origin to the ellipse. The tangent directions θ_1, θ_2 such that $\eta(\theta_1)$ and $\eta(\theta_2)$ are tangent to the ellipse are called the conjugate directions. The conjugate directions satisfy $\mathfrak{S}(\cos\theta, \sin\theta) = 0$ and they are the only solutions mod 180°. Furthermore,

$$\tan^2(\theta_1 - \theta_2) = \frac{\Delta}{N^2}.$$

If we let Ω be the angle at the origin subtended by the ellipse then we may show that

$$\tan^2\Omega = \frac{\Delta}{K^2}.$$

The above formulas may be found in WONG [19].

Let us next consider the quadratic forms $\mathfrak{S}$ and $\mathfrak{F}$. We ask whether $\mathcal{H}$, $\mathfrak{S}$ and $\mathfrak{F}$ are enough to determine the second fundamental form. That is, do the coefficients of $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{F}$ determine a, b, c, e, f, g? Without some additional assumptions this is false. To see this, consider the case when $a = f = -c = \cos\theta$ and $b = -e = g = \sin\theta$. Then $\mathcal{H} = 0$, $\mathcal{S}(x, y) = x^2 + y^2$ and $\mathcal{F}(x, y) = -x^2 - y^2$ so that $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{F}$ are independent of θ. Also consider the case when the surface really lies in some 3-space. Take e_4 normal to that 3-space so that $e = f = g = 0$. In this case we have $\mathcal{H} = \frac{1}{2}(a + c)e_3$, $\mathcal{S}(x, y) = 0$, $\mathcal{F}(x, y) = (ac - b^2)x^2$ so that $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{F}$ determine only the mean and GAUSS curvatures which in general do not determine a, b and c.

Notice that we have shown that if M lies in a 3-space then $\mathcal{S} \equiv 0$ for a specific normal frame, but since the coefficients of $\mathcal{S}$ are defined on F_τ, $\mathcal{S} \equiv 0$ in any frame. The question of whether the converse is true will be taken up later.

THEOREM 1.1. - *At a point where* $\mathcal{S} \not\equiv 0$ *and* $\mathcal{H} \neq 0$ *the second fundamental form is determined by* $\mathcal{H}$, $\mathcal{S}$ *and* $\mathcal{F}$.

PROOF. - Take e_3 in the direction of $\mathcal{H}$. Then $\mathcal{H} = \frac{1}{2}(a + c)e_3$ and $e + g = 0$. Using the fact that $e = -g$, we may write e, f and g in terms of the coefficients of $\mathcal{S}$ and $\mathcal{H}$, namely

$$e = \frac{-(ag - ce)}{a + c}, \qquad g = \frac{ag - ce}{a + c}, \qquad f = \frac{af - be - (bg - cf)}{a + c}.$$

If $e = f = 0$ then also $g = -e = 0$ and hence $\mathcal{S} \equiv 0$. Thus we may suppose $e^2 + f^2 \neq 0$. Now

$$ag + ce - 2bf = -(a - c)e - 2bf$$
$$af - be + bg + cf = (a - c)f - 2be$$

is a system of equations with unknowns $a - c$ and b. The coefficients are determined by $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{F}$. We may solve it for $a - c$ and b, since the determinant, $e^2 + f^2$, is not zero. Thus $a - c$ and b are determined and they, together with $a + c$, give a, b and c. We have chosen special normal frames. In another choice of frames we could rotate to find the coefficients of $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{F}$ in the frames where e_3 is along $\mathcal{H}$. Then after finding a, b, c, e, f, g in these frames, we could rotate back to find a, b, c, e, f, g in the original frames.

We have seen that if the surface lies in a 3-space then $\mathcal{S} \equiv 0$ at every point. The following theorem gives equivalent local conditions for the vanishing of $\mathcal{S}$ and $\mathcal{F}$ at a point.

THEOREM 1.2. - *Let* $p \in M$. *The following four conditions are equivalent.*

a) $\mathcal{S} \equiv 0$ *at* p;

b) *rank* $\alpha \leq 1$ at p, *where* $\alpha = \begin{bmatrix} a & b & c \\ e & f & g \end{bmatrix}$;

c) *p is an inflection point;*

d) $\Delta = 0$ *and* $N = 0$ *at* p.

Also, the following five conditions are equivalent and imply the first set of conditions.

a) $\mathcal{F} \equiv 0$ *at* p;

b) *rank* $\beta \leq 1$ *at* p *where* $\beta = \begin{bmatrix} a & b \\ b & c \\ e & f \\ f & g \end{bmatrix}$;

c) *the tangential map,* $\varphi\colon M \to G_{2,4}$, *fails to be an immersion at* p, *where* $G_{2,4}$ *is the Grassmanian of 2-planes through the origin in* E^4 *and* φ *maps a point of* M *to its tangent plane translated to the origin;*

d) $\mathcal{S} \equiv 0$ *and* $K = 0$ *at* p;

e) $\Delta = 0$ *and* $K = 0$ *at* p.

For the proof of the first part of Theorem 1.2 recall that an inflection point is a point at which the second-order osculating space drops at least one dimension. The second-order osculating space is generated by the vectors $X''(s)$ and $X'(s)$ where $X(s)$ is a curve through $X(p)$ and where the prime indicates differentiation with respect to arc length. We have earlier seen that the normal components of $X''(s)$ are $a\cos^2\theta + 2b\cos\theta\sin\theta + c\sin^2\theta$ and $e\cos^2\theta + 2f\cos\theta\sin\theta + g\sin^2\theta$ where θ is the argument of the tangent vector X' with respect to some reference direction in the tangent plane. But if rank $\alpha \leq 1$ the direction of the normal component of X'' is independent of θ and hence the second order osculating space has dimension at most three. On the other hand, if the point is an inflection point then the vector $\eta(\theta)$ must move on a line through the origin so that its argument must be constant, i.e. there exists a constant λ such that

$$a\cos^2\theta + 2b\cos\theta\sin\theta + c\sin^2\theta = \lambda(e\cos^2\theta + 2f\cos\theta\sin\theta + g\sin^2\theta)$$

which is impossible unless rank $\alpha \leq 1$ because $\cos^2\theta$, $\cos\theta\sin\theta$, $\sin^2\theta$ are independent functions. The remainder of the first part of Theorem 1.2 is easily seen.

For the proof of the second part of Theorem 1.2 consider the following diagram.

$$\begin{array}{ccc} F(M) & \longrightarrow & 0(4) \\ \downarrow & & \downarrow \\ F_\tau(M) & \longrightarrow & V_{2,4} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi} & G_{2,4} \end{array}$$

$0(4)$ is the orthogonal group and $V_{2,4}$ is the STIEFEL manifold of 2-frames in E^4. A choice of frame e_1e_2 in the tangent plane and e_3e_4 normal to the plane gives a cross-section σ: $G_{2,4} \to 0(4)$. Let σ' be the induced cross-section, σ': $M \to F(M)$. $G_{2,4}$ is $0(4)/H$ where H is the isotropy group defined by $\omega_{13} = \omega_{23} = \omega_{14} = \omega_{24} = 0$. Therefore ω_{13}, ω_{23}, ω_{14}, ω_{24}, when pulled down via σ, become independent forms on $G_{2,4}$. The forms ω_{13}, ω_{23}, ω_{14}, ω_{24}, which are defined on $0(4)$, may be pulled back to $F(M)$ and via σ to M. Since the diagram commutes, they will be the same as the forms carried from $G_{2,4}$ to M via φ^*. But equations (1) express ω_{13}, ω_{23}, ω_{14}, ω_{24} as linear combinations of ω_1 and ω_2 and hence the rank of φ^* is equal to the rank of the matrix β. But the rank of φ is equal to the rank of φ^* because a matrix and its transpose have the same rank. Consequently, φ fails to be an immersion if and only if rank $\beta \leq 1$. The remainder the second part of Theorem 1.2 is not difficult.

KOMMERELL [9] also discusses the local invariants of a surface in E^4. He shows that *the polar conjugate of the ellipse with respect to the unit circle in the normal plane is the locus of consecutive normal planes.* The polar conjugate of an ellipse A with respect to an ellipse B is the locus of the poles of the tangent lines of A with respect to B. In the case B is a circle this is the inverse in the circle of the pedal curve to A. It is a well-known fact of projective geometry that the polar conjugate is a conic. In our case this conic is known as the conic of KOMMERELL. Noteworthy is the fact that the polar conjugate is a circle if and only if the center of the circle B is a focus of the ellipse A. Thus the conic of KOMMERELL is a cirle if and only if the origin is a focus of the ellipse. Such a point is called a focal point. We might also mention that the conic of KOMMERELL is an ellipse, parabola or hyperbola, as the origin is respectively inside, on or outside the ellipse.

3. - Local theory of surfaces in E^4.

So far we have been concerned with the local invariants of surfaces in E^4. One might call this the theory «at a point». In this section we shall be concerned with local theorems, or theorems true in a neighborhood of a point.

THEOREM 1.3.

a) $\mathcal{S} \equiv 0$ *at every point of* M *if and only if the surface is locally either developable or lies in a* 3-*space.*

b) *$\mathcal{F} \equiv 0$ at every point of M if and only if the surface is developable.*

(LANE [10] has shown that every point is an inflection point if and only if the surface is developable or lies in a 3-space).

For the proof of Theorem 1.3 suppose $\mathcal{S} \equiv 0$ at every point. Since each point is an inflection point, we may choose e_3 so that the span of $e_1 e_2 e_3$ is equal to or contains the second order osculating space. In these frames it is easy to see that $e = f = g = 0$. Now choose a fixed tangent frame $e_1 e_2$ so that $b = 0$. The equations (1) then become

$$\omega_{13} = a\omega_1,$$

$$\omega_{23} = c\omega_2,$$

$$\omega_{14} = \omega_{24} = 0.$$

First assume that neither a nor c is zero. Then using the structure equations and the fact that $d\omega_{14} = 0$, and $d\omega_{24} = 0$, we see that $\omega_{34} = \rho\omega_1$ and $\omega_{34} = \sigma\omega_2$ for some ρ and σ. But ω_1 and ω_2 are independent so $\omega_{34} = 0$. Hence $de_4 = 0$ so that e_4 is constant, and therefore the surface lies in a 3-space perpendicular to e_4.

Consider now the case that one of a or c is zero and the other is not, for definiteness $c = 0$ and $a \neq 0$. In this case $K = 0$ and so by Theorem 1.2 $\mathcal{F} \equiv 0$. Now $\omega_{23} = 0$ and using the structure equations together with $d\omega_{23} = 0$, we have $\omega_{12} = \tau\omega_1$ for some τ. On the integral curves of $\omega_1 = 0$ we have $de_1 = 0$ and $de_2 = 0$, that is, both e_1 and e_2 are constant. Since e_2 is tangent to the curve and constant, the curve is a straight line; also, since e_1 and e_2 are tangent vectors to the surface and constant along the line, the tangent plane is constant along the line, so the surface is developable. Conversely, if the surface is developable then tangential map $\varphi\colon M \to G_{2,4}$ fails to be an immersion at every point so that $\mathcal{F} \equiv 0$ everywhere. Also, if the surface lies in a 3-space then certainly every point is an inflection point so that $\mathcal{S} \equiv 0$.

If $a = c = 0$ it is not difficult to check that the surface is a plane. Consider, for example, $d(e_1 \wedge e_2)$. This completes the proof the theorem.

We investigate, next, conditions which imply that two surfaces in E^4 are congruent. Let us first review conditions for them to be isometric. By definition two surfaces M and M' are isometric if there exists a map $\varphi\colon M \to M'$ such that $\varphi^*(\omega_1')^2 + (\omega_2')^2) = (\omega_1)^2 + (\omega_2)^2$. Here the prime indicates a quantity defined for the manifold M'. M and M' are isometric if and only if there is a map $\varphi_\tau\colon F_\tau M \to F_\tau M'$ such that $\varphi_\tau^*(\omega_1') = \omega_1$, $\varphi_\tau^*(\omega_2') = \omega_2$, $\varphi_\tau^*(\omega_{12}') = \omega_{12}$. φ_τ must then be a bundle map covering the isometry. This discussion does not in fact depend on an immersion. However now assume that both M and M' are immersed. We discuss first the case when they are immersed in E^3.

$$X : M \to E^3$$
$$X' : M' \to E^3 .$$

Let us now assume that M and M' are isometric. We say that φ perserves the second fundamental form if

$$\varphi_T^*(d^2X' \cdot e_3') = d^2X \cdot e_3 .$$

This is equivalent to

$$\varphi_T^*(\omega_{13}') = \omega_{13} ,$$
$$\varphi_T(\omega_{23}') = \omega_{23} .$$

Hence if φ is an isometry which perserves the second fundamental form then $\varphi_T^*(\omega_i') = \omega_i$ and $\varphi_T^*(\omega_{ij}') = \omega_{ij}$. From this we conclude that φ is a congruence.

In the case where M and M' are immersed in E^4,

$$X : M \to E^4$$
$$X' : M' \to E^4,$$

in order to define preservation of the second fundamental form we must be able to extend φ to a map between the frame bundles,

$$\varphi_F : F(M) \to F(M').$$

In 3-space, of course, this problem did not arise. Once we have a bundle map we may define preservation of the second fundamental form as for surfaces in E^3; namely, $\varphi_F^*(d^2X' \cdot e_3') = d^2X \cdot e_3$ and $\varphi_F^*(d^2X' \cdot e_4') = d^2X \cdot e_4$. This is equivalent to requiring $\varphi_F^*(\omega_{13}') = \omega_{13}$, $\varphi_E^*(\omega_{23}') = \omega_{23}$, $\varphi_F^*(\omega_{14}') = \omega_{14}$, $\varphi_F^*(\omega_{24}') = \omega_{24}$. As opposed to the situation in E^3, even though we have a bundle map which preserves both first and second fundamental forms, we do not know that $\varphi_F^*(\omega_{34}') = \omega_{34}$, i.e. that the connection in the normal bundle is preserved and hence we can not be sure of congruence. However we do have the following.

LEMMA 1.4 - *If φ_F: $FM \to FM'$ is a bundle map covering φ which preserves both first and second fundamental forms and if $\mathcal{F}$ never vanishes then φ preserves the connection in the normal bundle and is consequently a congruence.*

PROOF. - We need only show that $\varphi_F^*(\omega_{34}') = \omega_{34}$. Now $d\omega_{13} = \omega_{12} \wedge \omega_{23} + \omega_{14} \wedge \omega_{43}$ and also $d\omega_{13} = \varphi_F^*(d\omega_{13}') = \omega_{12} \wedge \omega_{23} + \omega_{14} \wedge \varphi_F^*(\omega_{43}')$. Subtracting, we have $\omega_{14} \wedge (\omega_{43} - \varphi_F^*(\omega_{43}')) = 0$, so that $\omega_{43} - \varphi_F^*(\omega_{43}')$ depends on ω_{14}. In a similar fashion, beginning with $d\omega_{23}$, $d\omega_{14}$, $d\omega_{24}$ we conclude that $\omega_{43} - \varphi_F^*(\omega_{43}')$ depends on ω_{13}, ω_{23}, ω_{14}, ω_{24}. If any two of these are independent we see that $\omega_{34} = \varphi_F^*(\omega'_{34})$. If they are all dependent then the β of Theorem 1.2 falls in rank, i.e., $\mathcal{F} = 0$.

THEOREM 1.5. - *If $\varphi : M \to M'$ is an orientation preserving isometry of M and M' and if neither M nor M' is locally minimal then φ can be extended to a bundle map φ_F, over at least a dense submanifold of M.*

a) *If φ_F preserves the second fundamental form and if M is not locally developable then φ is a congruence.*

b) *If φ_F preserves both $\mathcal{S}$ and $\mathcal{F}$ and if M is neither locally developable nor lies locally in a 3-space then φ is a congruence.*

PROOF. - Let $A = \{ x \in M \mid \mathcal{H}(x) \neq 0 \}$ and $A' = \{ y \in M' \mid \mathcal{H}'(y) \neq 0 \}$. Because M and M' are not locally minimal, A and A' are dense sets of M and M' respectively. They are also open. Because φ is an isometry $\varphi^{-1}(A')$ is a dense open set of M. Let $C = A \cap \varphi^{-1}(A')$. Then C is a dense open set of M. We extend φ to a bundle map over C by requiring that the unit vector along $\mathcal{H}$ map into the unit vector along $\mathcal{H}'$. That is, choose frames e_3e_4 where $e_3 = \mathcal{H}/\mathcal{H}\,|$ and $e_1e_2e_3e_4$ agrees with the orientation of E^4. In a similar way, because $\mathcal{H}' \neq 0$ on $\varphi(C)$, we may choose frames $e'_3e'_4$ over $\varphi(C)$. Then define φ_F over C by sending e_3 to e'_3 and e_4 to e'_4 and extending linearly.

Suppose that φ_F preserves the second fundamental form and that M is not locally developable. Let $B = \{ x \in M \mid \mathcal{F} \neq 0 \text{ at } x \}$. Because M is not locally developable and by Theorem 1.3 B must be dense in M. Also B is open. Thus $B \cap C$ is a dense open set of M on which the second fundamental form is preserved and on which $\mathcal{F} \neq 0$. By Lemma 1.4 φ restricted to $B \cap C$ is a congruence. However, since $B \cap C$ is dense, M and M' must be congruent.

Suppose that φ_F preserves both $\mathcal{S}$ and $\mathcal{F}$ and M is neither locally developable nor lies locally in a 3-space. Define $D = \{ x \in M \mid \mathcal{S} \neq 0 \text{ at } x \}$. Then because $\mathcal{S} \equiv 0$ and $K = 0$ if and only if $\mathcal{F} \equiv 0$ we see that $\mathcal{F} \neq 0$ on D. Also, since M is neither locally developable nor locally is in a 3-space, by Theorem 1.3 D must be dense in M. Obviously D is open. Thus $D \cap C$ is a dense open set on which $\mathcal{H} \neq 0$, $\mathcal{S} \neq 0$, $\mathcal{F} \neq 0$ and $\mathcal{S}$ and $\mathcal{F}$ are preserved. By Theorem 1.1 since $\mathcal{S} \neq 0$ and $\mathcal{H} \neq 0$ on $D \cap C$ the second fundamental form must be preserved. Thus by Lemma 1.4, since $\mathcal{F} \neq 0$, φ restricted to $D \cap C$ is a congruence. However since $D \cap C$ is dense, M and M' must be congruent.

Let us touch on the theory of minimal surfaces in E^4. A surface is a minimal surface if $\mathcal{H} = 0$ everywhere. Interesting examples of minimal surfaces are the graphs of analytic functions. EISENHART [5], has proved.

THEOREM 1.6 – *A surface immersed in E^4 is locally congruent to the graph of an analytic function if and only if at each point the ellipse is a circle with the origin as center.*

There are a great many other results known concerning minimal surfaces in E^4. One should be referred especially to WONG [19].

4. – It is well known that the EULER characteristic of a plane bundle over an oriented surface is equal to the sum of the indices of a cross-section. One can define indices for a line element field and for a field of pairs of orthogonal lines. In this section we shall show that in these cases also the sum of the indices is the EULER characteristic.

Perhaps it would be well to review some of the definitions of the index of a cross-section of a plane bundle. Suppose that $\pi: B \to M$ is an oriented plane bundle over a compact oriented surface. Let $\theta: M \to B$ be the zero cross-section and let $\sigma: M \to B$ be a cross-section transversal to $\theta(M)$. Then $\sigma(M)$ meets $\theta(M)$ at isolated points, say $p_1, \ldots, p_r$. These points where $\sigma(p) = \theta(p)$ we call singular points of σ. The intersection number of $\sigma(M)$ and $\theta(M)$ at p_i is then the index of σ at p_i.

We give another definition of the index of a cross-section σ which has isolated singular points. Suppose p is an isolated singular point of σ. Let C be a circle about p such that p is the only singular point inside or on C, and suppose that C lies in a neighborhood over which the bundle is trivial. Thus locally $\sigma: \mathrm{U} \to \mathrm{U} \times \mathbb{R}^2$ and via projection on the second factor we have a map $\sigma: C \to \mathbb{R}^2$ such that $\sigma(p) \neq 0$ for $p \in C$. By normalizing we have a map $C \to S^1$ given by

$$p \to \frac{\sigma(p)}{|\sigma(p)|}\,.$$

The degree of this map is the index of σ at p.

We give yet a third definition of the index of a cross-section. This time we assume only that the singular locus L can be written $L = \cup L_i$, L_i disjoint, where each L_i lies in a neighborhood U_i over which a non-zero cross-section exists and where $L \cap_i \mathrm{U} = L_i$.

Let B_F be the associated frame bundle of B, and let ω be the connection form on B_F. (ω_{12} for F_τ and ω_{34} for F_ν). The cross-section σ of B induces a cross-section σ_F of B_F over $M - L$. σ_F is defined as follows: For $p \in M - L$, $\sigma(p)$ is a nonzero vector. Therefore, we may choose a vector $\tau(p)$ normal to $\sigma(p)$ so that the frame

$$\frac{\sigma(p)}{|\sigma(p)|} \qquad \frac{\tau(p)}{|\tau(p)|}$$

agrees with the orientation of B. The index of σ at L_i is

$$\frac{1}{2\pi}\int_{\partial D_i}\sigma_F^*\omega-\frac{1}{2\pi}\int_{D_i}d\omega.$$

Here D_i is oriented to agree with M and ∂D_i takes its orientation from D_i. In the case that L_i is an isolated point we take $D_i(\varepsilon)$ to be a disk of radius ε about the point. Then the index is given by

$$\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{\partial D_i(\varepsilon)}\sigma_F^*\omega.$$

As an application of these ideas let us show that

$$\chi(B)=-\frac{1}{2\pi}\int_M d\omega$$

where $d\omega$ is the curvature form of B_F. $\chi(B)$ is the EULER characteristic of B, i.e. the EULER class of B evaluated on the fundamental class of M. To show the above formula take a cross-section σ_F of B_F over $M-\{p_i\}$. $\{p_i\}$ is a finite set of points. Such a cross section always exists. Then we have

$$\chi(B)=\sum_i\frac{1}{2\pi}\int_{C_i}\sigma_F^*\omega$$

where $C_i=\partial D_i(\varepsilon)$. But $\partial(M-\cup D_i)=\cup C_i$. Thus by STOKES' Theorem

$$\sum_i\int_{C_i}\sigma_F^*\omega=-\int_{M-\cup D_i}d\sigma_F^*\omega$$

and hence

$$\chi(B)=-\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{M-\cup D_i}d\sigma_F^*\omega.$$

But $d\sigma_F^*\omega$ is the curvature form of B. It is a well defined 2-form on M independent of σ.

