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On Size-Biased Double Weighted Exponential Distribution (SDWED)

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Abstract

This paper introduces a new distribution based on the exponential distribution, known as Size-biased Double Weighted Exponential Distribution (SDWED). Some