We now wish to discuss the case of line element fields and fields of pairs of lines. In order to discuss both these cases together let us define a more general r-cross field.

An r-cross is a set of r unit vectors in the plane such that their tips form a regular polygon. Define $C_r(B)$, the associated bundle of r-crosses, as follows: $C_r(B) = \{(x, C_x) \mid x \in M$ and C_x is an r-cross in the plane over $x \cdot \}$. A cross field is then defined to be a cross-section of this bundle.

Note that a 1-cross field is a vector field, a 2-cross field is a line element field and a field of pairs of lines is a 4-cross field.

Let C_r be the family of all r-crosses in the plane. Note that C_r is homeomorphic to S^1.

A singular point of a cross field is a point where the cross field is not defined. We may define the index of a singular point of a cross field analogously to that of a vector field. Choose a neighborhood U about p containing no singular points other than p, and such that over U the bundle is trivial. Let C be a circle about p contained in U. Then the cross field gives a map

$$U - \{p\} \to (U - \{p\}) \times C_r,$$

and by projection on the second factor, a map $C \to C_r$. We define the index to be $1/r$ times the degree of this map. The definition can be shown to be independent of the neighborhood and circle.

Suppose that the cross field has isolated singular points p_i. Let D_i be a disk about p_i and $C_i = \partial D_i$. Here D_i is a neighborhood about p_i which contains only the one singular point. Assume the radius of D_i is ϵ.

We construct an r-fold covering of $M - \cup D_i$ as follows: $C(M - \cup D_i) = \{(x, v_{ix}) \mid x \in M - \cup D_i$ and v_{ix} is a leg of the cross at $x\}$. Assume now that B has a connection form ω. Define π: $C(M - \cup D_i) \to M - \cup D_i$ by $\pi(x, v_{ix}) = x$. Then $\pi^{-1}(x) = \{(x, v_{1x}), \dots (x, v_{rx})\}$ and $\pi^{-1}(C_i)$ is homeomorphic to r or fewer disjoint circles. We may, via π, draw the bundle B back to a bundle, $\pi^{-1}B$, over $C(M - \cup D_i)$. Let ω_c be the connection on this bundle induced from ω on B. Let $d\omega$ be the curvature form of B. $d\omega$ is defined on M. Let $d\omega_c$ be the induced form on $\pi^{-1}B$; $d\omega_c$ is defined on $C(M - \cup D_i)$.

The cross field on $M - \cup D_i$ lifts to a vector field on $C(M - \cup D_i)$. We call this vector field v. It is a consequence of the definition that the index of the cross field at p_i is equal to

$$\frac{1}{r} \lim_{\varepsilon \to 0} \int_{\pi^{-1}(C_i)} v^* \omega_c$$

where $v^*\omega_c$ is the pull down of ω_c defined on $\pi^{-1}B$ via the vector field v.

We have

$$\sum_i \operatorname{Ind}(p_i) = \frac{1}{r} \lim_{\varepsilon \to 0} \sum_i \int_{\pi^{-1}(C_i)} v^* \omega_c$$

and because $\partial(C(M - \cup D_i)) = \cup\, \pi^{-1}(C_i)$

$$\sum_i \int\limits_{\pi^{-1}(C_i)} v^*\omega_c = \int\limits_{C(M-\cup D_i)} dv^*\omega_c .$$

Also

$$\int\limits_{C(M-\cup D_i)} dv^*\omega_c = \int\limits_{C(M-\cup D_i)} d\omega_c = r \int\limits_{M-\cup D_i} d\omega.$$

Thus

$$\sum_i \operatorname{Ind}(p_i) = \lim_{\varepsilon\to 0} \int\limits_{M-\cup D_i} d\omega = \int\limits_M d\omega.$$

But

$$\int\limits_M d\omega = \chi(B)$$

so that we have shown

PROPOSITION 1.7. - $\sum_i$ Index $(p_i) = \chi(B)$. Note that if $\chi(B) = 0$ then B has a nonzero vector field and therefore certainly a crossfield.

Let us mention a lemma which will prove useful in the evaluation of the index in the case that B is the tangent bundle. Let D be a disk about an isolated singular point p_0 over which TM is trivial. Let $e_1 e_2$ be a field of orthonormal tangent frames in this neighborhood, and let v_1 be one leg of the cross. Define arg cross field $= \angle\ (v_1, e_1)$. Arg cross field is well defined modulo $2\pi/r$, although it depends on the frames $e_1 e_2$.

LEMMA 1.8. - *Suppose that* $\varphi\colon D \to R^2$ *and that* $n \cdot \arg\varphi = \arg$ *cross field modulo* $2\pi/r$. *Then if Jacobian of* φ *is not zero at* p_0 *the index of the cross field at* p_0 *is* $\pm n$. *Moreover, the sign depends on the sign of the determinant of the Jacobian of* φ *at* p_0. *Here* n *may be a rational number.*

5. - Transversality.

This section is largely a summary of some of the results of Chapter II which in turn is based on the work of FELDMAN [6, 7]. Recall the notation established in Section 1. $X\colon M \to E^4$ is an immersion of a compact oriented 2-dimensional manifold, e_1, e_2 are tangent vectors, and e_3, e_4 are normal vectors at each point. Also, the second fundamental form is a vector valued quadratic form defined on the frame bundle F. It may be written

$$(a\omega_1^2 + 2b\omega_1\omega_2 + c\omega_2^2)e_3 + (e\omega_1^2 + 2f\omega_1\omega_2 + g\omega_2^2)e_4$$

where a, b, c, e, f, g are real valued functions defined on the frame bundle F. At each point p of F the second fundamental form gives a pair of quadratic forms $(a(p)x^2 + 2b(p)xy + c(p)y^2,\ e(p)x^2 + 2f(p)xy + g(p)y^2)$. Let Z be the set of all such pairs of quadratic forms. (We may identify Z and R^6). Then the second fundamental form gives a map $\mu\colon F \to Z$ by sending a point p of F into the pair of quadratic form.

The group $S^1 \times S^1$ acts on Z. We describe the action as follows. Let $(\theta, \varphi) \in S^1 \times S^1$, and let $(u(x, y), v(x, y)) \in Z$. Write

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix}, \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Then $(\theta, \varphi)(u(x, y), v(x, y)) = (u'(x', y'), v'(x', y'))$. Recall that the fibre of F is the torus, $S^1 \times S^1$, so that the group $S^1 \times S^1$ acts on each fibre. It is easy to see that μ commutes with the action of $S^1 \times S^1$, i.e.

$$\mu((\theta, \varphi)(p)) = (\theta, \varphi)(\mu(p)).$$

Thus in particular a fibre over a point of M is mapped, under μ, onto an orbit of Z under $S^1 \times S^1$.

Let us now make several definitions. An algebraic subvariety, K, of Z, invariant under the action of $S^1 \times S^1$, is called a *model singularity*. A point p of M such that μ (fibre of F over p) meets K is called a *K singular point of M*. If $\mu(F)$ meets K transversally in Z all along the fibre over p, we say that p is a *K geometrically trasversal singular point of M*.

FELDMAN has given definitions different from the above using jets. We shall give his definitions in Chapter II. The model singularities he uses are not the same as those we have defined; however in Proposition 29 we show that there is a $1-1$ correspondence

$$\rho\colon \{\text{jet model singularities}\} \to \{\text{geometric model singularities}\}.$$

Also, it is easy to see that his definition of a singular point is equivalent to the one given above, i.e. p is a K singular point in the sense of the jets if and only if it is a ρK singular point. FELDMAN has also defined a *K jet transversal singular point*. (His terminology is K generic singular point). We have not been able to show that these definitions are equivalent but in Chapter II we show.

THEOREM 1.9. - *If a point is a K jet transversal singular point then it is also a ρK geometrically transversal singular point.*

We say that a map $X\colon M \to E^4$ is K geometrically transversal ($\rho^{-1}K$ jet transversal) if each $K(\rho^{-1}K)$ singular point of M is a geometrically transversal (jet transversal) singular point.

Let $C^\infty(M, E^4)$ be the set of C^∞ maps from M into E^4 with a topology to be defined in Chapter II. Convergence in this topology implies that the functions and all their derivatives up to any given order converge uniformly on the compact M. We may now state a result due to FELDMAN [7, p. 194].

THEOREM 1.10. - *The K jet transversal (and hence the ρK geometrically transversal) immersions of M into E^4 are dense in $C^\infty(M, E^4)$.*

REMARK. - A remark is due on what it means for μ to be transversal to K if $\mu(p)$ is a singular point of the algebraic variety K. We offer one possible definition. Let sing K be the set of singular points of the algebraic variety K. It is a fact that sing K is a proper algebraic subvariety of K. Let $\mathcal{K}$ be a collection of varieties satisfying the following conditions.

1. $K \in \mathcal{K}$.
2. If $K_1 \in \mathcal{K}$ then all the irreducible components of K_1 are in $\mathcal{K}$.
3. If $K_1 \in \mathcal{K}$ then sing $K_1 \in \mathcal{K}$.
4. If K_1, $K_2 \in \mathcal{K}$ then $K_1 \cap K_2 \in \mathcal{K}$.

$\mathcal{K}$ is a well-defined finite collection of algebraic varieties. We say that μ is transversal to K at p if μ is transversal to each $K_1 \in \mathcal{K}$ such that $\mu(p)$ is a regular point of K_1.

If K is a manifold or if codimension sing $K > \dim M$ then the K generic immersions are open. In the case K is a manifold this is well enough known. In the case that codimension sing $K > \dim M$, by Theorem 1.11 and the fact that transversality to K implies transversality to sing K we see that the generic maps would not meet sing K at all. For the model singularities we will consider, this is enough to establish that the generic maps are both open and dense. However, it is hoped that the generic maps are open and dense for any variety K without the above dimensionality restriction.

Another result from Chapter II is

THEOREM 1.11. - *If X is a K geometrically transversal immersion of M into E^4 then the codimension of the locus of K singular points in M is equal to the codimension of K in Z.*

Suppose that in a neighborhood U of K singular point p we choose frames $e_1e_2e_3e_4$ on M. Let σ: $M \to F$ be the cross-section giving these frames. Then functions a, b, c, e, f, g defined on F may be pulled down via these frames to functions on M which we call a', b', c', e', f', g'. Let us define μ': $U \to Z$ by $\mu' = (a', b', c', e', f', g')$. Then we may state the following.

LEMMA 1.12 - *p is a K geometrically transversal singular point if and only if μ' is transversal to K at p. This is true for any choice of frame $e_1e_2e_3e_4$ on U.*

PROOF. - Since $\mu'(U) \subset \mu(F(U))$, μ' transversal at p implies μ is transversal at $\sigma(p)$. However, σ can be chosen arbitrarily so that μ is transversal at every point in the fibre over p. Thus we need only show that if μ is transversal at $p \in F$ then μ' is transversal at $\pi(p)$. Let us take coordinates u, v in U and θ, ψ in the fibres over U. Assume that at p $\theta = \psi = 0$. Let us assume that $\mu(p)$ is a regular point of K. Suppose that K is given by the polynomials

$$\varphi_i(a, b, c, e, f, g) = 0, \qquad i = 1, \dots, r.$$

By this we mean that φ_i are a basis for $I(K)$, the ideal of all polynomials vanishing on K. Let α be the column vector (a, b, c, e, f, g), and let $H(\theta, \psi)$ be the 6×6 matrix representing the action of $S^1 \times S^1$ on Z. Define functions

$$\varphi_{i\theta\psi}(\alpha) = \varphi_i(H(\theta, \psi)\alpha)$$

on Z. Then since K is invariant under $S_1 \times S_1$, $\alpha \in K$ implies $\varphi_{i\theta\psi}(\alpha) = 0$. Thus $\varphi_{i\theta\psi} \in I(K)$, and hence there exist functions $L_{ij}(\theta, \psi)$ such that

$$2) \quad \varphi_{i\theta\psi}(\alpha) = \sum_j L_{ij}(\theta, \psi)\varphi_j(\alpha).$$

Let J be the $2 \times r$ matrix with first row $\partial\varphi_i/\partial u(\mu(p))$ and second row $\partial\varphi_i/\partial v(\mu(p))$. Then since μ is transversal at p, J must be full rank at p. Let J' be the Jacobian of $\varphi_i(\mu'(u, v))$ at $\pi(p)$. If we can show that J' has full rank at $\pi(p)$ then we are finished. But from 2) we see that

$$J = J'(L_{ij}(\theta, \psi))$$

and since at p, where $\theta = \psi = 0$, we have $L_{ij}(0, 0) =$ identity, J' must have full rank. We have been assuming that $\mu(p)$ is a regular point of K; if not, then we must apply the above argument to one, or several, proper subvarieties.

6. - Global Theory.

When is the ellipse a circle? From the fact that $|N| = 2|\mathfrak{A}||\mathfrak{B}|$ and $\mathcal{H}^2 - K = \mathfrak{A}^2 + \mathfrak{B}^2$ it is not difficult to see that $|\mathfrak{A}| = |\mathfrak{B}|$ if and only if $\mathcal{H}^2 - K = |N|$. Thus a necessary and sufficient condition that the ellipse be a circle is that $\mathcal{H}^2 - K = |N|$. We include a point as a circle.

If the ellipse is not a circle then the two ends of the major axis determine two well-defined, and in fact orthogonal lines in the tangent plane. Remember each tangent line determines a point on the ellipse because $\eta(\theta) = \eta(\theta + 180^\circ)$. But a compact surface has a field of pairs of lines if and only if the EULER characteristic is zero. Hence we have

THEOREM 1.13. - *On any compact surface of non-vanishing Euler characteristic there must be a point where* $\mathcal{H}^2 - K = |N|$.

The line through the mean curvature vector will meet the ellipse in two points. These points determine a pair of lines, in fact orthogonal lines, in the tangent plane. The only time this construction fails is either when $\mathcal{H} = 0$ or else when the ellipse degenerates to a radial line segment, i.e. the point is an inflection point. Observe that if the ellipse is a nonradial line segment then the tip of $\mathcal{H}$ intersects the segment and this point also picks out a pair of orthogonal lines. Again, because if a manifold has a field of pairs of lines then the EULER characteristic must be zero, we have the

THEOREM 1.14. - *On any compact surface of non-vanishing Euler characteristic there must either be a point where the mean curvature vector vanishes or else an inflection point.*

The flat torus is an example of an inflection-free surface with everywhere non-zero mean curvature vector. To give a more general example, suppose that $\varphi(t) = (\varphi_1(t), \varphi_2(t))$ and $\chi(t) = (\chi_1(t), \chi_2(t))$ are two inflection-free immersed circles in the plane. The surface $\varphi \times \chi(u, v) = (\varphi_1(u), \varphi_2(u), \chi_1(v), \chi_2(v))$ is easily seen to be an inflection-free immersion of the torus with everywhere nonzero mean curvature vector.

The normal curvature vector η maps antipodal points of the unit tangent sphere into the same point of the ellipse. If the ellipse does not degenerate to a line segment each point of the ellipse determines a unique pair of antipodal points of the unit tangent circle and hence a unique tangent line.

η maps four points on the unit circle which form a cross, i.e. the vertices of an inscribed square, into two diametrical points of the ellipse.

For any pair of diametrical points of an ellipse there is what we might call a conjugate pair of diametrical points, namely points whose tangents are parallel to the diameter line of the first pair.

Consider two crosses, one a rotation by 45° of the other; then the images of the two crosses will be conjugate.

When the ellipse is a line segment the center of the segment is the image of a cross. Each endpoint of the segment is the image of a pair of antipodal points, both pairs forming a cross. The two crosses picked out by the center and the endpoints differ by 45°.

Thus any method of uniquely choosing a point of the ellipse, which depends continuously on the configuration, determines a tangent line element field. Points where the construction fails will be singular points of this field.

Also, any method of uniquely choosing a pair of diametrical points of the ellipse will determine a field of tangent crosses.

A surface in E^3 may be viewed as a special case of a surface in E^4. The ellipse is of course always a line segment and the principal curvatures are the distances from the origin to the endpoints of the segment. Since every point is an inflection point Theorem 1.4 does not provide much information. However, Theorem 1.3 does indeed show that there exists a unique cross field with singularities only when the line segment degenerates to a point, i.e. an umbilic. Note that for a surface in E^4 there is no good way to distinguish one end of the major axis and therefere pick out a line element field. However, for the special case of a surface in E^3 we may choose the smaller (or larger) principal curvature. This does pick out an endpoint of the line segment which determines a unique tangent line.

It is, however, possible to obtain information about a surface in E^3 from Theorem 1.13. To do this we need only project the surface stereographically into S^3. Let $\varphi\colon E^3 \to S^3$ be the sterographic projection,

$$\varphi(x,\ y,\ z) = \frac{1}{1+t}(x,\ y,\ z,\ t),$$

where $t = 1/2(x^2 + y^2 + z^2 - 1)$. Since φ is conformal, $d\varphi$ is a similarity. Thus given orthonormal frames $e_1e_2e_3$ in E^3, $d\varphi(e_1)d\varphi(e_2)d\varphi(e_3)$ are orthogonal frames tangent to S^3. We may normalize them to obtain orthonormal frames $\varepsilon_1\varepsilon_2\varepsilon_3$ and then choose ε_4 to be the outward unit normal to S^3.

Let $X\colon M \to E^3$ be an immersion of a surface in E_3 and let $X' = \varphi \circ X$ be its stereographic projection. Suppose that $e_1e_2e_3$ are orthonormal frames on $X(M)$ so that e_1e_2 are tangent and e_3 is normal. We choose $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$ as frames on $X'(M)$.

Suppose that the second fundamental from for $X(M)$ is

$$d^2X \cdot e_3 = a\omega_1^2 + 2b\omega_1\omega_2 + c\omega_2^2$$

and the second fundamental form for $X'(M)$ is

$$(a'\omega_1'^2 + 2b'\omega_1'\omega_2' + c'\omega_2'^2)\varepsilon_3 + (e'\omega_1'^2 + 2f'\omega_1'\omega_2' + g'\omega_2'^2)\varepsilon_4$$

where as usual $\omega_i = dX \cdot e_i$, $\omega_{ij} = de_i \cdot e_j$ and $\omega_i' = dX' \cdot \varepsilon_i$, $\omega_{ij}' = d\varepsilon_i \cdot \varepsilon_j$.

Then it is possible to show that

$$\begin{bmatrix} a' & b' & c' \\ e' & f' & g' \end{bmatrix} = \begin{bmatrix} (1+t)a - h & (1+t)b & (1+t)c - h \\ 1 & 0 & 1 \end{bmatrix},$$

where $h = X \cdot e_3$ is the support function for $X(M)$. Note that $1 + t \geq 1/2$ so that it is never zero. Also note that umbilics are carried over into inflection points and they give the only inflection points. The mean curvature vector is seen to have ε_4 component equal to 1 so that it can never be zero. Thus applying Theorem 1.14, we have the well-known result that a surface in E_3 of nonzero EULER characteristic must have an umbilic.

It is also interesting to observe what happens to the principal frames under stereographic projection. Let $\mathcal{H}$, η be the mean and normal curvature vectors for $X(M)$ and $\mathcal{H}'$ η' be those for $X'(M)$. Then we see that

$$\eta(\theta) - \mathcal{H} = (1/2(a-c)\cos 2\theta + b\sin 2\theta)e_3 \quad \text{and}$$

$$\eta'(\theta) - \mathcal{H}' = (1+t)(1/2(a-c)\cos 2\theta + b\sin 2\theta)\varepsilon_3 .$$

Thus we see that the directions picked out by a point of the one segment are the same as those picked out by the corresponding point of the other segment, the correspondence given by multiplication by $1+t$. In particular, the directions picked out by the midpoints of each ellipse are the same. The principal axes chosen by the point where the mean curvature vector meets the ellipse become, when the ellipse is a line segment, the axes picked out by the midpoint. Thus the principal axes used in Theorem 1.13 give, for surfaces in E^3, directions such that the curvatures in those directions are equal to the mean curvature. These directions differ by 45° from the usual directions.

Let us now study the singular loci we have found in more detail. According to the previous section, we must study the model singularities. For instance, to study the locus of points where $\mathcal{H} = 0$ let K be the model singularity given by

$$a + c = 0, \qquad e + g = 0.$$

It is trival that this model singularity has codimension 2. Thus, by Theorem 1.11, if the immersion is generic then the locus where $\mathcal{H} = 0$ is of dimension zero, and consists of isolated points.

As another example, let the model singularity be given by $\mathcal{H}^2 - K = |N|$. This means that the ellipse is a circle so that the configuration depends on two parameters $\mathcal{H}^2$ and N. The map η depends on four parameters, $\mathcal{H}^2$, N, a choice of frame in the tangent space and a choice of frame in the normal space. Thus the model singularity has dimension four. Consequently, the generic locus is of dimension zero and consists of isolate points. Convenient polynomials may be found if we observe thet the ellipse is a circle if and only if the matrix

$$\begin{bmatrix} \frac{1}{2}(a-c) & b \\ \frac{1}{2}(e-g) & f \end{bmatrix}$$

is a dilatation. Thus we may write the model singularity as

$$(a-c)b+(e-g)f=0$$

$$\left(\frac{1}{2}(a-c)\right)^2+\left(\frac{1}{2}(e-g)\right)^2=b^2+f^2.$$

This model singularity is reducible, namely

$$b=\frac{1}{2}(e-g), \qquad f=-\frac{1}{2}(a-c),$$

and

$$b=-\frac{1}{2}(e-g), \qquad f=\frac{1}{2}(a-c).$$

Using the previous theorems, it is easy to see that the locus where $H^2-K=N=0$ is generically empty.

As a further example, let the model singularity be the inflection points, namely where

$$\operatorname{rank}\begin{bmatrix} a & b & c \\ e & f & g \end{bmatrix}<2.$$

Polynomials are then

$$af-be=0,$$

$$bg-cf=0,$$

$$ag-ce=0.$$

The maximum rank of the Jacobian on this model singularity is two so that the codimension is two. Hence the locus of inflection points consists, generically, of isolated points.

Let us now return to our consideration of the ellipse. We have seen that, except at inflection points or points where $\mathcal{H}=0$, there is a field of pairs of pairs of lines. Just as before, only being more careful this time, we shall see that we may pick out a line element field. Let q_1 and q_2 be the two points, in the normal space in which the line through $\mathcal{H}$ meets the ellipse. If the ellipse is not a segment, i.e. if $N\neq 0$, then $q_1\neq q_2$, Moreover, exactly one point, say q_1, is farther from the origin. Let us now assume a generic situation; namely, the locus where $N=0$ is one dimensional; the locus where $\mathcal{H}=0$ consists of isolated points; and the inflection points are also isolated points. Since being an inflection point implies $N=0$, the isolated inflection points lie on the curves $N=0$. Let us now pick out a line element field as follows. At a point where $N>0$ (and $\mathcal{H}\neq 0$) let us choose the point q_1 to pick the line, and at a point where $N<0$ (and $\mathcal{H}\neq 0$) let us choose the point q_2. Then we have determined a line element field everywhere except on the locus $N=0$. We may extend the line element field from

$N > 0$ to the curve $N = 0$ (minus inflection points and from $N < 0$ to $N = 0$. The extensions must either agree or differ by 90^o because we have a cross field throughout. However, the choice of q_1 for $N > 0$ and q_2 for $N < 0$ makes them agree. Note that the sign of N gives the sense in which η traverses the ellipse. When we cross the curve $N = 0$ this sense changes, interchanging q_1 and q_2. Thus we have proved the:

THEOREM 1.15. - *There exists on M a line element field with singularities only at points where $\mathcal{H} = 0$ and at inflection points.*

We have shown the result in the generic case but by continuity this is enough for the general case.

Let us show next, as one might guess, that the generic index of the singularities of this line element field is $\pm\frac{1}{2}$. It is simpler to work with the cross field, even though we have seen that it is composed of two line element fields. If we show the generic index of the cross field is $\pm\frac{1}{2}$ then certainly the same will be true of the line element field.

Suppose p_0 is an isolated singular point of the cross field, i.e. either a point where $\mathcal{H} = 0$ or an inflection point. Let U be a coordinate neighborhood containing no other singular points andlet e_1e_2 be a field of orthonormal tangent frames in this neighborhood. Let θ be the argument of the cross field; θ is defined modulo 90^o. (Actually since the cross field is composed of two line element fields, we know that θ could be defined modulo 180^o, but no matter). From our definition of the cross field we know that θ satisfies $\mathcal{H} \wedge \eta(\theta) = 0$. This gives the expression

$$(ag - ce)\cos 2\theta - (af - be - (bg - cf))\sin 2\theta = 0,$$

and is well defined modulo 90^o unless

$$\begin{aligned} ag - ce &= 0 \\ af - be &= bg - cf. \end{aligned}$$

These conditions are equivalent to

$$\begin{array}{ccc} a + c = 0 & & ag - ce = 0 \\ e + g = 0 & \text{or} & af - be = 0 \\ & & bg - cf = 0, \end{array}$$

and hence it is the model singularity of points where either $\mathcal{H} = 0$ or points that are inflectional. They are generically isolated. To be specific, we assume the manifold is generic with respect to the model singularity defined by

$$\text{3)}\qquad \begin{aligned} ag - ce &= 0 \\ af - be &= bg - cf. \end{aligned}$$

Thus points where $\mathcal{H} = 0$ and inflection point are distinct isolated points. Notice also that $af - be$, $ag - ce$ and $bg - cf$ are coefficients of the form $\mathcal{S}(x, y)$ and so are defined on F_τ. In fact, it is not difficult to check, by differentiating $\mathcal{S}(\cos\theta, \sin\theta)$ with respect to θ, that the cross field is given by the eigenvectors of the quadratic form $\mathcal{S}$. Now let us regard $af - be$, $ag - ce$, $bg - cf$ as defined on U via the field of frames e_1e_2 we have chosen. Let φ: $U \to R^2$ be defined as follows:

$$\varphi = (ag - ce,\ af - be - (bg - cf)).$$

Because X: $M \to E^4$ is generic with respect to the model singularity given by 3), the map φ is regular at every point which maps to $(0, 0)$, namely at p_0. Here we used Lemma 1.12. We are nearly ready to apply Lemma 1.8; i.e. if we can show that

$$n \cdot \arg\varphi = \theta \quad \text{modulo } 90^\circ$$

then the index will be $\pm n$. But

$$\tan 2\theta = \frac{ag - ce}{af - be - (bg - cf)} = \tan\arg\varphi.$$

Therefore $\arg\varphi = 2\theta$ modulo 90° so that we have shown:

THEOREM 1.16. - *The generic index of the singularities of the line element field described in Theorem* 1.15 *is* $\pm 1/2$.

We next turn to our other cross field, the one picked out by the major axis of the ellipse, and ask what'its generic index may be. Just as before, it is simpler to consider the 8-cross field picked out by both the major and minor axis. So suppose p_0 is an isolated singularity of this field, and let U be a coordinate neighborhood containing no other singularities. Let e_1e_2 be a field of frames in this neghborhood, and let θ be the argument of the 8-cross field, defined modulo 45°. From our definition of the 8-cross field we know that its argument, θ, must satisfy $d(|\eta(\theta) - \mathcal{H}|^2)/d\theta = 0$. This says that the major and minor axes are extremes of $|\eta(\theta) - \mathcal{H}|$. Using the fact that $d(|\eta(\theta) - \mathcal{H}|^2)/d\theta = 2d(\eta(\theta) - \mathcal{H})/d\theta \cdot (\eta(\theta) - \mathcal{H})$, we find, upon simplifying, that the argument must satisfy

$$(b(a - c) + f(e - g))\cos 4\theta - \left(\left(\frac{1}{2}(a - c)\right)^2 + \left(\frac{1}{2}(e - g)\right)^2 - b^2 - f^2\right)\sin 4\theta = 0.$$

Thus θ is well-defined, modulo 45^o, except on the model singularity given by $\mathcal{H}^2 - K = |N|$ which we have studied before. Because the vectors $ae_3 + ee_4$, $be_3 + fe_4$, $ce_3 + ge_4$ are normal vectors invariantly defined on F_τ, we see that the functions

$$b(a - c) + f(e - g),$$

$$\left(\frac{1}{2}(a - c)\right)^2 + \left(\frac{1}{2}(e - g)\right)^2 - b^2 - f^2$$

are defined on F_τ, and, as before, they may be regarded as defined on U via the frames, e_1e_2. Let φ: $U \to R^2$ be defined by

$$\varphi = \left(b(a - c) + f(e - g), \left(\frac{1}{2}(a - c)\right)^2 + \left(\frac{1}{2}(e - g)\right)^2 - b^2 - f^2\right).$$

Then because we are assuming X: $M \to E^4$ generic with respect to the model singularity $\mathcal{H}^2 - K = |N|$, we see, by Lemma 1.12 that φ is regular at p_0. Also $\arg\varphi = 4\theta$ modulo 45^o, and thus by Lemma 1.8 we have shown:

THEOREM 1.17. - *The generic index of the 4-cross field defined by picking the points on the tangent circle which map into the major axis is* $\pm\frac{1}{4}$.

We now turn out attention to the mean curvature vector, $\mathcal{H}$, as a normal vector field. By the discussion in Section 4, we know that the index of the vector field $\mathcal{H}$ at a point, p_i, where $\mathcal{H} = 0$, is equal to

$$\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{C_i}\omega_{34},$$

where ω_{34} is the pull down of ω_{34} via $\mathcal{H}^*$ to M and C_i is a circle about p_i of radius ε, small enough to contain no other singularity. We have also noted that if $\mathcal{H}$ is transversal to the zero section then the singularities are isolated and their index is ± 1. In order to examine more fully transversality to the zero section, let U be a neighborhood of p such that the normal bundle over U is trivial, and assume C_i is contained in this neighborhood. Thus we may regard $\mathcal{H}$ as giving locally the map

$$\mathcal{H}:\ U \to U \times R^2.$$

It is transversal to the zero section if and only if $\pi \circ \mathcal{H}$: $U \to R^2$ is transversal to the origin, where π is projection of $U \times R^2$ onto the second factor. Let e_3e_4 be normal frames giving the local triviality, i.e. such that $\pi(x,\ e_3) =$

$(1, 0)$ and $\pi(x, e_4) = (0, 1)$. Then if we write $\mathcal{H} = \frac{1}{2}(a + c)e_3 + \frac{1}{2}(e + g)e_4$, we see that $\mathcal{H}$ is transversal to the zero section if and only if the Jacobian

$$\frac{\partial(a + c,\ e + g)}{\partial(u,\ v)}$$

has full rank at p. Hence, by the results of Section 5, we see that $X\colon M \to E^4$ is generic with respect to the model singularity $a + c = 0$, $e + g = 0$, if and only if $\mathcal{H}$ is transversal to the zero section. Consequently, the generic index of $\mathcal{H}$ as a normal vector field is ± 1.

Since the sum of the indices of a normal vector field is the EULER characterisitc of the normal bundle, we have the:

THEOREM 1.18. - $\chi(N) = \Sigma$ indices of $\mathcal{H}$. *In particular if* $\mathcal{H}$ *is never zero then* $\chi(N) = 0$.

Conversely, any orientable manifold may be imbedded with everywhere nonzero $\mathcal{H}$. Just imbed in E^3 and use stereographic projection to imbed in S^3. Then since it lies in S^3, $\mathcal{H}$ is never zero.

We come now to a rather interesting situation. Points where $\mathcal{H} = 0$ are singularities of the normal vector field $\mathcal{H}$, and also singularities of the tangent line element field discussed in Theorem 1.15. In the one case the generic index is ± 1, and in the other case it is $\pm 1/2$. Let us ask how the signs might be related. As a normal vector field, the sign is determined by the sign of the determinant of the Jacobian

$$\frac{\partial(a + c,\ e + g)}{\partial(u,\ v)},$$

and in the other case by the sign of the determinant of the Jacobian

$$\frac{\partial(ag - ce,\ af - be - (bg - cf))}{\partial(u,\ v)}.$$

Here we have chosen frames in a neighborhood of the singular point p where $\mathcal{H} = 0$, and we regard a, b, c, e, f, g as defined on M. Denote the first Jacobian above by J and the second by J'. Then at p we see that

$$J' = J\begin{pmatrix} g & f \\ a & -b \end{pmatrix}.$$

Just differentiate and substitute $c = -a$, $e = -g$. The determinant of

$$\begin{bmatrix} g & f \\ a & -b \end{bmatrix}$$

is equal to $-\frac{1}{2}N$ at p. If $\mathcal{H}=0$ and $N=0$, we have an inflection point. If we assume that inflection points and points where $\mathcal{H}$ vanishes are distinct, which is certainly a generic situation, then we may assume $N \neq 0$. Thus we have shown

THEOREM 1.19. - *A point p where $\mathcal{H}=0$ is a singularity of the normal vector field with generic index ± 1 and of a tangent line element field with generic index $\pm 1/2$. The signs agree if $N<0$ and disagree if $N>0$. Generically, we may assume that $\mathcal{H}=0$ and $N=0$ do not occur at the same point.*

THEOREM 1.20. - *Let $X: M \to E^4$ be an immersion of a compact orientable surface in E^4. Suppose that N, the curvature of the connection im the normal bundle, is everywhere positive (negative). Then X is an inflection free immersion and furthermore $\chi(N) = -2\chi(M)$ $(\chi(N) = 2\chi(M))$.*

PROOF. - A point where $N>0$ is certainly not an inflection point. Let us suppose that there are only a finite number of points where $\mathcal{H}$ vanishes. This is the generic situation. Let p be such a point. Let $\mathrm{Ind}_1(p)$ be the index where p is regarded as a singular point of the normal vector field, and let $\mathrm{Ind}_2(p)$ be the index when p is regarded as a singular point of the tangent line element field. Since $N>0$ we see by the previous theorem that $\mathrm{Ind}_1(p) = -2\,\mathrm{Ind}_2(p)$. Here again the previous theorem requires a generic situation. But $\chi(N) = \Sigma\,\mathrm{Ind}_1(p)$. Also, since X is inflection free, the only singularities of the tangent line element field are at points where $\mathcal{H}=0$. Thus by Proposition 1.7 $\chi(M) = \Sigma\,\mathrm{Ind}_2(p)$. The result now follows in the generic case. Hence by continuity it is true in general. The case when $N<0$ is similar.

We state the following theorem for its general interest. Let X: $M \to E^4$ be an immersion of a compact oriented surface. The tangential degree is the degree of the map e_1: $F_\tau \to S^3$ given by translating unit tangent vectors to the origin.

THEOREM 1.21. - *The tangential degree is equal to the Euler characteristic of the normal bundle. It is also equal to twice the algebraic number of double points.*

If the map X has only transversal double points and no triple points then the algebraic number of double points is just the double points counted with sign determined by their intersection numbers. With a suitable intersection theory, in order to be able to define the algebraic number of double points, the theorem is true for an arbitrary immersion X. WHITNEY [15] has made some of the original investigations concerning the algebraic number of double points. For generalizations of this theorem and further references consult LASHOF and SMALE [11] and WHITE [16].

We see using Theorem 1.21 that Theorem 1.20 may be rephrased to state that if $N<0$ everywhere on a surface M in E^4 then $\chi(M)$ is equal to the algebraic number of double points and if $N>0$ everywhere then $\chi(M)$ is minus the algebraic number of double points.

COROLLARY 1.22. - *Every immersion of the torus or the sphere must have a point where* $N=0$.

PROOF. - Since $\chi(N)=\frac{1}{2\pi}\int NdA$, we see that if $N>0$ (or $N<0$) everywhere then $\chi(N)>0$ (or $\chi(N)<0$). By Theorem 1.20, if $N>0$ (or $N<0$) everywhere then $\chi(M)<0$. Consequently, we obtain a contradiction if M is a torus or a sphere.

In the light of Theorem 1.20 it would be interesting to know of examples of immersions with everywhere positive N. We have not found any yet.

A sphere immersed in E^3 must have generically 4 umbilics [8]. The Caratheodory conjecture, which is only known in the case of real analytic surfaces, states that even in the non-generic case the surface must have 2 umbilics. Our computations of the generic indices give the generic number of singular points. Namely, a sphere immersed in E^4 must have generically 8 points where the ellipse is a circle and 4 points which are either inflection points or points where $\mathcal{H}=0$. One could then pose the Caratheodory type of question and ask how many singular points there must be even in the nongeneric case.

CHAPTER II.

General Position.

This chapter is concerned with general position or transversality arguments. Motivation and a preliminary discussion has been provided in Section 5 of the preceding chapter. Our treatment will, however, be more general than the discussion in Section 5. In particular we shall consider manifolds of arbitrary dimension and singularities of arbitrary order. We wish to acknowledge a heavy debt to FELDMAN [6, 7] throughout this entire chapter. Also we again assume, unless it is otherwise explicitly stated to the contrary, that all maps and manifolds are C^∞.

1. - Introduction and review of Feldman's results.

The abstract *pth* order tangent bundle, T_pM^n, of a differentiable manifold M^n is a vector bundle over M with fibre dimension

$$\nu(n, p) = \sum_{i=1}^{p} \binom{n+i-1}{i}.$$

If $x_1, \ldots, x_n$ are local coordinates on M then

$$\partial/\partial x_1|_q, \ \ldots, \ \partial/\partial x_n|_q, \ \partial^2/\partial x_1^2|_q, \ \partial^2/\partial x_1 \partial x_2|_q, \ \ldots, \ \partial^2/\partial x_n^2|_q, \ \ldots, \ \partial^p/\partial x_1^p|_q, \ \ldots, \ \partial^p/\partial x_n^p|_q$$

form a basis for the fibre over q. If $u_1, \ldots, u_n$ is a second coordinate system in a neighborhood of q then the coordinate transformations of the bundle are given by

$$\begin{aligned} &\partial/\partial u_i = \sum_j \partial x_j/\partial u_i \partial/\partial x_j, \\ (4)\qquad &\partial^2/\partial u_i \partial u_j = \sum_{k,l} \partial x_k/\partial u_i \partial x_l/\partial u_j \partial^2/\partial x_k \partial x_l + \sum_k \partial^2 x_k/\partial u_i \partial u_j \partial/\partial x_k, \end{aligned}$$

and so on, deriving the transformation law by successive differentiation. The structural group of T_pM^n is the group of linear transformations in the fibre induced by all possible coordinate changes on the base. This group we call $J^p(n)$. It is the group of invertible p jets from R^n to R^n with source and target the origin. This construction is functorial, namely if

$$f: M \longrightarrow N$$

is a map between manifolds, then there exists an induced map, $T_p(f)$, such that the diagram

$$\begin{array}{ccc} T_p(f): T_pM & \longrightarrow & T_pN \\ \downarrow & & \downarrow \\ f: M & \longrightarrow & N \end{array}$$

commutes. Also the following sequence of vector bundles in exact and natural.

$$0 \longrightarrow T_{p-1}M \longrightarrow T_pM \longrightarrow 0^p T_1 M \longrightarrow 0.$$

Suppose that M possesses a symmetric connection, for instance the LEVI-CIVITA connection in case M is a Riemannian manifold. Then this connection induces a map

$$D_{(i)}: T_iM \longrightarrow T_{i-1}M$$

which splits the sequence

$$0 \to T_{i-1}M \to \mathrm{T}_i M \to O^i T_1 M \to 0.$$

By composing the maps $D_{(2)} o D_{(3)} o \dots o D_{(p)}$ we get a map

$$D_p : T_p M \to T_1 M.$$

Consider the bundle $\mathrm{Hom}\,(T_p M,\ T_1 N)$, a vector bundle with base $M \times N$ and fibre the linear transformations from $T_p M_q$ to $T_1 N_r$ where $(q, r) \in M \times N$. Given a map $f: M \to N$, the induced map $D_p T_p(f): T_p M \to T_1 N$ is a vector bundle homomorphism covering f. Let $f^{-1}\,\mathrm{Hom}\,(T_p M,\ T_1 N)$ be the bull-back bundle induced by the map $id \times f: M \to M \times N$. It satisfies the commutative diagram

$$\begin{array}{ccc} f^{-1}\,\mathrm{Hom}\,(T_p M,\ T_1 N) & \to & \mathrm{Hom}\,(T_p M,\ T_1 N) \\ \downarrow & & \downarrow \\ M & \xrightarrow{id \times f} & M \times N. \end{array}$$

The fact that $D_p T_p(f)$ is a vector bundle homomorphism shows that f induces a cross-section of $f^{-1}\,\mathrm{Hom}\,(T_p M,\ T_1 N)$. This cross-section composed with the map into $\mathrm{Hom}\,(T_p M,\ T_1 N)$ we call $\widehat{f}$.

$$\widehat{f}: M \to \mathrm{Hom}\,(T_p M,\ T_1 N).$$

Assume $f: M \to E^k$. We note that the image of $T_p M$ under $D_p T_p(f)$ is the *pth* order osculating bundle and the image of each fibre is the *pth* order osculating space. (For a map $f: M \to N$ we take this as the definition of the osculating bundle). The *pth* order osculating space at q is defined to be the span of the $(p-1)$-*st* order osculating space at q together with all the *pth* derivatives at q of curves through q.

Since $T_p M$ is a bundle with group J^p and $T_1 N^k$ may be taken as a bundle with group $O(k)$ we see that $\mathrm{Hom}\,(T_p M,\ T_1 N)$ is a bundle with group $J^p \times O(k)$. Let the fibre, $\mathrm{Hom}\,(T_p M,\ T_1 N)_{(q,r)}$, be called F. F is a vector space of dimension $\nu(n, p) \cdot k$ on which the group $J^p \times O(k)$ acts. Suppose that K is a subvariety of F invariant under $J^p \times O(k)$. Such a subvariety is called *model singularity*. By picking out this same subvariety in each fibre we have a bundle $K(M \times N)$ over $M \times N$ with fibre K.

We define a point $q \in M$ to be a K *singular point* of $\widehat{f}$ if $\widehat{f}(q)$ meets $K(M \times N)$. A point q is called K *jet transversal* if $\widehat{f}$ is transversal to $K(M \times N)$ at q. (Feldman's terminology is «K generic»). If every point is K jet transversal we say that the map f is K *jet transversal*.

The following lemma will be of use; its proof is trivial.

LEMMA 2.1. - *Let K be a submanifold of F and G a manifold. Let $\pi: F \times G \to F$ be the projection and let $f: U \to F \times G$ be a map of manifolds. Then $\pi o f(U)$ is transversal to K if and only if $f(U)$ is transversal to $K \times G$.*

Now let q be *a* K singular point of f. Since being a singular point is only local we may choose a neighborhood U of q such that there is a neighborhood V of $f(q)$ such that $U \times V$ trivializes the bundle $\mathrm{Hom}(T_pM, T_1N)$ Then over $U \times V$ the bundle has the form $U \times V \times F$, F the fibre. If we let $\pi: U \times V \times F \to F$ be the projection we see by Lemma 2.1 that q is a K jet transversal singular point if and only if $\pi\widehat{f}$ is transversal to K at q.

Something should be said about what it means for $\pi\widehat{f}$ to be transversal if $\pi\widehat{f}(q)$ is a singular point of the algebraic variety. Let us, however, give only a few vague comments and reference FELDMAN [7, pp. 194] for a discussion and further references. Transversality at a singular point of an algebraic variety is defined by choosing in some way (for example, as was described in Chapter I, Section 5) a finite collection of submanifolds, and then defining $\pi\widehat{f}$ to be transversal to the variety at the singular point if and only if it is transversal to each submanifold.

A result found in FELDMAN [7, pp. 185-186] but not stated as a theorem is:

THEOREM 2.2. - *If f is K jet transversal then the locus of K singular points has codimension in M equal to the codimension of K in F, where F is the fibre of $Hom(T_pM, T_1N)$.*

We next review some topological notions. Let $C^0(M, N)$ be the continuous functions from M to N with a topology to be described below. Here we still assume that M and N are C^∞ manifolds. Choose a metric D on N. For each continuous povitive real valued function, δ, on M, let $N_\delta(f) = \{g \mid D(g(x), f(x)) < < \delta(x)\}$. The topology is defined to be that given by taking the $N_\delta(f)$ as basis. It is independent of the choice of metric D on N, compatible of course with the manifold topology. In contrast with the topology of uniform convergence on compact sets, or, if you prefer, the compact open topology, we might call this the *fine* topology of $C^0(M, N)$. It is equal to the compact open topology if and only if M is compact.

We remark that $f_i \to f$ in this topology means that the sets $U_i = \{x \mid f_i(x) \neq \neq f(x)\}$, from some i on, are all contained in some compact set and that the convergence is uniform on this set.

For each $f \in C^\infty(M, N)$ we have the map $T_p(f): T_pM \to T_pN$. The fact that this is a vector bundle homomorphism implies that there is a map

$$e_p: C^p(M, N) \to C^0(M, \mathrm{Hom}(T_pM, T_pN))$$

defined by $e_p(f)(x) = T_p(f)_x$. e_p is clearly one to one. On $C^0(M, \mathrm{Hom}(T_pM, T_pN))$ place the fine topology described above, and let $C^p(M, N)$ have the topology induced under the map e_p.

Note that $C^{p+1}(M, N) \subset C^p(M, N)$ and that the inclusions are continuous in the above described topologies. This gives an inverse limit system. We define the topology on $C^\infty(M, N)$ to be the inverse limit topology.

This discussion of function space topologies is taken, with minor changes from the appendix in FELDMAN [6, pp. 220–223]. We are now in a position to state the following theorem due to FELDMAN [7, Prop. 3 2, p. 194].

THEOREM 2.3. – *The set of functions in $C^\infty(M, N)$ which is K jet transversal is dense in $C^\infty(M, N)$. K is, of course, a model singularity.*

2. – The group J^p.

Suppose that $x_1, \ldots, x_n$ are coordinates in a neighborhood of a point $q \in M$. These coordinates give rise to a basis for the fibre of T_pM,

$$\partial/\partial x_1, \ \ldots, \ \partial^p/\partial x_n^p,$$

where we order the basis by taking the first derivatives, the second derivatives, and so on, and among the derivatives of a given order we order them lexicographically. Let us write

$$X_1 = \begin{bmatrix} \partial/\partial x_1 \\ \cdot \\ \cdot \\ \cdot \\ \partial/\partial x_n \end{bmatrix} \quad X_2 = \begin{bmatrix} \partial^2/\partial x_1^2 \\ \partial^2/\partial x_1 \partial x_2 \\ \cdot \\ \cdot \\ \cdot \\ \partial^2/\partial x_n^2 \end{bmatrix} \quad \ldots \quad X_p = \begin{bmatrix} \partial^p/\partial x_1^p \\ \cdot \\ \cdot \\ \cdot \\ \partial^p/\partial x_n^p \end{bmatrix}$$

and call the basis $X_1, \ldots, X_p$. Then an element $L \in J^p$ may be written as a matrix which we also call L. The matrix L may be broken into blocks (l_{ij}), where l_{ij} are defined by

$$L(X_i) = \sum_j l_{ij} X_j .$$

Because $L \in J^p$ we know that $L(X_i)$ will also form a basis, in fact, a basis induced from coordinates, say $u_1, \ldots, u_n$. The blocks l_{ij} may be found from equations 4) and the successive equations generated by application of the chain rule. From this it is obvious that $l_{ij} = 0$ for $i < j$. Now, of course, $X_1, \ldots, X_r$, $r \leq p$ is a basis for T_rM, which may be regarded as a subspace of T_pM. That $l_{ij} = 0$ for $i < j$ reflects the fact that each T_rM is an invariant subspace of T_pM under the action of J^p. Another fact, readily seen by induction, is that $l_{r1} = (\partial^r x_j \ \partial u_{i_1} \ldots \partial u_{i_r})$. Not quite so obvious is the fact that if $l_{21}, \ldots, l_{r-1 1}$ are all zero at the point then $l_{r2}, \ldots, l_{r\, r-1}$ are also all zero. To see this note that we may express the entries of l_{rs} as a polynomial in the partial derivatives, $\partial x^k/\partial_{i_1} \dots \partial u_{i_k}$, of orders upto and including r, i.e. in the entries of $l_{11}, l_{21}, \dots, l_{r1}$. These polynomials have integer coefficients. They, of course, depend on which entry is chosen but on nothing else. The may be found by an inspection of equations 4) and the equations generated from them by successive differentiation. Let a polynomial for some entry of l_{rs} be

$$\mathfrak{F}(\partial x_k/\partial u_i,\ \partial^2 x_k/\partial u_{i_1}\partial u_{i_2},\ \dots,\ \partial^r x_k/\partial u_{i_1} \dots \partial u_{i_r}).$$

It may be shown by induction that these polynomials are homogeneous of degree s and of weight r. That is

$$\mathfrak{F}(t\partial x_k/\partial u_i,\ \dots,\ t\partial^r x_k/\partial u_{i_1} \dots \partial u_{i_r}) = t^s \mathfrak{F}(\partial x_k/\partial u_i,\ \dots,\ \partial^r x_k/\partial u_{i_1} \dots \partial u_{i_r})$$

and

$$\mathfrak{F}(t\partial x_k/\partial u_i,\ t^2\partial^2 x_k/\partial u_{i_1}\partial u_{i_2},\ \dots,\ t^r\partial x_k/\partial u_{i_1} \dots \partial u_{i_r}) =$$

$$= t^r \mathfrak{F}(\partial x_k/\partial u_i,\ \dots,\ \partial^r x_k/\partial u_{i_1} \dots \partial u_{i_r}).$$

Now if $l_{21}, \dots, l_{r-1\,1}$ are all equal to zero then the partial derivatives from the second through the $(r-1)$-*st* order are all zero at the point. Thus the entry of l_{rs} has the form

$$\mathfrak{F}(\partial x_k/\partial u_i,\ 0,\ \dots,\ 0,\ \partial^r x_k/\partial u_{i_1} \dots \partial u_{i_r}),$$

but since it is homogeneous of degree s and of weight r it must be zero.

The matrix l_{11} may be regarded as a linear transformation of T_1M_q via the basis X_1, namely the restriction of L to T_1M_q. Let l^i_{11} be the induced linear transformation of the i-fold symmetric product of T_1M_q.

$$l^i_{11} : O^i T_1 M_q \to O^i T_1 M_q .$$

Via the basis $\partial/\partial x_{j_1} o \dots o \partial/\partial x_{j_i}$, $j_1 \leq \dots \leq j_i$, ordered lexicographically, on $O^i T_1 M_q$, the linear transformation l^i_{11} is given by a matrix which is equal to l_{ii}. This shows that $l_{11} =$ identity implies that $l_{ii} =$ identity, and that if l_{11} is nonsingular then l_{ii} is nonsingular for $1 \leq i \leq p$.

3. - A-special choice of basis for F.

Recall that each point A of F is a linear transformation from T_pM_q to T_1N_r, $(q, r) \in M \times N$. We ask if we may choose a basis for T_pM_q and for T_1N_r so that the matrix for A relative to these bases has an especially simple form. Let us choose an orthonormal basis $e_1, \dots, e_k$ on T_1N_r in such a way that $e_1, \dots, e_{n_i}$ is a basis for $A(T_iM_q)$ for each $1 \leq i \leq s$, where $s \leq p$ and

$n_i = \dim A(T_i M_q)$. $e_{n_s+1}, \ldots, e_k$ just complete $e_1, \ldots, e_{n_s}$ to a basis for $T_1 N_r$. We partition the basis into $E_1, \ldots, E_{s+1}$ where $E_1 = (e_1, \ldots, e_{n_1})$, $E_i = (e_{n_{i-1}}, \ldots, e_{n_i})$, $i = 2, \ldots, s$ and $E_{s+1} = (e_{n_s+1}, \ldots, e_k)$ and refer to the basis as $E_1, \ldots, E_{s+1}$. A choice of coordinates $x_1, \ldots, x_n$ in a neighborhood of $q \in M$ induces a basis $X_1, \ldots, X_p$ for $T_p M_q$. In terms of these bases we may split the matrix for A into blocks

$$A(X_i) = \sum_{j=1}^{s+1} A_{ij} E^j .$$

We have now established the notation for

THEOREM 2.4. - *For any element $A \in F$ such that A restricted to $T_1 M_q$ is $1-1$ and also for any orthonormal frame $E_1, \ldots, E_{s+1}$ on $T_1 N_r$ such that $E_1, \ldots, E_i$ spans $A(T_i M_q)$ $i = 1, \ldots, s$ it is possible to choose a basis for $T_p M_q$ which is induced from coordinates in a neighborhood of q and such that the matrix for A relative to this pair of bases takes the form*

$$\begin{bmatrix} I_d & 0 & \cdot & \cdot & \cdot & 0 \\ & & \cdot & & & \cdot \\ & & & \cdot & & \cdot \\ & & & & \cdot & \cdot \\ 0 & A_{22} & & & & 0 \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & A_{s+1\, s+1} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & A_{p2} & \cdot & \cdot & \cdot & A_{p\, s+1} \end{bmatrix}.$$

Furthermore, the basis for $T_p M_q$ is unique.

PROOF. - The fact that $T_j M_q$ is spanned by $X_1, \ldots, X_j$ and $A(T_j M_q)$ by $E_1, \ldots, E_j$, $j = 1, \ldots, s$ shows that $A_{ij} = 0$ for $i < j$. Because A restricted to $T_1 M_q$ is $1-1$, $n = n_1$ and the block A_{11} is an invertible $n \times n$ matrix. We choose any coordinates on M and then make successive coordinate changes until the desired coordinates have been found. Let us, therefore, examine what happens to the matrix for A when the coordinates are changed. Suppose $X_1, \ldots, X_p$ and $E_1, \ldots, E_{s+1}$ are «old» bases for $T_p M_q$ and $T_1 N_r$ respectively. Let L be a linear transformation of $T_p M_q$ induced by a coordinate change on M. Then, relative to the old bases, A and L are given by matrices we call by te same name. We ask what is the matrix for A in the bases $L(X_1), \ldots$ $\ldots, L(X_p)$, $E_1, \ldots, E_{s+1}$. It is just LA.

Notice that in this new basis $A_{ij}=0$ for $i<j$ for exactly the same reason. In block form LA looks like

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ & & \ddots & \vdots \\ l_{21} & l_{22} & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & & 0 \\ l_{p1} & l_{p2} & \cdots & l_{pp} \end{bmatrix} \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & & 0 \\ A_{s+1\,1} & & \cdots & A_{s+1\,s+1} \\ \vdots & & & \vdots \\ A_{p1} & & \cdots & A_{ps+1} \end{bmatrix}$$

Since A_{11} is invertible (A restricted to T_1M_q is 1-1) we may choose $l_{11}=A_{11}^{-1}$. Then in the new matrix for A the A_{11} block will be the identity. Under a further change of coordinates since we wish to retain $A_{11}=$identity it is easy to see that it is necessary that the block l_{11} of the new matrix L be the identity. This implies also that $l_{22}, \ldots, l_{pp}$ are all the identity. Under a basis change such that $l_{11}=$identity, the A_{21} block becomes $l_{21}+A_{21}$. Also we may choose l_{21} freely. It is the matrix of second partial derivatives and at a point is independent of the first partials Thus let us choose $l_{21}=-A_{21}$. (They are the same dimension). In the new coordinates the matrix for A has $A_{11}=$ identity and $A_{21}=0$.

Suppose now that we have, by successive coordinate changes, achieved $A_{11}=$identity, $A_{21}=0, \ldots, A_{j-11}=0$. In order to retain this much, it is not difficult to see that it is necessary that $l_{11}=$ indentity, $l_{21}=0, \ldots, l_{j-1\,1}=0$. From our earlier remarks on L in Section 2 we note that if $l_{21}, \ldots, l_{j-11}$ are all zero then $l_{\alpha\beta}$ are zero for $1<\beta<\alpha\leq j$. Thus in block form the matrix LA looks like

$$\begin{bmatrix} Id & 0 & \cdots & & & & \cdots & 0 \\ 0 & \ddots & & & & & & \vdots \\ \vdots & & \ddots & & & & & \vdots \\ \vdots & & & \ddots & & & & \vdots \\ l_{j1} & 0 & \cdots & 0 & Id & & & \vdots \\ l_{j-1\,1} & \cdots & & \cdots & l_{j+1\,j} & Id & & \vdots \\ \vdots & & & & & \ddots & \ddots & \vdots \\ \vdots & & & & & & \ddots & 0 \\ l_{p1} & \cdots & & & & \cdots & l_{pp-1} & Id \end{bmatrix}$$

$$\begin{bmatrix} Id & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ 0 & A_{22} & \cdot & \cdot \\ \vdots & & \ddots & \\ 0 & & \cdot & 0 \\ A_{j1} & \cdot & & A_{s+1\ s+1} \\ \vdots & \cdot & & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{p\ s+1} \end{bmatrix}$$

We see that under such an L the block A_{j1} becomes $l_{j1} + A_{j1}$. Again we are free to choose $l_{j1} = -A_{j1}$ so that we have now achieved, after this coordinate change, $A_{j1} = 0$. Also in order to retain this much under the next coordinate change, it is easy to check that we must have $l_{11} =$ identity, $l_{21}, \ldots, l_{j1} = 0$ Consequently continuing in this fashion the theorem is established.

Note that if we change not only the basis $X_1, \ldots, X_p$ by L but also the basis $E_1, \ldots, E_{s+1}$ by an orthogonal transformation O of T_1N_r the new matrix for A in terms of $L(X_1), \ldots, L(X_p)$ and $O(E_1), \ldots, O(E_{s+1})$ is

$$LA^tO,$$

where L, A and O are matrices for the linear transformations L, A and O in the old basis.

4. - The orbits of $J^p \times O(k)$.

At this point we switch our point of view. We have until now considered a fixed linear transformation A and tried to choose bases so that the corresponding matrix had a particularly simple form. We now fix a pair of bases $X_1, \ldots, X_p$ and $E_1, \ldots, E_{s+1}$, for short (X, E), and consider the linear tranformations whose matrices relative to this fixed choice of bases are particularly simple. Let $\tilde{F} = \tilde{F}(n_1, \ldots, n_s)$ be the subset of F consisting of those linear transformations, A, such that $\dim A(T_iM_q) = n_i$, $i = 1, \ldots, s$. We assume that $n_1 = n$ and of course that $n_i \leq \nu(n, i)$, where $\nu(n, i)$ is the dimension of the fibre of T_iM^n. Let $Z = Z(n_1, \ldots, n_s)$ be the subset of $\tilde{F}$ consisting of elements A such that the block matrix for A relative to the bases (X, E) is of the form $A_{11} =$ $=$ identity, $A_{ij} = 0$ for $i < j$, and $A_{j1} = 0$ for $j = 2, \ldots, p$. Then the above theorem may be restated as follows.

COROLLARY 2.5. – *Every orbit under $J^p \times O(k)$ of a point of $\tilde{F}(n, n_2, \ldots, n_s$ meets $Z(n, n_2, \ldots, n_s)$. Conversely if an orbit meets $Z(n, n_2, \ldots, n_s)$ it must lie in $\tilde{F}(n, n_2, \ldots, n_s)$.*

The converse follows because every point of F belongs to $\tilde{F}$ for some choice of $n_1, n_2, \ldots, n_s$ and because the numbers $n_1, n_2, \ldots, n_s$ are invariant under the action of $J^p \times O(k)$.

Let us now show

THEOREM 2.6. – *If a member of $J^p \times O(k)$ maps an element of Z again into Z it leaves Z setwise fixed.*

PROOF. – Take $A, B \in Z$ and suppose for $(L, O) \in J^p \times O(k)$ we have $LA\,{}^tO = B$, or in block form

$$\begin{bmatrix} l_{11} & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ l_{p1} & \cdots & \cdots & \cdots & l_{pp} \end{bmatrix} \begin{bmatrix} Id & 0 & \cdots & \cdots & 0 \\ & & \ddots & & \vdots \\ & & & \ddots & 0 \\ 0 & A_{22} & \cdot & & \\ \vdots & \vdots & & \ddots & A_{s+1\,s+1} \\ \vdots & \vdots & & & \vdots \\ 0 & A_{p2} & \cdots & \cdots & A_{p\,s+1} \end{bmatrix}$$

$$\begin{bmatrix} \theta_{11} & \cdots & \cdots & \theta_{1\,s+1} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \theta_{s+1\,1} & \cdots & \cdots & \theta_{s+1\,s+1} \end{bmatrix} = \begin{bmatrix} Id & 0 & \cdots & \cdots & 0 \\ 0 & B_{22} & \cdot & & \vdots \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ \vdots & \vdots & & & B_{s+1\,s+1} \\ \vdots & \vdots & & & \vdots \\ 0 & B_{p2} & \cdots & \cdots & B_{p\,s+1} \end{bmatrix}$$

where we let $\theta = {}^tO$ for convenience. θ is also orthogonal. The top row of blocks of $LA\theta$ is

$$(l_{11}\theta_{11},\ l_{11}\theta_{12},\ \ldots,\ l_{11}\theta_{1\,s+1}).$$

Thus $l_{11}\theta_{11} =$ identity, and since l_{11} is nonsingular so is θ_{11}. Also we see that $l_{11}\theta_{1j} = 0$, $2 \leq j \leq s+1$ so that, since l_{11} is nonsingular, $\theta_{1j} = 0$. But since θ is orthogonal this implies that $\theta_{21}, \ldots, \theta_{s+1\,1}$ are also zero. Using these facts we see that the left hand column of $LA\theta$ is

$$\begin{bmatrix} l_{11}\ \theta_{11} \\ l_{21}\ \theta_{11} \\ \cdot \\ \cdot \\ \cdot \\ l_{p1}\ \theta_{11} \end{bmatrix}.$$

Therefore we must have $l_{j1}\theta_{11} = 0,\ 2 \leq j \leq p$. But θ_{11} is nonsingular so that $l_{j1} = 0,\ 2 \leq j \leq p$. Hence by the property of L discussed in Section 2 we also have $l_{j1} = 0$ for $2 \leq j \leq i$.

We now show that $\theta_{ij} = 0$ for $i \neq j$. Let us proceed by induction. Suppose $\theta_{ij} = 0$ for i or $j < t,\ i \neq j$. The matrix $LA\theta$ now looks like

$$\begin{bmatrix} l_{11} & 0 & \cdot & \cdot & \cdot & 0 \\ & \cdot & & \cdot & & \\ 0 & & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & l_{tt} & \cdot & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & l_{pp} \end{bmatrix} \begin{bmatrix} I_d & 0 & \cdot & \cdot & \cdot & 0 \\ & & \cdot & & & \cdot \\ 0 & A_{22} & \cdot & \cdot & & \cdot \\ \cdot & \cdot & A_{tt} & & \cdot & 0 \\ \cdot & \cdot & & & \cdot & \\ \cdot & \cdot & & & & A_{s+1s+1} \\ 0 & A_{p2} & \cdot & \cdot & \cdot & A_{ps+1} \end{bmatrix}$$

$$\begin{bmatrix} \theta_{11} & \cdot & 0 & \cdot & \cdot & \cdot & 0 \\ & \cdot & & \cdot & & & \cdot \\ 0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 \\ \cdot & & \cdot\ 0 & \theta_{tt} & \cdot & \cdot & \theta_{ts+1} \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ 0 & \cdot & \cdot\ 0 & \theta_{s+1t} & \cdot & \cdot & \theta_{s+1s+1} \end{bmatrix}$$

Consider the *t-th* row of blocks in $LA\theta$. It is

$$(0,\ l_{tt}A_{t2}\theta_{22},\ \ldots,\ l_{tt}A_{tt}\theta_{tt},\ l_{tt}A_{tt}\theta_{tt+1},\ \ldots,\ l_{tt}A_{tt}\theta_{ts+1}).$$

Hence we must have $l_{tt}A_{tt}\theta_{ti} = 0$ for $t < i \leq s + 1$. But l_{tt} is nonsingular since l_{11} was nonsingular. So we must have $A_{tt}\theta_{ti} = 0$ for $t < i \leq s + 1$. Since the maximum possible rank of A_{ii} is $n_i - n_{i-1}$ and since the matrix

$$\begin{bmatrix} Id & 0 & . & . & . & 0 \\ & & . & & & . \\ 0 & A_{22} & . & & & . \\ . & . & . & & . & . \\ . & . & & . & . & . \\ . & . & & & . & 0 \\ 0 & A_{s2} & . & . & . & A_{ss} \end{bmatrix}$$

has rank n_s and is in block triangular form, we must have rank $A_{ii} = n_i - n_{i-1}$, $i = 2, \ldots, s$. Thus A_{tt} has full rank. So $\theta_{ti} = 0$ for $t < i \leq s + 1$. Because θ is orthogonal, we must have $\theta_{it} = 0$ for $t < i \leq s + 2$, and so we have $\theta_{ij} = 0$ for i or $j < t + 1$, $i \neq j$. The induction ends with $t = s$, however, this is far enough to show that $\theta_{ij} = 0$ if $i \neq j$. Thus we see that L and θ are block diagonal matrices and furthermore that $l_{11}\theta_{11} =$ identity. This demonstrates the theorem.

Since we showed that θ must be in block diagonal form, all the blocks themselves must be orthogonal. Since $l_{11}\theta_{11} =$ identity, the matrix L is completely determined by θ_{11}, namely $l_{ii} = (\theta_{11}^{-1})^i$ and $l_{ij} = 0$, $i \neq j$. In fact the action of $J^p \times O(k)$ is completely described by saying that $(\theta_{11}^{-1})^i A_{ij}\theta_{jj} = B_{ij}$, or in terms of O

$$O_{11}A_{ij}^t O_{jj} = B_{ij}.$$

Here $1 < j < i$. Thus on Z the action of $J^p \times O(k)$ reduces to an action of

$$O(n) \times O(n_2 - n) \times \ldots \times O(n_s - n_{s-1}) \times O(k - n_s)$$

which we, for short, call $H(n, n_2, \ldots, n_s) = H$. Therefore, we may state the following

COROLLARY 2.9. - *The subgroup of $J^P \times O(k)$ which leaves Z setwise fixed is H.*

Also notice that the elements of $J^p \times O(k)$ which leave a point of Z fixed are a subgroup of H. Therefore, we have another

COROLLARY 2.8. - *At a point of Z the isotropy group with respect to $J^P \times O(k)$ is equal to the isotropy group with respect to H.*

By Corollary 2.5 and Theorem 2.6 we see that if K is an orbit of $\tilde{F}$ under $J^p \times O(k)$ then $K \cap Z$ is an orbit of Z under H.

Also, if K is an invariant subvariety of $\tilde{F}$ then $K \cap Z$ is an invariant subvariety of Z under H. Let us call the correspondence between invariant subvarietes of $\tilde{F}$ under $J^p \times O(k)$ and invariant subvarieties of Z under H ρ. ρ is given by $\rho(K) = K \cap Z$. We claim that ρ is $1-1$ and onto as a correspondence of invariant subvarieties. To see that ρ is onto, let K_Z be an invariant subvariety of Z under H. Let K be the union of all orbits under $J^p \times O(k)$ which pass through K_Z. K is the «closure» of K_Z. This idea of closure may be made precise as follows. Let all the invariant subvarieties of F under $J^p \times O(k)$ be closed sets. Then one may check that the union of two closed sets is again closed and the intersection of any family of closed sets is closed. Thus we have a topology. The closure above is in the sense of this topology. ρ is $1-1$ because the closure of a set is unique. We may summarize these facts in:

PROPOSITION 2 9. - *The subvarieties of $\tilde{F}(n, n_2, \ldots, n_s)$, invariant under $J^p \times O(k)$, and the subvarieties of $Z(n, n_2, \ldots, n_s)$, invariant under $H(n, n_2, \ldots, n_s)$ are in $1-1$ correspondence by a correspondence ρ given by $\rho K = K \cap Z$. Also, if K is an orbit of $\tilde{F}$ under $J^p \times O(k)$ then $K \cap Z$ is an orbit of Z under H.*

Let us next turn to the infinitesimal version of Theorem 2.6, namely, that if an infinitesimal element of $J^p \times O(k)$ maps a point of Z into an infinitesimally nearby point of Z then that element is already an infinitesimal element of H. More precisely, we wish to prove:

THEOREM 2.10. - *If an infinitesimal transformation of $J^p \times O(k)$ maps a point of Z into a tangent vector to Z then the infinitesimal transformation is an infinitismal transformation of H.*

PROOF. - Let $d(L, O)$ be an infinitesimal transformation of $J^p \times O(k)$ and let $A \in Z$. By definition

$$d(L, 0)(A) = \lim_{t \to 0} \frac{1}{t} [L(t) A^t O(t) - A],$$

where $(L(t), O(t))$ is a one-parameter subgroup of $J^p \times O(k)$, beginning at the identity, i.e. $L(0) =$ indetity, $O(0) =$ identity.

The theorem and proof are very similar to the macroscopic version and may possibly be a consequence of it on general principles.

We begin the proof with an infinitesimal version of Section 2.

Consider a one-parameter subgroup of J^p, say $L(t)$. Suppose $L(0) =$ identity. Write the infinitesimal transformation $dL = \lim_{t \to 0} \frac{1}{t} (L(t) - Id.)$; in block form

$$\begin{bmatrix} dl_{11} & 0 & \cdot & \cdot & \cdot & 0 \\ & \cdot & & & & \cdot \\ dl_{21} & dl_{22} & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & 0 \\ dl_{p1} & dl_{p2} & \cdot & \cdot & \cdot & dl_{pp} \end{bmatrix}.$$

dl_{11} is the infinitesimal transformation along $l_{11}(t)$, where we regard $l_{11}(t)$ as a one-parameter subgroup of J^1. Also dl_{ii} is the infinitesimal transformation along $l_{ii}(t) = (l_{11}(t))^i$.

Hence if we can show that dl_{11} is an infinitesimal transformation in $O(n)$ then dl_{ii} will be an infinitesimal transformation of $(O(n))^i$; in fact, the image of the same one parameter subgroup, only under a different representation.

If $dl_2, \ldots, dl_p$ are all zero then also $dl_{\alpha\beta} = 0$ for $1 < \beta < \alpha \leq p$. To see this, let a be a typical element of $l_{\alpha\beta}$.

Then by the discussion in Section 2, $a = \mathfrak{F}(l_{11} \ldots, l_{\alpha 1})$. By this we mean that a is a polynomial in the entries of $l_{11}, \ldots, l_{\alpha 1}$.

We know that $\mathfrak{F}$ is homogeneous of degree β and weight α.

Consequently no term may contain only entries from l_{11}. Let a' be a typical term of $\mathfrak{F}$. Hence $a' = b\pi$ where b is an entry of one of $l_{21}, \ldots, l_{\alpha 1}$, and π is a monomial in the entries of $l_{11}, \ldots, l_{\alpha 1}$. Since $L(0) =$ identity, the elements of $l_{21}, \ldots, l_{p1}$ are zero at 0 and thus $b(0) = 0$. Consequently $a'(0) = 0$. Thus it is euough to show that

$$\lim_{t \to 0} \frac{a'(t)}{t} = 0.$$

But $a'(t) = b(t)\pi(t)$ and if we linearize we have

$$b(t) = tdb,$$

$$\pi(t) = K + td\pi,$$

where db, K, $d\pi$ are just constants. Thus

$$\lim_{t \to 0} \frac{a'(t)}{t} = Kdb.$$

Since $dl_{21}, \ldots, dl_{p1}$ are all zero and since db is an entry of one of them we have $db = 0$ which by the above suffices to show that $dl_{\alpha\beta} = 0$.

To continue the proof let us suppose that $(L(t), O(t))$ is a one-parameter subgroup of $J^p \times O(k)$ beginning at the identity so that $L(t)$ is a one-parameter subgroup of J^p and $O(t)$ is a one-parameter subgroup of $O(k)$. Let dL and dO be the corresponding infinitesimal transformations, namely

$$dL = \lim_{t\to 0} \frac{L(t) - Id}{t} \quad dO = \lim_{t\to 0} \frac{O(t) - Id}{t}.$$

Then one knows because our group acts like

$$A \to LA\theta ,$$

where $\theta = {}^tO$ that

$$d(L,\ O)(A) = dLA + Ad\theta .$$

$d\theta$ is the infinitesimal transformation defined by ${}^tO(t)$, or $d\theta = - dO$. Suppose dC is a tangent vector to Z. Then the hypothesis is

$$dLA + Ad\theta = dC,$$

or in block form:

$$\begin{bmatrix} dl_{11} & 0 & \cdots & 0 \\ dl_{21} & dl_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & & 0 \\ dl_{p1} & dl_{p2} & \cdots & dl_{pp} \end{bmatrix} \begin{bmatrix} Id & 0 & \cdots & 0 \\ 0 & A_{22} & \ddots & 0 \\ \vdots & \vdots & & A_{s+1s+1} \\ \vdots & \vdots & & \vdots \\ 0 & A_{p2} & \cdots & A_{ps+1} \end{bmatrix}$$

$$+ \begin{bmatrix} Id & 0 & \cdots & 0 \\ 0 & A_{22} & \ddots & 0 \\ \vdots & \vdots & & A_{s+1s+1} \\ \vdots & \vdots & & \vdots \\ 0 & A_{p2} & \cdots & A_{ps+1} \end{bmatrix} \begin{bmatrix} d\theta_{11} & \cdots & d\theta_{1s+1} \\ \vdots & & \vdots \\ d\theta_{s+11} & \cdots & d\theta_{s+1s+1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \vdots & dC_{22} & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ \vdots & \vdots & & dC_{s+1s+1} & \\ \vdots & \vdots & & & \vdots \\ 0 & dC_{p2} & \cdots & dC_{ps+1} & \cdot \end{bmatrix}$$

$d\theta$ is skew symmetric since θ was orthogonal and since $d\theta_{ii}$ is a square block on the diagonal, $d\theta_{ii}$ is also skew. Very similarly now to the previous proof equate the top row of dLA, $Ad\theta$, and dC obtaining

$$dl_{11} + d\theta_{11} = 0$$

and $d\theta_{1i} = 0$, $i = 2, \ldots, s+1$.
Since $d\theta$ is skew symmetric this implies that $d\theta_{i1} = 0$, $i = 2, \ldots, s+1$. Next equate the left hand columns of dLA, $Ad\theta$, and dC, obtaining $dl_{i1} = 0$, $i = 2, \ldots, p$. From this we conclude that $dl_{ij} = 0$ for $2 \leq j < i \leq p$. Next, as before, proceed by induction, assuming that $d\theta_{ij} = 0$ for i or $j < t$, $i \neq j$. Then by a consideration of the t^{th} row of blocks we see that

$$A_{tt} d\theta_{ti} = 0 \quad \text{for} \quad i < t \leq s+1.$$

and hence, as before, since A_{tt} is full rank, $d\theta_{ti} = 0$ for $i = t+1, \ldots, s+1$. By the skew symmetry of $d\theta$, $d\theta_{it} = 0$ for $i = t+1, \ldots, s+1$ and thus $d\theta_{ij} = 0$ for i or $j < t+1$, $i \neq j$. Hence dL is a block diagonal matrix with diagonal blocks $(dl_{11})^i$, $d\theta$ is a block diagonal matrix with diagonal blocks $d\theta_{ii}$, all of which are skew symmetric, and also $dl_{11} + d\theta_{11} = 0$. From this we conclude that $d(L, O)$ is an infinitesimal transformation of H.

Using the fact that $K \cap Z$ is an orbit under H (Proposition 2.9) this theorem may be restated as:

COROLLARY 2.11. - *$TK \cap TZ = T(K \cap Z)$ at any point of $K \cap Z$. Here T means the tangent space at the point.*

We now ask if the orbit of any point of $\tilde{F}$ does not meet Z transversally. Both Z and an orbit which meets Z lie in $\tilde{F}$; recall Corollary 2.5. This means that the only possibility of transversality is transversality in $\tilde{F}$. However, jet transversality requires transversality in F. If $\tilde{F}$ is open in F then, of course, transversality in F and $\tilde{F}$ are the same. For a choice of $n, n_2, \ldots, n_s$ such that $\tilde{F}$ is not open in F there is no possibility of transversality in F because $\dim \tilde{F} < \dim F$.

THEOREM 2.12. - *If $n, n_2, \ldots, n_s$ are chosen so that $\tilde{F}(n, n_2, \ldots, n_s)$ is open in F then any orbit of F is transversal to $Z(n, n_2, \ldots, n_s)$.*

PROOF. - Note that if for some j, $n_j = \nu(n, j)$ then $n_i = \nu(n, i)$ for $1 < i \leq j$, and that if for some j, $n_j = k$ then $n_i = k$ for $j \leq i \leq s$. Thus the possible choices of $n, n_2, \ldots, n_s$ which make $\tilde{F}$ open in F are

$$\text{for some } j,\ 1 \leq j \leq s,\ n_j \leq \nu(n, j) \text{ and } n_{j+1} = k.$$

Here if $j = s$ the condition $n_{j+1} = k$ is vacuous.

Let K be an orbit of F. If K does not meet Z then transversality is trivial. So suppose $K \cap Z \neq \emptyset$. By Corollary 2.11 it remains only to show that

$$\dim F - \dim K = \dim Z - \dim (Z \cap K).$$

Now $\dim K \cap Z = \dim H - \dim$ (isotropy group of a point of $Z \cap K$ with respect to H) and $\dim K = \dim J^p \times O(k) - \dim$ (isotropy group at the same point with respect to $J^p \times O(k)$). Since these isotropy groups are equal they certainly have the same dimension. Thus it is enough to show that

$$\dim F - \dim J^p \times O(k) - \dim Z + \dim H = 0.$$

Let us use induction to check this. Suppose first that $s = p = 1$.
Then

$$\dim F = n \cdot k;$$

$$\dim J^p \times O(k) = n^2 + \dim O(k);$$

$$\dim Z = 0;$$

$$\dim H = \dim O(n) + \dim O(k - n);$$

and by a simple computation the result is true.

Now suppose that $s = p$. We raise s and p together, leaving n and k fixed. As a preparation for the induction step we write

$$\dim Z(p) = \dim Z(p-1) + \binom{n+p-1}{p}(n_p - n);$$

$$\dim J^p \times O(k) = \dim J^{p-1} \times O(k) + \binom{n+p-1}{p} n;$$

$$\dim F(p) = \dim F(p-1) + \binom{n+p-1}{p} k;$$

$$\begin{aligned}\dim H(p) = \dim H(p-1) &+ \dim O(k_p - n_{p-1}) \\ &+ \dim O(k - n_p) - \dim O(k - n_p).\end{aligned}$$

Here $H(p-1) = H(n, n_2, \ldots, n_{p-1})$ and $H(p) = H(n, n_2, \ldots, n_p)$.
Similarly for $Z(p)$ and $Z(p-1)$. By the induction it is enough to show that

$$\begin{aligned}&\binom{n+p-1}{p} k - \binom{n+p-1}{p} - \binom{n+p-1}{p}(n_p - n) \\ &+ \dim O(n_p - n_{p-1}) + \dim O(k - n_p) - \dim O(k - n_{p-1}) = 0.\end{aligned}$$

This may be simplified to

$$\left[\binom{n+p-1}{p}(n_p - n_{p-1})\right](k - n_p) = 0.$$

That is,

$$\text{either } \binom{n+p-1}{p} = n_p - n_{p-1}$$

$$\text{or } n_p = k.$$

If $n_{p-1} = k$ then also $n_p = k$ so the induction step is valid. If $n_{p-1} = \nu(n, p-1)$ the above two conditions become

$$n_p = \nu(n, p) \text{ or } n_p = k.$$

These are just the condition that $\tilde{F}$ is open in F so that again the induction step is valid.

Now we use induction on p leaving s, k and n fixed. At the first step of the induction, s is equal to p, which is the case we have just shown. We use induction to raise p leaving s fixed. We may write

$$\dim Z(p) = \dim Z(p-1) + \binom{n+p-1}{p}(k-n);$$

$$\dim J^p \times O(k) = \dim J^{p-1} \times O(k) + \binom{n+p-1}{p} n;$$

$$\dim F(p) = \dim F(p-1) + \binom{n+p-1}{p} k;$$

$$\dim H(p) = \dim H(p-1).$$

One easily checks that the induction step is satisfied which completes the proof.

From now on we assume $n, n_2, \dots, n_s$ chosen so that $\tilde{F}$ is open in F.

COROLLARY 2.13. - *Any invariant subvariety, K of F under $J^p \times O(k)$ meets $Z(n, n_2, \dots, n_s)$ transversally. The conditions of Theorem* 2.12 *are assumed for $n, n_2, \dots, n_s$.*

PROOF. - If K does meet Z the corollary is trivial. Thus suppose K meets $Z(n, n_2, \dots, n_s)$. Let $q \in K \cap Z$ be a regular point of K. Let L be the orbit of q under $J^p \times O(k)$. Then $L \subset \tilde{F}$ by Corollary 2.5 and, by Theorem 2.12, L meets Z transversally. But $L \subset K$. Consequently, K also meets Z transversally. If q is a singular point of K then transversality is equivalent to being transversal to a finite number of varities which are regular ot q. Thus, by the above, the result is again true.

COROLLARY 2.14. - *The codimension of $K \cap Z$ in Z is equal to the codimension of the locus of K singular points for any map $f: M \to N$ which is K jet transversal. K is a subvariety of some $\tilde{F}$ where the conditions of Theorem 2.12 apply to $n, n_2, \ldots, n_s$.*

PROOF. - This is a consequence of FELDMAN'S result, stated here as Theorem 2.2, and Corollary 2.13.

NOTE - In the case $s = 1$ the hypotheses of Theorem 2.12 are satisfied whenever f is an immersion.

In the case $s = 1$, $p = 2$, $n = 2$, $k = 4$ we see that Z consists of matrices of the form

$$\begin{bmatrix} Id & 0 \\ 0 & A_{22} \end{bmatrix}$$

where A_{22} is a 3×2 matrix. We will show in Proposition 2.16 that if $N = E^4$ the matrix A_{22} is equal to

$$\begin{bmatrix} a & e \\ b & f \\ c & g \end{bmatrix},$$

where a, b, c, e, f, g are the coefficients of the second fundamental form as described in Chapter I. Thus Corollary 2.14 contains our Theorem 1.11 of the first chapter. Bear in mind the correspondence ρ given in Proposition 2.9.

5. - Consideration of a map $f: M \to N$; N a manifold with symmetric connection.

The map f induces a map

$$D_p T_p : T_p M \to T_1 N$$

and also the map

$$\hat{f} : M \to \mathrm{Hom}\,(T_p M;\ T_1 N)$$

so that $\hat{f}(q) \in F$, where F is the fibre of $\mathrm{Hom}\,(T_p M,\ T_1 N)$ at $(q,\ f(q))$. If we assume f is an immersion then $\hat{f}(q)$ is $1-1$ on $T_1 M_q$. Hence we may apply Theorem 2.4. This says that given any orthonormal basis $E_1, \ldots, E_{s+1}$ of $T_1 N_{f(q)}$ such that $E_1, \ldots, E_i$ spans $\hat{f}(q)(T_i M_q)$ for $i = 1, \ldots, s$ and E_{s+1} completes $E_1, \ldots, E_s$ to an orthonormal basis for $T_1 N_{f(q)}$, then there exists a unique basis $X_1, \ldots, X_p$ of $T_p M_q$ induced from coordinates in a neighborhood of q such that the matrix for $\hat{f}(q)$ relative to the bases (X, E) has the form prescribed by the theorem. The family of matrices of this form we call Z. (This Z is a family of matrices. The previous Z is a family of linear transformations). Because the basis X is unique the matrix for $\widetilde{f}(q)$ of form Z will also be unique.

For the purpose of studying transversality at a point q we may assume that M is just a small neighborhood of q and that the bundle $\mathrm{Hom}\,(T_pM,\ T_1M)$ is trivial over this neighborhood. In fact, according to Lemma 2.1 we may, in order to study transversality at q, replace the map $\widetilde{f}: M \to \mathrm{Hom}\,(T_pM, T_1 M)$ be a map $\widetilde{f}: M \to F$, where F is the fibre of $\mathrm{Hom}\,(T_pM,\ T_1M)$. We make these assumptions for the remainder of the chapter.

Let us suppose that for every point $q \in M$ the $\dim \widetilde{f}(q)(T_iM) = n_i$ for $i = 1, \dots, s$. Then by the above argument we have shown.

PROPOSITION 2.15. - *There exists a map $\mu: \mathcal{O} \to Z$, where $\mathcal{O} = \mathcal{O}(n, n_2, \dots, n_s)$ is the osculating frame bundle, such the diagram*

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\mu} & Z \\ \pi\big\downarrow & \widetilde{f} & \big\uparrow\rho \\ M & \longrightarrow & \widetilde{F} \end{array}$$

is commutative in the sense that a fibre of $\mathcal{O}$ is mapped, either way, onto an orbit of Z under H. The map μ is defined by saying that $\mu(q, E_1, \dots, E_{s+1})$ is the unique matrix picked out by Theorem 2.4. Here $(q, E_1, \dots, E_{s+1}) \in \mathcal{O}$.

By $\rho: \widetilde{F} \to Z$ is meant a map of points of $\widetilde{F}$ into orbits of Z under H. It is defined by sending a point of $\widetilde{F}$ into the intersection of Z with the orbit, under $J^p \times O(k)$, of that point. This map ρ induces a map of orbits of $\widetilde{F}$, under $J^p \times O(k)$ which is just the $1-1$ correspondence of Proposition 2.9.

The fact that the entries of Z are defined on $\mathcal{O}$ indicates that we ought to be able to express the entries of Z in terms of the differential forms ω_i and ω_{ij} by the method of E. CARTAN. (Assume for this discussion that $N^k = E^k$, Euclidean k-space). The map $\mu: \mathcal{O} \to Z$ given by Proposition 2.15 is defined as follows. Given a point $(q, e_1, \dots, e_k)$ of $\mathcal{O}$, pick nice coordinates $u_1, \dots, u_n$ by Theorem 2.4 and define $\mu(q, e_1, \dots, e_k)$ to be the matrix for $D_pT_p(f)_q$ in terms of the bases $X_1, \dots, X_p$, induced from $u_1, \dots, u_n$, and $e_1, \dots, e_k$. Let the matrix $\mu(q, e_1, \dots, e_k)$ be called A. A has the block form described in Proposition 2.4. We now show:

PROPOSITION 2.16. - *The block A_{22} of A has coefficients which are nothing but the coefficients of the second fundamental form.*

Recall that the second fundamental form is

$$\sum_{i=n+1}^{k} (d^2f \cdot e_i)e_i.$$

Sine $df \cdot e_i = 0$ for $i > n$ we see that $\omega_i = 0$ for $i > n$; also, since $de_i \cdot e_j = 0$ for $1 \leq i \leq n$ and $j > n_2$, we see that $\omega_{ij} = 0$ for $i \leq n, j > n_2$. Now $d^2f \cdot e_i = - df \cdot de_i$, so

$$d^2f \cdot e_i = - df \cdot \sum_{j=1}^{k} \omega_{ij} e_j = \sum_{j=1}^{n} \omega_{ij} \omega_j .$$

Thus we may write the second fundamental form as

$$\sum_{i=n+1}^{n_2} \sum_{j=1}^{n} \omega_{ij} \omega_j e_i .$$

Here i is summed only to n_2 because $\omega_{ij} = 0$ for $i > n_2$, $j \leq n$. Since $\omega_j = 0$ for $j = n+1, \ldots, n_2$ we have $d\omega_j = \sum \omega_{ji} \wedge \omega_i = 0$ and thus, since $\omega_1, \ldots, \omega_n$ are independent forms on M, we have, by a lemma of CARTAN,

$$\omega_{ij} = \sum_{k=1}^{n} a^i_{jk} \omega_k .$$

The second fundamental for may then be written

$$\sum_{i=n+1}^{n_2} \sum_{j,\, k=1}^{n} a^i_{jk} \omega_j \omega_k e_i .$$

We wish to show

$$A_{22} = (a^i_{jk}) .$$

Now $df = \sum \partial f / \partial u_i \, du_i$ where $u_1, \ldots, u_n$ are coordinates which are nice at q. Let us write $\partial f / \partial u_i = \sum f_{ij} e_j$. Thus

$$df = \sum_{i,\, j=1}^{n} f_{ij} e_j du_i .$$

At q $\partial f / \partial u_i = e_i$ and the matrix f_{ij} is the identity. Also $du_i = \omega_i$ at q. Differentiating the above we have

$$d^2f = \sum_{i,\, j=1}^{n} df_{ij} e_j du_i + \sum_{i,\, j=1}^{n} f_{ij} de_j du_i$$

but

$$de = \sum_{k=1}^{n_2} \omega_{jk} e_k$$

so

$$d^2f = \sum_{i,\, k=1}^{n} (df_{ik} + \sum_{j=1}^{n} f_{ij} \omega_{ik}) e_k du_i$$

$$+ \sum_{k=n+1}^{n_2} \sum_{i,\, j=1}^{n} f_{ij} \omega_{jk} e_k du_i .$$

At q, since d^2f has no components in the tangent direction because of the special properties of the coordinates $u_1, \dots, u_n$ at q, we must have

$$df_{ik} = -\omega_{ik} \qquad i, k = 1, \dots, n$$

and

$$d^2f = \sum_{i=n+1}^{n_2} \sum_{j=1}^{n} \omega_{ij}\omega_j e_i .$$

Since

$$\omega_{ij} = \sum_k a^i_{kj}\omega_k$$

everywhere and in particular at q, we have

$$d^2f = \sum_{i=n+1}^{n_2} \sum_{i,\, k=1}^{n} a^i_{jk}\omega_j\, \omega_k\, e_i .$$

But at q $\omega_i = du_i$ so

$$d^2f = \sum_{i=n+1}^{n_2} \sum_{j,\, k=1}^{n} a^i_{jk} du_j\, du_k e_i \text{ at } q .$$

This shows that

$$\frac{\partial^2 f}{\partial u_j\, \partial u_k} = (a^1_{jk}, \dots, a^n_{jk}) \text{ at } q,$$

which establishes the result.

One may also describe the entries of the higher order blocks of A in terms of the forms ω_i and ω_{ij}. The calculations are not very inspiring even for the third order case however, for curves in E^3 the calculations are not so uninteresting.

If $e_1 e_2 e_3$ are the Frenet frames, s the arc length and t a coordinate which is good at q then for the curve $X(t)$ we have at q:

$$\frac{dX}{dt} = e_1,$$

$$\frac{d^2X}{dt^2} = \varkappa e_2,$$

$$\frac{d^3X}{dt^3} = \frac{d\varkappa}{ds} e_2 + \varkappa\tau e_3,$$

where $\varkappa$ and τ are the curvature and torsion.

6. - In this section we prove that jet transversality implies geometric transversality. The following rather technical lemma will be useful.

LEMMA 2.17. - *Let I^f be the unit cube and $\{I^m\}$, $m < f$, the parallel linear spaces given by* $x_{m+1} = \text{const.}, \ldots, x_f = \text{const.}$

If a manifold Z meets each I^m transversally then we may choose coordinates in some neighborhood of a point $q \in Z$ so that each I^m remains parallel linear and in addition Z is linear.

PROOF. - Take coordinates about q so that Z is a linear space, say $y_1, \ldots, y_f$. Z is given by $z_{r+1} = \ldots = y_f = 0$, where $r = \dim Z$. We will show that in some neighborhood of q, $x_1, \ldots, x_l, y_{r+1}, \ldots, y_f, x_{m+1}, \ldots, x_f$ are a system of coordinates, where $l = \dim Z \cap I^m$. They will then obviously do the job. Note that by transversality, $m + r = l + f$ and $\partial/\partial x_1, \ldots, \partial/x_l$ are tangent to each $Z \cap I^m$. We may, by an affine change of coordinates which preserves parallelism and linearity, assume that at q $\partial/\partial x_{l+1}, \ldots, \partial/\partial x_m$ are tangent to I^m and normal to Z. To show that $x_1, \ldots, x_l, y_{r+1}, \ldots, y_f, x_{m+1}, \ldots, \ldots, x_f$ are a coordinate system we compute their Jacobian with respect to $x_1, \ldots, x_f$. This will be nonsingular if

$$\begin{bmatrix} \dfrac{\partial y_{r+1}}{\partial x_{l-1}} & \cdots & \dfrac{\partial y_f}{\partial x_{l+1}} \\ \vdots & & \vdots \\ \dfrac{\partial z_{r+1}}{\partial x_m} & \cdots & \dfrac{\partial y_f}{\partial x_m} \end{bmatrix}$$

is nonsingular. Note that since $m + r = l + f$, the above matrix is square. Now

$$\frac{\partial}{\partial y_{l+1}} = \frac{\partial y_1}{\partial x_{l+1}}\frac{\partial}{\partial y_1} + \ldots + \frac{\partial y_r}{\partial x_{l+1}}\frac{\partial}{\partial y_r} + \frac{\partial y_{r+1}}{\partial x_{l+1}}\frac{\partial}{\partial y_{r+1}} + \ldots + \frac{\partial y_k}{\partial x_{r+1}}\frac{\partial}{\partial y_f}$$

$$\vdots$$

$$\frac{\partial}{\partial x_m} = \frac{\partial y_1}{\partial x_m}\frac{\partial}{\partial y_1} + \ldots + \frac{\partial y_r}{\partial x_m}\frac{\partial}{\partial y_r} + \frac{\partial y_{r+1}}{\partial x_m}\frac{\partial}{\partial y_{r+1}} + \ldots + \frac{\partial y_f}{\partial x_m}\frac{\partial}{\partial y_f}.$$

At q, since $\partial/\partial x_{l-1}, \ldots, \partial/\partial x_m$ are normal to Z and since $\partial/\partial y_1, \ldots, \partial/\partial y_r$ are vectors tangent to Z, we have

$$\frac{\partial}{\partial x_{l+1}} = \frac{\partial y_{r+1}}{\partial x_{l+1}}\frac{\partial}{\partial y_{r+1}} + \ldots + \frac{\partial y_f}{\partial x_{l+1}}\frac{\partial}{\partial y_f}$$

$$\vdots$$

$$\frac{\partial}{\partial x_m} = \frac{\partial y_{r+1}}{\partial x_m}\frac{\partial}{\partial y_{r+1}} + \ldots + \frac{\partial y_f}{\partial x_m}\frac{\partial}{\partial y_f}.$$

Since obviously both $\partial/\partial x_{l+1}, \ldots, \partial/\partial x_m$ and $\partial/\partial y_1, \ldots, \partial/\partial y_r$ are independent sets of vectors, the needed Jacobian, at q, is nonsingular. Hence it is nonsingular in a neighborhood of q, and thus our desired coordinate change is valid.

For any LIE transformation group G which acts on a space F there is a map from the LIE algebra of G to the family of continuous vector fields on F, given by sending each $X \in$ LIE algebra of G into the infinitesimal transformation,

$$[X^+f](p) = \lim_{t \to 0} \frac{f(\exp tX \cdot p) - f(p)}{t}.$$

This is a LIE algebra homomorphism; i.e. $[X^+, Y^+] = [X, Y]^+$. See for instance PALAIS [11]. Thus the family of infinitesimal transformations form a completely integrable system and their integrals, the orbits, are the leaves of a differential system.

THEOREM 2.18. - *Suppose that M is a submanifold of $\tilde{F}$, not necessarily invariant, which meets an invariant manifold K transversally. Then ρM meets ρK transversally.*

PROOF. Take $q \in \rho M \cap \rho K$. Let L be the orbit under $J^p \times O(k)$ through q. Then $M \cap L \neq \emptyset$. So take $r \in M \cap L$. Since $r, q \in L$ there is an element of the group $J^p \times O(k)$, say α, which sends r to q. α is a diffeomorphism from a neighborhood of r to a neighborhood of q. Therefore M is transversal to K at r if and only if $\alpha(M)$ is transversal to K at q. So, by replacing M by $\alpha(M)$ if necessary, we may assume that $q \in M$.

Since the orbits are leaves of a differential system we may, in a neighborhood of q, choose coordinates so that the orbits are parallel linear spaces. Since each orbit meets Z transversally, by Theorem 2.12, we may apply Lemma 2.17. Thus we choose coordinates in a neighborhood of q so that the orbits are parallel and linear and also Z is linear. We may, by an affine transformation, assume that the orbits intersect Z orthogonally. Let π_Z be the normal projection onto Z. Then, because π_Z is projection along the orbits and because $\rho K = K \cap Z$, we see that

$$\pi_Z K = \rho K.$$

Also because of the definition of ρM and the nice choice of coordinates we have

$$T\rho M = T\rho L + T\pi_Z M,$$

where L is the orbit through q, and where T means the tangent space at q. Since M is transversal to K in $\tilde{F}$, $\pi_Z M$ is transversal to $\pi_Z K$ in Z. Thus

$$TZ = T\pi_Z M + T\pi_Z K$$

but $\pi_Z K = \rho K$ so

$$TZ = T\pi_Z M + T\rho K.$$

Also $T\rho L \subset T\rho K$ because $L \subset K$. Thus $TZ = T\rho L + T\pi_Z M + T\rho K$ and, because $T\rho L + T\pi_Z M = T\rho M$, we see $TZ = T\rho M + T\rho K$, which implies that ρM is transversal to ρK in Z.

Let K be a subvariety of $\tilde{F}$ invariant under J^p and let ρK be the corresponding subvariety of Z invariant under H. A point $q \in M$ is called a K *geometrically transversal singular* point of M if $\mu(\mathcal{O})$ meets ρK transversally all along the fibre of $\mathcal{O}$ over q.

COROLLARY 2.19. - *Let K be a subvariety of $\tilde{F}$ invariant under J^p and suppose $\tilde{F}$ is open in F. Then if q is a K jet transversal singular point it is also a ρK geometrically transversal singular point.*

PROOF. - Since q is a jet transversal singular point $\widehat{f}(M)$ meets K transversally at q. By Theorem 2.18, since $\tilde{F}$ is open in F, $\rho\widehat{f}(M)$ meets ρK transversally at $\rho\widehat{f}(q)$. But μ (fibre over $q) = \rho\widehat{f}(q)$ and $\mu(\mathcal{O}) = \rho\widehat{f}(M)$. Hence $\mu(\mathcal{O})$ meets ρK transversally all along the fibre over q.

Consider the case $p = 2$, $s = 1$, (the frame bundle F of Chapter I is now $\mathcal{O}$). Assume that $N^k = E^k$. By Corollary 2.16 Z may be taken to be the space of second fundamental forms at a point, i.e. $Z = \{(\Sigma a_{ij}^{n+1} x_i x_j, \ldots, \Sigma a_{ij}^k x_i x_j)\}$. In this case $H = 0(n) \times 0(k-n)$ and its action on Z is induced from rotations in the tangent and normal space. The map μ: $\mathcal{O} \to Z$ is just the second fundamental form, i.e., $\mu(xe_1 \ldots e_k)$ is the second fundamental form in the frame $e_1 \ldots e_k$ evaluated at the point x. Let us summarize the results of this chapter in the second order case.

THEOREM 2.20. - *Let f: $M^n \to E^k$ be an immersion. Let K be any subvariety of Z invariant under H, where Z is the space of second fundamental forms.*

The K geometrically transversal functions are dense in $C^\infty(M^n, E^k)$.

If f is K geometrically transversal then the codimension of the locus of K singular points is equal to the codimension of K in Z.

PROOF. - Since $s = 1$ $\tilde{F}$ is open in F, so, by Corollary 219, jet transversality implies geometric transversality. Thus, by Theorem 2.3, the geometrically transversal maps are dense in $C^\infty(M^n, E^k)$. The second part is a restatement of Corollary 2.14 bearing in mind the correspondence ρ given in Proposition 2.9.

CHAPTER III.

The Veronese Manifold.

In Chapter I we began by studying the local invariants of a surface in E^4. The invariants of the second fundamental form can be very well understood by means of the curvature ellipse. (See Figure 1 in Chapter I). Studying the invariants is equivalent to studying this configuration. It is therefore quite natural to try and extend these ideas to higher dimensions. Are the second order invariants the invariants associated to some configuration? If so, can we use this configuration to choose either tangent or normal frames as was done for surfaces in E^4? It is with these questions that this chapter is concerned.

We shall indeed find a configuration. In the case $n = 3$ it is very well known in classical algebraic geometry as the VERONESE surface. Our interest is in the affine and metrical properties of the VERONESE surface, so that our treatment must be independent of that of classical algebraic geometry. We are also able to pick out « in general » principle axes in the tangent space. This construction was inspired by the classical treatment. Thus although we do not use algebraic geometric proofs, we do wish to acknowledge the large inspirational debt. We refer to BAKER [1] and SEMPLE and ROTH [14] for the very rich literature on the VERONESE surface.

Our discussion admits a purely algebraic treatment. The motivation is, of course, the study of the second fundamental form. We postpone, however, a differential geometric interpretation until the concluding chapter.

DEFINITION 3.1. - Let

$$X : S^{n-1} \to E^N$$

be the map given by

$$X(x_1, \dots, x_n) = (\Sigma a^1_{ij} x_i x_j, \dots, \Sigma a^N_{ij} x_i x_j)$$

where $N = \frac{1}{2} n(n+1)$, $x_1^2 + \dots + x_n^2 = 1$, and $(a^1_{ij}), \dots, (a^N_{ij})$ are N symmetric matrices. The indices i and j run from 1 to n. Let $a_{ij} = (a^1_{ij}, \dots, a^N_{ij})$ so that we may write

$$X(x_1, \dots, x_n) = \Sigma a_{ij} x_i x_j .$$

If the N vectors $a_{11}, a_{22}, \dots, a_{nn}, a_{12}, \dots, a_{n-1n}$ are independent we call the image of S^{n-1} a VERONESE $n-1$ manifold.

NOTE 3.2. - Since $(x_1, \ldots, x_n)$ and $(-x_1, \ldots, -x_n)$ are mapped to the same point, X gives a map

$$X : P^{n-1} \to E^N,$$

of the real projective $n-1$ plane.

PROPOSITION 3.3. - *All Veronese $n-1$ manifolds are affinely equivalent in E^N.*

PROOF. - Let $A : E^N \to E^N$ be the affine (actually, linear) transformation defined by sending

$$a_{11} \to e_1, \ldots, a_{nn} \to e_n, \qquad a_{12} \to e_{n+1}, \ldots, a_{n-1n} \to e_N,$$

where $e_1, \ldots, e_N$ are the standard basis in E^N. Then the image of $\Sigma a_{ij} x_i x_j$ is

$$(x_1^2, \ldots, x_n^2, 2x_1x_2, \ldots, 2x_{n-1}x_n).$$

Since all VERONESE $n-1$ manifolds are affinely equivalent to this particular manifold, they are all equivalent to one another.

Thus to demonstrate that the VERONESE $n-1$ manifolds have some affine property it is necessary only to show it for some particular $n-1$ manifold.

PROPOSITION 3.4. - *A Veronese $n-1$ manifold lies in an $N-1$ dimensional linear space.*

PROOF. - $x_1^2 + \ldots + x_n^2 = 1$. So

$$\Sigma a_{ij} x_i x_j = a_{11} + (a_{22} - a_{11})x_2^2 + \ldots + (a_{nn} - a_{11})x_n^2 + 2 \sum_{i<j} a_{ij} x_i x_j .$$

Thus it lies in the space spanned by $a_{22} - a_{11}, \ldots, a_{nn} - a_{11}, a_{12}, \ldots, a_{n-1n}$ which is an $N-1$ dimensional space.

Let us, from here on in, require only that $a_{22} - a_{11}, \ldots, a_{nn} - a_{11}, a_{12}, \ldots, a_{n-1n}$ be independent. Note, under this assumption, that if $a_{11} + \ldots + a_{nn} = 0$ then any $n-1$ among $a_{11}, \ldots, a_{nn}$ are independent.

THEOREM 3.5. - *The Veronese manifold*

$$X(x_1, \ldots, x_n) = (x_1^2, \ldots, x_n^2, \sqrt{2}x_1x_2, \ldots, \sqrt{2}x_{n-1}x_n),$$

hereafter called the standard manifold, lies on S^{N-1}. Consequently since it lies in a hyperplane it lies in an S^{N-2} sphere. It also has the property that a rotation of S^{n-1} gives a Euclidean motion of E^N.

PROOF. - $X \cdot X = (x_1^2 + ... + x_n^2)^2 = 1$ proves the first assertion.

Note that $E^n \circ E^n$, the symmetric tensor product of E^n with itself, has inner product

$$(v \circ w,\ r \circ s) = \frac{1}{2}((v,\ r)(w,\ s) + (v,\ s)(w,\ r))$$

induced from E^n. If $e_1, ..., e_n$ is the standard basis in E^n then $e_1 \circ e_1, ..., e_n \circ e_n$, $\sqrt{2} e_1 \circ e_2, ..., \sqrt{2} e_{n-1} \circ e_n$ is an orthonormal basis for $E^n \circ E^n$. Thus the map which sends the above basis for $E^n \circ E^n$ into the standard basis $e_1, ..., e_N$ for E^N gives an isometry between $E^n \circ E^n$ and E^N.

Note also that if $S : E^n \to E^n$ is an isometry then $S \circ S : E^n \circ E^n \to E^n \circ E^n$, defined by $S \circ S(v \circ w) = S(v) \circ S(w)$, is also an isometry. The standard VERONESE manifold in terms of the orthonormal basis for $E^n \circ E^n$ becomes $X(x_1, ..., x_n) = (x_1 e_1 + ... + x_n e_n) \circ (x_1 e_1 + ... + x_n e_n)$ or, if $v = x_1 e_1 + ... + x_n e_n$,

$$X(v) = v \circ v.$$

Consider now an isometry $S : E^n \to E^n$. Then $X(S(v)) = S(v) \circ S(v) = S \circ S(v \circ v) = S \circ S(X(v))$. Hence the fact that $S \circ S$ is an isometry gives the result.

COROLLARY 3.6. - *For the standard manifold any point and tangent frame may be sent into any other point and frame by a motion which maps the Veronese manifold onto itself.*

THEOREM 3.7. - *The Veronese manifold is an inflection-free imbedding of* P^{n-1} in E^{N-1}.

PROOF. - It is enough to show this for the standard manifold. Note that $x_i x_j = y_i y_j$, $1 \leq i \leq j \leq n$ implies $(x_1, ..., x_n) = \pm (y_1, ..., y_n)$ which shows that the map is $1 - 1$.

To check that X is an inflection free imbedding, we need only check that the first and second derivatives are independent at one point, say $X(e_1)$. The previous theorem then gives the conclusion. Regard $x_2, ..., x_n$ as local coordinates at e_1 so $\partial x_1 / \partial x_i = - x_i / x_1$, $i \geq 2$. The tangent space at $X(e_1)$ is spanned by

$$\partial X / \partial x_i (e_1), \qquad i \geq 2,$$

which is equal to $2\ e_i \circ e_i$, and the first normal space by

$$\partial^2 X / \partial x_i^2 (e_1) \quad \text{and} \quad \partial^2 X / \partial x_i \partial x_j (e_1),$$

which are $2(e_i \circ e_i - e_1 \circ e_1)$ and $2 e_i \circ e_j$. Since these vectors are all independent we have the result.

Since $X : S^{n-1} \to E^N$ we may also regard X as mapping a frame in E_n into n points in E^N.

PROPOSITION 3.8. - *The image of every frame is a set of n independent points with centroid* $\mathcal{H} = \frac{1}{n}(a_{11} + \dots + a_{nn})$.

PROOF. - For the standard manifold $X(v) \cdot X(w) = (v \cdot w)^2$ so that the image of a frame is a frame. Thus for an arbitrary VERONESE manifold the image of a frame is a set of independent points.

Let $\alpha_1, \dots, \alpha_n$ be a frame in E^n where $\alpha_i = (\alpha_{i1}, \dots, \alpha_{in})$. Then

$$X(\alpha_1) + \dots + X(\alpha_n) = \Sigma_j a_{ij}\alpha_{ki}\alpha_{kj} = \sum_{i,j} (\Sigma_k \alpha_{ki}\alpha_{kj}) a_{ij} = a_{11} + \dots + a_{nn},$$

because column vectors of the matrix (α_{ij}) are also orthonormal. Thus $\mathcal{H} = \frac{1}{n}(X(\alpha_1) + \dots + X(\alpha_n))$.

REMARK 3.9. - The map $\rho : O(n) \to O(N)$ given by $\rho(S) = S \circ S$ is a representation of $O(n)$ by elements of $O(N)$ with representation space $E^n \circ E^n$ or, because they are isometric, E^N. The standard VERONESE manifold is an orbit.

We ask if the representation is irreducible and if not what are the invariant subspaces.

PROPOSITION 3.10. - *The line through* $\frac{1}{n}(e_1 \circ e_1 + \dots + e_n \circ e_n)$ *which is* $\mathcal{H}$ *for the standard Veronese manifold is left pointwise fixed.*

PROOF. - $\rho(S)(\mathcal{H}) = S \circ S(\mathcal{H}) = S \circ S\left(\frac{1}{n}(e_1 \circ e_1 + \dots + e_n \circ e_n)\right) = \frac{1}{n}(S(e_1) \circ S(e_1) + \dots + S(e_n) \circ S(e_n)) = \frac{1}{n}(X(S(e_1)) + \dots + X(S(e_n)))$. But since S is orthogonal $S(e_1) \dots S(e_n)$ is a frame and thus by the previous proposition $\rho(S)(\mathcal{H}) = \mathcal{H}$. Thus the line through $\mathcal{H}$ is left pointwise fixed.

Therefore, any orthogonal hyperplane to the line is also left fixed, in particular the one containing the tip of $\mathcal{H}$. This hyperplane we call $\mathfrak{L}$. If we restrict ρ to $\mathfrak{L}$ with origin now the tip of $\mathcal{H}$ we obtain a representation of $O(n)$ by elements of $O(N-1)$. We ask if this restricted representation is irreducible.

PROPOSITION 3.11. - *ρ restricted to $\mathfrak{L}$ is irreducible. Hence the only point of $\mathfrak{L}$ left fixed by all elements of $\rho(O(n))$ is $\mathcal{H}$.*

This proposition is a very well known fact in the theory of group representations, see foristance, BOERNER [2].

THEOREM 3.12. - *The centroid of the Veronese manifold is* $\mathcal{H} = \frac{1}{n}(a_{11} + \dots + a_{nn})$.

PROOF. - Because the centroid is an affine invariant, it is enough to show that $1/n(e_1 \circ e_1 + \dots + e_n \circ e_n)$ is the centroid of the standard manifold. The linear space $\mathfrak{L}$ is the hyperplane in which the standard manifold lies. The motions of E^N induced by rotations of S^{n-1} all leave $\mathcal{H}$ fixed, by Proposition 3.10. They are also motions which map the standard manifold onto itself. Thus the centroid must be a point of $\mathfrak{L}$ left fixed by all motions induced from rotations by S^{n-1}. By the previous proposition such a point must be $\mathcal{H}$.

PROPOSITION 3.13. - *Let S^{i-1} be a great $i-1$ dimensional sphere contained in S^{n-1}, $i < n$. Then X restricted S^{i-1} is a Veronese $i-1$ manifold.*

PROOF. - We may rotate any great S^{i-1} sphere into the one given by $x_{i+1} = \dots = x_n = 0$, and for this one the result is obvious.

Let us call $X(S^{i-1})$ a sub-VERONESE manifold of $X(S^{n-1})$.

PROPOSITION 3.14. - *$\mathcal{H}$ is not contained in the span of any strictly sub-Veronese manifold.*

This need only be checked for the standard manifold and the image of the S^{i-1} given by $x_{i+1} = \dots = x_n = 0$.

We have previously seen that V^{n-1} is an inflection-free imbedding of P^{n-1} in E^{N-1} Thus each VERONESE submanifold of V^{n-1} say V^{i-1}, $i < n$, is an $i-1$ dimensional submanifold contained in a $\frac{1}{2}i(i+1)-1$ dimensional linear space and infact imbedded in an inflection-free manner. The converse is also true.

THEOREM 3.15. - *Any $i-1$ dimensional submanifold of V^{n-1} which is immersed in a $\frac{1}{2}i(i+1)-1$ dimensional linear space must be a sub-Veronese manifold.*

PROOF. - We may assume the VERONESE manifold, V^{n-1}, is the standard one, V^{n-1}_{st}. By a rotation if necessary we may assume that $X(e_1)$ is a point of the submanifold and that the tangent space to the submanifold at $X(e_1)$ is spanned by

$$\partial X/\partial x_2(e_1), \ \dots, \ \partial X/\partial x_i(e_1).$$

Regard $x_2, \dots, x_n$ as coordinates for V^{n-1} at $X(e_1)$ and denote the submanifold by M.

Since $\partial X/\partial x_2(e_1), \dots, \partial X/\partial x_i(e_1)$ span the tangent space to M at $X(e_1)$ we may take $x_2, \dots, x_i$ as coordinates for M. Furthermore, the second order osculating space at $X(e_1)$ is contained in the a spanned by

$$\partial X/\partial x_j(e_1), \ j = 2, \ \dots, \ n \quad \text{and} \quad \partial^2 X/\partial x_k \partial x_m(e_1), \ 2 \leq k, \ m \leq i.$$

Call this space L. Also the second order osculating space has maximon dimension. The reason is that a submanifold of an inflection free manifold is also inflection free.

Thus the dimension of the second order osculating space to M at $X(e_1)$ is $\frac{1}{2}i(i+1)-1$. But M lies in a linear space of that dimension and hence it must lie entirely in its second order osculating space at $X(e_1)$.

We will now show that

$$V_{st}^{n-1} \cap L = X(S^{i-1})$$

where S^{i-1} is the sphere given by $x_{i+1} = \dots = x_n = 0$. The vectors $\partial^2 X/\partial x_k^2(e_1)$ for $k > i$ do not lie in L. But

$$\partial^2 X/\partial x_k^2(e_1) = 2(e_k \circ e_k - e_1 \circ e_1).$$

Now

$$X(x_1, \dots, x_n) = \Sigma x_i x_j e_i \circ e$$

$$= e_1 \circ e_1 + \Sigma x_j^2(e_j \circ e_j - e_1 \circ e_1) + 2 \sum_{j<k} x_j x_k e_j \circ e_k.$$

Thus the component of $X(x_1, \dots, x_n)$ along $e_j \circ e_j - e_1 \circ e_1$ is x_j^2. So, we must have $x_{i+1} = \dots = x_n = 0$ at a point of $V_{st}^{n-1} \cap L$.

Thus the manifold M is immersed in the sub-VERONESE manifold $X(S^{i-1})$. Since they are both of dimension $i-1$ they must be equal, and this concludes the proof.

COROLLARY 3.16. - *An $i-1$ dimensional submanifold of V^{n-1} lies in a linear space of dimension* $\geq \frac{1}{2}i(i+1)-1$.

PROPOSITION 3.17. - *If an $n-1$ plane through the tip of $\mathcal{H}$ meets a Veronese manifold V^{n-1} in n points either they are the image of a frame or they lie on a sub-Veronese manifold.*

PROOF. - Assume the n points do not lie on any sub-VERONESE manifold. Thus the n points are images of n independent points of S^{n-1}, say $\alpha_1, \dots, \alpha_n$. Let $\alpha_i = (\alpha_{i1}, \dots, \alpha_{in})$. Notice that by a rotation of S^{n-1}, which just changes the VERONESE manifold to some other one, we may assume that (α_{ij}) has triangular form.

$$\alpha_1 = (1, 0 \dots 0),$$
$$\vdots$$
$$\alpha_i = (\alpha_{i1}, \dots, \alpha_{ii} 0 \dots 0),$$
$$\vdots$$
$$\alpha_n = (\alpha_{n1}, \dots, \alpha_{nn}).$$

Since $\alpha_1, \ldots, \alpha_n$ are independent we must have $\alpha_{ii} \neq 0$ for $i = 1, \ldots, n$. Now

$$X(\alpha_i) - \mathcal{H} = \sum_{j=1}^{i} a_{jj}(\alpha_{ij})^2 + 2 \sum_{j,\,k=1}^{i} \alpha_{jk}\alpha_{ij}\alpha_{ik} - \frac{1}{n}(a_{11} + \ldots a_{nn}).$$

Also $\alpha_{i1}^2 = 1 - \alpha_{i2}^2 - \ldots - \alpha_{ii}^2$ so

$$\begin{aligned} X(\alpha_i) - \mathcal{H} &= a_{11}(1 - \alpha_{i2}^2 - \ldots - \alpha_{ii}^2) - \sum_{j=2}^{i} a_{jj}(\alpha_{ij})^2 \\ &+ 2 \sum_{j,\,k=1}^{i} a_{jk}\alpha_{ij}\alpha_{ik} - \frac{1}{n}(a_{11} + \ldots + a_{nn}) \\ &= \sum_{j=2}^{i} (a_{jj} - a_{11})\left((\alpha_{ij})^2 - \frac{1}{n}\right) + 2 \sum_{j,\,k=1}^{i} a_{jk}\alpha_{ij}\alpha_{ik} \\ &- \frac{1}{n} \sum_{k=i+1}^{n} (a_{kk} - a_{11}). \end{aligned}$$

But $a_{jj} - a_{11}$, $j = 2, \ldots, n$ and a_{jk}, $j, k = 1, \ldots, n$, $j < k$ are independent and $X(\alpha_1) - \mathcal{H}, \ldots, X(\alpha_n) - \mathcal{H}$ are dependent. Thus the matrix

$$\begin{bmatrix} -\frac{1}{n} & \ldots & -\frac{1}{n} & 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \alpha_{i2}^2 - \frac{1}{n} \ldots \alpha_{ii}^2 - \frac{1}{n} & - \frac{1}{n} \ldots & -\frac{1}{n} & 2\alpha_{i1}\alpha_{i2} \ldots 2\alpha_{ii-1}\alpha_i 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \alpha_{n2}^2 - \frac{1}{n} & \ldots & \alpha_{nn}^2 - \frac{1}{n} & 2\alpha_{n1}\alpha_{n2} & \ldots & \alpha_{nn-1}\alpha_{nn} \end{bmatrix}$$

falls in rank. Next, consider the square matrix of the first $n-1$ columns together with one of the last, say

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \alpha_{nj}\alpha_{nn} \end{bmatrix} j < n.$$

Its determinant, expanding via the last columns, is

$$\alpha_{nj}\alpha_{nn}\begin{vmatrix} -\frac{1}{n} & & -\frac{1}{n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \alpha^2_{n-1\,2} & -\frac{1}{n} \dots & -\frac{1}{n} \end{vmatrix},$$

which is $-1/n\alpha_{nj}\alpha_{nn}\alpha_{n-1\,n-1}\dots\alpha_{22}$. But since the determinant is equal to zero and $\alpha_{ii} \neq 0$ we must have $\alpha_{nj} = 0$, $j < n$. So $\alpha_n = e_n$.

Similarly, by induction if you wish, we prove that $\alpha_{n-1} = e_{n-1}$, ..., $\alpha_2 = e_2$.

COROLLARY 3.18. - *If $\mathcal{H}$ lies on the span of $j \leq n$ points of V^{n-1} then $j = n$ and the points are images of a frame.*

PROOF. - If $j < n$ then the j points lie in a sub-VERONESE manifold, and so $\mathcal{H}$ cannot lie in their span. Thus $j = n$. If the points lay in a sub-VERONESE manifold then $\mathcal{H}$ could not lie in their span; hence they lie in no sub-VERONESE manifold and so, by the theorem above, the points are images of a frame.

PROPOSITION 3.19. - *A Veronese manifold has no trisecants.*

PROOF. - Note that the standard manifold lies on a sphere.

COROLLARY 3.20. - *If a 2-plane through the tip of $\mathcal{H}$ meets V^2 in 3 points, they are the image of a frame.*

PROOF. - If the 3 points are dependent they lie on a line, but the VERONESE surface has no trisecants. Thus, they are independent, so that $\mathcal{H}$ lies in their span. Thus, by the preceeding corollary, they are the image of a frame.

DEFINITION 3.21. - By an *n-gon* we mean the image of a frame, or n points of V^{n-1} whose centroid is $\mathcal{H}$.

THEOREM 3.22. - *The $n-1$ planes of all n-gons fill out the entire $N-1$ space in which V^{n-1} lies.*

PROOF. - It will be convenient for this proof to assume that the vectors a_{ij} are all independent. If this is not the case we may translate V^{n-1} so that its linear span, $\mathfrak{L}$, does not contain the origin. Let b_{ij} be a dual to a_{ij}, $i \leq j$. That is, $b_{ij} \cdot a_{kl} = 0$ unless $i = k$, $j = l$, in which case, $b_{ij} \cdot a_{ij} = 1$. Let v be the position vector of a point in $\mathfrak{L}$. We may write v as

$$v = \sum_{i \leq j} (b_{ij} \cdot v) a_{ij} .$$

Since the tip of v lies in $\mathfrak{L}$ we know that

$$\sum_{i=1}^{n} b_{ii} \cdot v = 1.$$

Thus we see that $(b_{ij} \cdot v)$ is a symmetric matrix with trace $= 1$.

Here we define $b_{ij} = b_{ji}$. Let

$$X^*(x) = \Sigma b_{ij} x_i x_j .$$

X^* also gives a VERONESE $n-1$ manifold which we call V^*, the dual VERONESE manifold. Consider

$$X^*(x) \cdot v = \Sigma (b_{ij} \cdot v) x_i x_j ,$$

this is a quadratic form with matrix $(b_{ij} \cdot v)$. We may, by a rotation, diagonalize this quadratic form. Let the diagonalizing frame be $e_1' \dots e_n'$. In terms of this frame let us write

$$X(x) = \Sigma a_{ij}' \cdot x_i x_j$$

and let b_{ij}' be dual to a_{ij}'. Then $(b_{ij}' \cdot v)$ will be in diagonal form. Also its trace will remain 1. Thus we have $b_{ij}' \cdot v = 0$, $i \neq j$ and $b_{ii} \cdot v = \lambda_i$, where $\Sigma \lambda_{ii} = 1$. Hence

$$\begin{aligned} v &= \sum_{i \leq j} (b_{ij}' \cdot v) a_{ij}' \\ &= \sum_{i=1}^{n} \lambda_i a_{ii}', \quad \text{where} \quad \sum_{i=1}^{n} \lambda_i = 1. \end{aligned}$$

This shows that the point v lies in the $n-1$ plane through the points a_{ii}'. Since of course $X(e_i') = a_{ii}'$ we see that the point v lies in the span of an $n-1$ plane through an n-gon.

THEOREM 3.23. - *Given a line through the centroid of V^{n-1} lying in $\mathfrak{L}$, the linear span of V^{n-1}, there is an n-gon whose $n-1$ plane contains the line. If the line does not pass through the span of a Veronese $n-3$ submanifold the n-gon is unique.*

PROOF. - As in the previous theorem we may assume that the a_{ij} are a basis for E^N. Let v be the position vector of a point on the line; $v \neq \mathcal{H}$. Then by the previous theorem we may assume that the point v lies in the $n-1$ plane of an n-gon. Suppose that $e_1 \dots e_n$ is a tangent frame which gives the n-gon. Let $x_1, \dots, x_n$ be coordinates written in terms of $e_1 \dots e_n$ and let

$$X(x) = \Sigma a_{ij} x_i x_j .$$

Thus the n-gon is just $a_{11}, \ldots, a_{nn}$. The $n-1$ plane, since it contains the points v and $\mathcal{H}$ must contain the line passing through these two points.

Now suppose that v lies in the $n-1$ plane of another n-gon, with frame $e'_1 \ldots e'_n$. In terms of this frame let us write

$$X(x) = \Sigma a'_{ij} x_i x_j .$$

The n-gon is then is then given by $a'_{11}, \ldots, a'_{nn}$ and since v lies in its $n-1$ plane we must have

$$v = \sum_{i=1}^{n} \lambda'_i a'_{ii}, \quad \text{where} \quad \sum_{i=1}^{n} \lambda'_i = 1.$$

Let b_{ij} and b'_{ij} be dual bases for the bases a_{ij} and a'_{ij} respectively. Then we see that the matrices $(b_{ij} \cdot v)$ and $(b'_i \cdot v)$ are both in diagonal form. Furthermore they are both matrices for the quadratic form $X^* \cdot v$ with respect to these two bases. X^* is the dual VERONESE manifold defined in the previous theorem. Thus up to the order of the diagonal entries the matrices $(b_{ij} \cdot v)$ and $(b'_{ij} \cdot v)$ are the same. By reordering, if necessary, the basis $e'_1, \ldots, e'_n$, we may assume the matrices are identical. Notice that reordering the basis $e'_1, \ldots, e'_n$ does not change the n-gon $a'_{11}, \ldots, a'_{nn}$. If the eigenvalues are distinct then the eigenvectors must be the same up to a sign, and hence the two n-gons identical. However, we have assumed the two n-gons are different.

Thus there must be at least two equal eigenvalues, say $\lambda_1 = \lambda_2 = \lambda$.

This means that we may write

$$v = \lambda a_{11} + \lambda a_{22} + \sum_{i=3}^{n} \lambda_i a_{ii},$$

where of course $2\lambda + \sum_{i=3}^{n} \lambda_i = 1$. But

$$n\mathcal{H} = a_{11} + a_{22} + \sum_{i=3}^{n} a_{ii}.$$

Thus

$$v = n\lambda\mathcal{H} + \sum_{i=3}^{n} (\lambda_i - \lambda) a_{ii},$$

where $n\lambda + \sum_{i=3}^{n} (\lambda_i - \lambda) = 1$. Let $\mu = \sum_{i=3}^{n} (\lambda_i - \lambda)$ and let $b = \frac{1}{\mu} \sum_{i=3}^{n} (\lambda_i - \lambda) a_{ii}$. Thus we may write

$$v = \lambda n\mathcal{H} + \mu b, \quad \text{where} \quad \lambda n + \mu = 1.$$

This means that v lies on the line joining $\mathcal{H}$ and b. But b is in the linear span of the VERONESE $n-3$ submanifold which is the image of the great $n-3$ sphere containing $e_3, \ldots, e_n$. Thus the desired conclusion is reached.

REMARK 3.24. - *Every point in the span of a Veronese $n-3$ submanifold lies on the span of two distinct n-gons.*

PROOF. - There is an $n-2$ gon of the submanifold whose span contains the point. Complete the $n-2$ frame to an n-frame in two different ways getting two distinct n-gons.

The standard manifold has the property that the image of a frame is a frame, see Proposition 3.8. Let us show the converse

PROPOSITION 3.25. - *If $X = \sum a_{ij} x_i x_j$ maps frames to frames then it is congruent to the standard manifold.*

PROOF. - Since $X(e_i) = a_{ii}$ and $e_i, \ldots, e_n$ is a frame, we see that $a_{ii}{}^2 = 1$ and $a_{ii} \cdot a_{jj} = 0$. Since any point belongs to some frame $X \cdot X = 1$, so the manifold lies on the unit S^{N-1}.

If we write out $X \cdot X$, using the fact that $x_1^2 = 1\, x_2^2 - \ldots - x_n^2$ to eliminate powers of x_1 greater than one, then the monomials in $x_1, \ldots, x_n$ are independent. Equating coefficients and using the fact that nothing is special about the index 1 we find

$$
\begin{aligned}
&a_{ij} \cdot a_{kl} = 0, \\
&a_{ii} \cdot a_{ij} = 0, \\
&a_{ii} \cdot a_{jk} + 2a_{ij} \cdot a_{ik} = 0, \\
&(a_{jj} - a_{ii})^2 = 4a_{ij}{}^2, \\
&a_{ii}{}^2 = 1,
\end{aligned}
$$

where the indices are distinct. Using the fact that $a_{ii} \cdot a_{jj} = 0$ we have $a_{ij}{}^2 = \frac{1}{2}$.

Again, since frames map to frames, $X(e_i)$ must be orthogonal to $X(\alpha)$, where α is perpendicular to e_1. Thus $a_{11} \cdot (\sum_{i,j=2}^{n} a_{ij} x_i x_j) = 0$, where $x_1^2 + \ldots + x_n^2 = 1$. Eliminating powers of x_2 greater than 1 we have an expression in which the monomials are independent. Thus we must have $a_{11} \cdot a_{ij} = 0$, $i, j \geq 2$. By symmetry of the indices, $a_{ii} \cdot a_{jk} = 0$. This, together with the above, shows that $a_{ij} \cdot a_{jk} = 0$, and now the result readily follows.

COROLLARY 3.26. - *All Veronese submanifolds of the standard manifold are congruent to the standard manifolds of their dimension.*

Another property of the standard manifold is stated in

PROPOSITION 3.27. - *Let V^{i-1} be a Veronese submanifold of the standard manifold. Then $S^{N-2} \cap$ span V^{i-1} is a sphere whose center is the centroid of V^{i-1}. S^{N-2} is the sphere in $\mathfrak{L}$ with center $\mathfrak{H}$ which contains V_{st}.*

PROOF. - The intersection is certainly a sphere since the intersection of a linear space and a sphere is again a sphere. Also, since the submanifold is congruent to a standard manifold, it lies on a sphere whose center is the centroid. Let $C =$ the center of the sphere $S^{N-2} \cap \text{span } V^{i-1}$. If C is not the centroid let L be the line through C and the centroid. Choose an i-gon whose $i-1$ plane contains the line L. Then C is equidistant from the vertices of a regular i-gon, lies in the $i-1$ plane of the i-gon, and is not its centroid. This is a contradiction.

PROPOSITION 3.28. - *Any Veronese manifold, V^{n-1}, lies in a hyperquadric Q^{N-2} which has the property that the span of any Veronese submanifold V^{i-1} meets Q^{N-2} in a hyperquadric whose center is the centroid of V^{i-1}. Also the center of Q^{N-2} is the centroid of V^{n-1}.*

PROOF. - The proposition is affine and the previous proposition verified this proposition for the standard manifold.

DEFINITION 3.29. - Let us say that such a hyperquadric and manifold belong to each other.

THEOREM 3.30. - *Every Veronese manifold has a unique hyperquadric belonging to it.*

PROOF. - Since V^{n-1} can be mapped to the standard manifold by an affine map of all of E^N, it is enough to show that S^{N-2} is the only hyperquadric belonging to the standard manifold. Again it is enough to show that if V_1 belongs to S^{N-2} then V_1 is congruent to V_{st}. For suppose V_{st} belongs to a Q^{N-2}, not a sphere. Let A be the affine map which takes Q^{N-2} to S^{N-2}. Then $A(V_{st})$ belongs to $A(Q^{N-2}) = S^{N-2}$. Also $A(V_{st})$ is not congruent to V_{st} because A is not a congruence. Thus if V_{st} belongs to two hyperquadrics then two noncongruent manifolds $A(V_{st})$ and V_{st} belong to S^{N-2}. The converse is proved similarly.

We now show that if V_1 belongs to S^{N-2} then V_1 is congruent to the standard manifold. To see this, consider a 3-frame in S^{n-1}, $\alpha\beta\gamma$. Let X be the map for V_1. Let $\mathcal{H}_1 = \frac{1}{3}(X(\alpha) + X(\beta) + X(\gamma))$. Let S^2 be the great two sphere containing α, β, γ. Then $\mathcal{H}_1$ is the centroid of the sub-Veronese manifold $X(S^2)$. The span of $X(S^2)$ is a 5-space which intersects S^{N-2} in a 4-sphere, say S^4. Then because S^{N-2} belongs to V_1, the center of S^4 is $\mathcal{H}_1$. Thus, because $X(\alpha)$, $X(\beta)$, $X(\gamma)$ lie on S^4, the vertices of the triangle $X(\alpha)$ $X(\beta)$ $X(\gamma)$ are equidistant from its centroid, and consequently it is equilateral.

Now given an n-gon $X(\alpha_1), \ldots, X(\alpha_n)$ of V_1, we see that it must be regular since any three vertices form an equilateral triangle. Since S^{N-2} belongs to V_1, the centroid of V_1 is the center of $S^{N-2} = \mathcal{H} = \frac{1}{n} \sum e_i \circ e_i$. Also, since

$V_1 \subset S^{N-2} \subset \mathfrak{L}$, the span $V_1 = \mathfrak{L}$ which is perpendicular to $\mathcal{H}$. From this one sees that $X(\alpha_1), \ldots, X(\alpha_n)$ is a frame. Consequently, V_1 has the property that the image of every is again a frame, and so V_1 is congruent to V_{st}. This completes the proof.

Let us call the map

$$X : S^{n-1} \to E^N$$

given by

$$X(x_1, \ldots, x_n) = \sum a_{ij} x_i x_j$$

a *configuration* if the vectors $a_{11}, \ldots, a_{n-1n}$ are all independent.

We make this definition because we will need to distinguish between a VERONESE manifold, $X(S^{n-1})$, and the map X.

Let us also consider the map

$$\chi : S^{n-1} \to E^{N-1}$$

given by

$$\chi(x_1, \ldots, x_n) = \sum a_{ij} x_i x_j ,$$

where $a_{ij} = (a^1_{ij}, \ldots, a^{N-1}_{ij})$ is a vector in E^{N-1}. (Notice E^{N-1}, not E^N). Assume that $a^1_{ij}, \ldots, a^{N-1}_{ij}$ are $N-1$ symmetric matrices. If the N vectors $a_{11}, \ldots, a_{n-1n}$ have maximal rank we call χ a *projected configuration.*

LEMMA 3.31. - *A projected configuration is the projection of a configuration (along e_N).*

PROOF. - We may extend the $N \times N-1$ matrix (a^k_{ij}) to an $N \times N$ matrix (a^k_{ij}) which again has maximal rank.

REMARK. - A projected configuration is of two types. Either the span of $X(S^{n-1})$ is all of E^{N-1}, in which case $X(S^{n-1})$ is a VERONESE $n-1$ manifold and the vector $\mathcal{H}$ is contained in its span, or else $X(S^{n-1})$ does not span E^{N-1}. In this case both $\mathcal{H}$ and $X(S^{n-1})$ together span E^{N-1}. In this second case we call $X(S^{n-1})$ a *Steiner* variety.

Let X' be a projected configuration and X a configuration such that $\pi \circ X = X'$, where π is projection along e_N of E^N onto E^{N-1}. Let $V = X(S^{n-1})$ and $\mathfrak{L} = \text{span } V$. Then X' is of the first type if and only if e_N is not a direction in $\mathfrak{L}$ and otherwise X' is of the second type.

THEOREM 3.32. - *Let X' be a projected configuration, with $\mathcal{H}' = \sum a_{ii}$, $V' = X'(S^{n-1})$ and $\mathfrak{L}' = \text{span } V'$. If the line through $\mathcal{H}'$ does not meet the span of $X'(S^{n-3})$ for any great S^{n-3} contained in S^{n-1} then we may uniquely choose axes and furthermore the axes depend continuously on the configuration.*

PROOF. - If X' is of the first type then V' is a VERONESE manifold, $\mathcal{H}'$ lies in $\mathfrak{L}'$ and the tip of $\mathcal{H}'$ is the centroid of V'. Also, by assumption, $\mathcal{H}'$

meets no $n-3$ sub-VERONESE manifold. Thus using Theorem 3.23 we may define a unique frame.

Suppose that X' is the projection of a configuration X, along e_N, and that π is the projection. $\pi: E^N \to E^{N-1}$. Suppose $\mathcal{H}$, $\mathfrak{L}$, V go with η. Then $\pi\mathcal{H} = \mathcal{H}'$, $\pi\mathfrak{L} = \mathfrak{L}'$, $\pi V = V'$, $\pi \circ X = X'$.

Suppose that X' is of the second type. Then e_N is along $\mathfrak{L}$. Let l be the line in the direction of e_N through the tip of $\mathcal{H}$. We claim that l meets the span of no VERONESE $n-3$ submanifold. For if l did meet the span of some $X(S^{n-3})$, $\pi l =$ tip $\mathcal{H}'$ would meet the span of some $X'(S^{n-3})$, which is not the case. Thus, by Theorem 3.23, l picks out a unique set of axes. It is these axes that we choose in the case X' is of the second type. We must show that they are independent of X and depend only on X'. Let $\mathcal{S}$ be the unique $n-1$ plane which meets V in n points and contains l. Since π is projection along l, $\pi\mathcal{S}$ is an $n-2$ plane which meets V' in n points. Since $\pi \circ X = X'$, these n points are the images of the axes picked out by l. Suppose there exists another $n-2$ plane $\mathcal{S}'$ through the tip of $\mathcal{H}'$ which meets V' in n points. Then $\pi^{-1}(\mathcal{S}')$ is an $n-1$ plane which contains l and meets V in n points. By the uniqueness of $\mathcal{S}$ we must have $\pi^{-1}(\mathcal{S}') = \mathcal{S}$. Hence $\mathcal{S}' = \pi(\mathcal{S})$. Thus by a construction depending solely on X' we have uniquely chosen axes.

We next show that the axes depend continuously on the projected configuration X'. With notation as before assume that X' is of either type and that $\pi \circ X = X'$. Let $\mathfrak{R}$ be the 2-plane spanned by $\mathcal{H}$ and the line parallel to e_N through the tip of $\mathcal{H}$. This 2-plane is well defined, because if $\mathcal{H}$ were along e_N then $\pi\mathcal{H} = \mathcal{H}'$ would be the zero vector, which by the definition of a projected configuration is not the case. Also $\pi\mathfrak{R}$ is the line through $\mathcal{H}'$, and hence $\pi\mathfrak{R}$ meets the span of no $X'(S^{n-3})$. Thus $\mathfrak{R}$ meets the span of no $X(S^{n-3})$. Also $\mathfrak{R}$ meets $\mathfrak{L}$ in a unique line, say W. This is so because $\mathfrak{R}$ does not lie in $\mathfrak{L}$, because $\mathfrak{R}$ contains $\mathcal{H}$ and $\mathcal{H}$ does not lie in $\mathfrak{L}$, and because $\mathfrak{R}$ meets $\mathfrak{L}$ at the tip of $\mathcal{H}$ and so for dimensional reasons must meet $\mathfrak{L}$ in at least a line. Thus, since $\mathfrak{R}$ meets the span of no sub-VERONESE $n-3$ manifold of V, neither does W. Hence, by Theorem 3.23, W picks out a unique set of axes. It is easy to see that these axes depend continuously on the configuration X, regardless of whether e_N is a direction in $\mathfrak{L}$ or not.

Thus it will be sufficient to show that under π the axes, picked out by W are equal to the previously defined axes on X'. If X' is of the second type this is trivial by the definition. If X' is of the first type then, since W is not along e_N, we have $\pi W = \pi\mathfrak{R} = \mathcal{H}'$ and from this it readily follows.

CHAPTER IV.

Principal Axes and Singularities.

For a surface in ordinary space the usual principal axes are defined everywhere except, of course, at umbilics. A consequence of this is the familiar theorem that a surface in ordinary space of non-zero EULER characteristic must have an umbilic. We have seen in the first chapter that one may also construct principal axes for surfaces in E^4 which generalize the construction made in E^3. As a consequence, we were able to prove several global theorems which generalize the situation for surfaces in E^3. It is the purpose of this chapter to define principal axes for manifolds of arbitrary dimension and to state resulting theorems concerning their singularities.

Let $X: M^n \to E^k$ be an immersion of a differentiable manifold in Euclidean space. Let $\mathcal{H}$ be the mean curvature vector of the manifold and let $\mathfrak{V}$ be the subset of the first normal space defined as follows: $\mathfrak{V}$ is the set of endpoints of curvature vectors of geodesics, parameterized by arc length, which pass through p. The first normal space is the sum of the mean curvature vector and the linear span of $\mathfrak{V}$, which we call $\mathfrak{L}$. If the point is not inflectional and if $k \geq \nu = \frac{1}{2} n(n+1) + n$, this sum is direct. In this case $\mathfrak{V}$ is a VERONESE manifold, and the results of Chapter III apply. In particular, $\mathcal{H}$ is the centroid of $\mathfrak{V}$.

For the case $k = \nu - 1$ the situation becomes more complicated.

At a non-inflectional point two different situations can obtain.

Either $\mathfrak{L}$ is all of the first normal space, in which case $\mathfrak{V}$ is a VERONESE manifold, or else both $\mathfrak{L}$ and $\mathcal{H}$ are needed to span the first normal space. In this latter case $\mathfrak{V}$ is the projection of a VERONESE manifold.

It will be helpful to introduce a map η from the space of tangent lines to M at p into the first normal space to M at p. Given a tangent line l at p, let γ be a geodesic, parameterized by arc length, whose tangent vector at p is along the given line. Then $\eta(l)$ is the curvature vector of γ at p. That η depends only on the line and not on the curve follows from a generalized MEUSNIER'S theorem.

Note that η maps the space of tangent lines onto the space $\mathfrak{V}$.

We now define principal axes for the case $k \geq \nu$, $\mathcal{H}$ will not « in general » be perpendicular to $\mathfrak{L}$. Let $\mathcal{H}'$ be the projection of $\mathcal{H}$ normally onto $\mathfrak{L}$. The vector $\mathcal{H}'$ will « in general » lie in a unique $n-1$ plane which meets the VERONESE manifold in exactly n distinct points. These points are the images of n mutually perpendicular tangent directions under the map η.

Notice that the principal axes are well defined at every point of the manifold except at inflection points, at points where $\mathcal{H}$ is perpendicular to $\mathfrak{L}$, and at points where $\mathcal{H}'$ does not contain a unique $n-1$ plane which meets the VERONESE manifold in n points.

We define principal axes also for the case $k = \nu - 1$. This includes surfaces in E^4. Assume that the point is not an inflection point. There are two cases. The first case in when $\mathfrak{L}$ is the entire first normal space. $\mathcal{H}$ is then contained in $\mathfrak{L}$ so that «in general» there is a unique $n-1$ plane through $\mathcal{H}$ which meets $\mathfrak{Y}$ in exactly n points. These points are images under η of n mutually orthogonal tangent lines. In the second case $\mathfrak{Y}$ is the projection of a VERONESE manifold and $\mathcal{H}$ does not lie in $\mathfrak{L}$ but meets $\mathfrak{L}$ at one point, namely the centroid of $\mathfrak{Y}$. «In general» there is a unique $n-2$ plane through the tip of $\mathcal{H}$ which meets $\mathfrak{Y}$ in exactly n points, and again these points are the images under η of a set of mutually orthogonal lines.

For $k = \nu - 1$ the principal axes are defined at each point except inflection points and except those points where the $n-1$ plane (in the second case the $n-2$ plane) through $\mathcal{H}$, which meets $\mathfrak{Y}$, is not unique.

It should be noted that the axes depend continuously on the (a_{ij}^k) and hence the axes, where defined on M, are continuous. See Theorem 3.32.

We come now to the generic dimension of the singular locus of the various types of singular points in question. For this discussion we use Theorem 2.20 of Chapter II.

The condition that $\mathcal{H}$ be perpendicular to $\mathfrak{L}$ requires $\frac{1}{2}n(n+1)-1$ conditions. Hence, except for $n=2$, generically $\mathcal{H}$ is never perpendicular to $\mathfrak{L}$. In the case $n = 2$, a surface in E^5 for example, the generic locus consists of isolated points.

Inflection points have been studied by FELDMAN [7]. We mention that for $k = \nu$ the generic locus is codimension 1, that for $k = \nu - 1$ it is of codimension 2 and that for $k \geq \nu + n$ the generic locus of inflection points is empty.

Consider next the locus of points where $\mathcal{H}'$, the projection of $\mathcal{H}$ onto $\mathfrak{L}$, fails to contain a unique $n-1$ plane which meets $\mathfrak{Y}$ in n points. In order to describe these points let $\mathcal{G}$ be a subset of the first normal space at p defined as follows

$$\mathcal{G} = \bigcup \ \operatorname{span} \eta(S^{n-3}),$$

where the union is taken over all great S^{n-3} contained in the unit tangent sphere to η at p. We may as well assume that p is not an inflection point, since if it is we treat p as part of the locus of inflection points. In this case $\mathfrak{Y}$ is a VERONESE manifold and each $\eta(S^{n-3})$ is an $n-3$ VERONESE submanifold of $\mathfrak{V}$. By Theorem 3.23 $\mathcal{H}'$ contains a unique $n-1$ plane which meets $\mathfrak{V}$ in n points if and only if the line through $\mathcal{H}'$ does not meet $\mathcal{G}$.

Let us now compute the dimension of $\mathcal{G}$. There is a $2n-4$ parameter family of $n-3$ spheres on the $n-1$ sphere. Also, span $\eta(S^{n-3})$ is the span of a VERONESE $n-3$ submanifold and is therefore of dimension $\frac{1}{2}(n-2)(n-1)-1$. Hence $\dim G = 2n-4+\frac{1}{2}(n-2)(n-1)-1$. The dimension of all lines through the tip of $\mathcal{H}$ is $\frac{1}{2}n(n+1)-2$. Thus it requires

$$\frac{1}{2}n(n+1)-2-(2n-4+\frac{1}{2}(n-2)(n-1)-1)=2$$

conditions to insure that a line through the tip of $\mathcal{H}$ does not pass through G. Hence the generic locus is of codimension 2.

Lastly, let us examine, in the case where $k=\nu-1$, the points where the frames are not uniquely defined. We do not consider inflection points as they have been already dealt with. Define $\mathcal{G}$ as before. There are two cases. In the case when $\mathcal{L}$ is equal to the first normal space, by Theorem 3.23, the frame is unique when $\mathcal{H}$ fails to pass through $\mathcal{L}$. This may be rephrased to say that $\mathcal{H}$ does not lie in the second order osculating space of any submanifold of codimension 2 which passes through the point. In the second case, by Theorem 3.32, the frames are unique when the tip of $\mathcal{H}$ fails to lie in $\mathcal{G}$. This again may be rephrased to say that $\mathcal{H}$ does not lie in the second order osculating space of any submanifold of codimension 2. That the frames are continuous where defined is also a consequence of Theorem 3.32.

We compute the dimension of the generic locus. In the first case, when $\mathcal{L}$ is the entire first normal space, by the same argument as before, the generic codimension is 2.

In the second case, as before,

$$\dim G = 2n-4+\frac{1}{2}(n-2)(n-1)-1.$$

The dimension of $\mathcal{L}$ is now $\frac{1}{2}n(n+1)-2$, and thus it requires

$$\frac{1}{2}n(n+1)-2-(2n-4+\frac{1}{2}(n-2)(n-1)-1)=2$$

conditions to insure that the tip of $\mathcal{H}$ does not lie in $\mathcal{G}$. Hence, in this case also the generic codimension is 2.

Define an $\mathcal{A}$ manifold to be a manifold which does not admit a field of axes, i.e. a field of n mutually orthogonal tangent lines.

Examples are simply connected nonparallelizable manifolds.

THEOREM 4.1. - *Let $X: M \to E^{\nu-1}$ be an immersion of an $\mathfrak{A}$ manifold in Euclidean space of dimension $\nu - 1$ where $\nu = \frac{1}{2}n(n+3)$. Then X has an inflection point or a point where the mean curvature vector lies in the second order osculating space of some submanifold of codimension 2. The generic codimension of the singular locus is 2.*

Let a point be called a G point if $\mathcal{H}'$, the projection of $\mathcal{H}$ onto $\mathfrak{L}$, lies in the subset $\mathcal{G}$ of the first normal space.

THEOREM 4.2. - *Let $X: M^n \to E^{\nu+k}$ be an immersion of an $\mathfrak{A}$ manifold into Euclidean $\nu + k$ space, where $\nu = \frac{1}{2}n(n+3)$ and $k \geq 0$. Then X has either an inflection point or a G point. The generic codimension of the locus of inflection points is k, and the generic codimension of the locus of G points is 2. This shows that G singular points are stable under raising the codimension. For the case $k \geq n$ generically there are no inflection points so that the singular points consist solely of those of type G.*

Let us now ask if it is possible to construct axes in the first normal space. We say that a hyperellipsoid, Q, «belongs» to $\mathcal{V}$ if the following are true. Let S^r be any great r-sphere contained in the unit tangent sphere. Let $\mathfrak{L}(S^r)$ be the linear span of $\eta(S^r)$. Then Q belongs to $\mathcal{V}$ if

$$\mathcal{V} \subset Q \subset \mathfrak{L},$$

$$\text{centroid of } \mathcal{V} = \text{center } Q,$$

$$\text{centroid } \eta(S^r) = \text{center } \mathfrak{L}(S^r) \cap Q \text{ for } r \leq n-1.$$

Note that $\mathfrak{L}(S^r) \cap Q$ is again a hyperellipsoid. We mention that if $n = 2$ or 3 the first two conditions imply the third.

We have seen in Theorem 3.30 that there is a unique hyperellipsoid Q which belongs to $\mathcal{V}$ in the case that $\mathcal{V}$ is a VERONESE manifold. «In general» Q will have a unique set of principal axes which span $\mathfrak{L}$. These together with the normal to $\mathfrak{L}$ in the first normal space give a set of axes which span the first normal space.

Let us call a point a T point if Q at that point fails to have a unique set of axes.

THEOREM 4.3. - *Let $X: M^n \to E^{\nu+k}$, $k \geq 0$ be an inflection free immersion of a simply connected n-manifold. The either the first normal bundle is trivial or else there exists a T point. Furthermore, the generic codimension of such points is 2. We mention that if $k = 0$, the condition that the first normal bundle is trivial implies that M is a π manifold.*

PROOF. - Since X is an inflection free immersion in $E^{\nu+k}$, we know that $\mathfrak{Y}$ is a VERONESE manifold and not the projection of one. Thus by the previous discussion these are a set of axes which span the first normal space at every point which is not a T point. Since M is simply connected, the existence of a set of such axes implies the existence of frames. Thus the first normal bundle must be trivial if there are no T points.

To compute the generic codimension of T points, it is necessary only to find the codimension in the family of all hyperellipsoids of those with two or more equal eigenvectors. This codimension is easily seen to be 2.

It may be shown that if $\mathfrak{Y}$ is the projection of a VERONESE manifold then it still belongs to a unique hyperellipsoid. We may define, as before, a T point to be a point at which Q fails to have a unique set of axes.

THEOREM 4.4. - *Let $X : M^n \to E^{\nu-1}$ be an inflection-free immersion of a simply connected manifold which is not a π manifold. Then there exists a T point. Furthermore, the generic codimension of such points is 2.*

«In general» Q will have a unique principal axis of greatest lenght. Since the center of Q is the tip of $\mathcal{H}$ this axis gives a line through the tip of $\mathcal{H}$. «In general» this line will meet no VERONESE $n-3$ submanifold. Hence «in general» there is a unique $n-1$ plane which contains this axis and which meets $\mathfrak{Y}$ in n points. These points are the images of a frame by Theorem 3.23. This gives an alternative construction of principal axes.

Let a point where such axes are not defined be called a U point.

THEOREM 4.5. - *Let $X : M \to E^{\nu+k}$, $k \geq 0$ be an immersion of an $\mathfrak{A}$-manifold. Then X has a U point. Furthermore, the generic codimension of the locus of U points is 2.*

PROOF. - We make a few comments about the generic dimension. Any configuration is the image of the standard configuration (for the definition see Theorem 3.5), and every affine map of the standard configuration gives a different configuration. Thus any point of S^{N-2} (S^{N-2} in the hyperellipsoid belonging to $\mathfrak{Y}_{st}$) may map into the major axis of Q if the proper affine map is chosen. But also $\mathcal{G}_{st}$ maps into $\mathcal{G}$ under this map. Thus if the preimage of the major axis does not pass through $\mathcal{G}_{st}$ then the major axis will not pass through $\mathcal{G}$. But $\mathcal{G}_{st} \cap S^{N-2}$ is a subset of S^{N-2} of codimension 2.

Also, the family of hyperellipsoids are determined by their principal axes, including the lengths. If two axes have equal length, the degree of freedom is reduced by 2, namely 1 because the axes are equal in length and 1 because in the 2-plane of the two axes no angle is needed to specify an orientation.

Since the model singularity for U points consists of the union of these two singularities, the model singularity for U points must be of codimension 2.

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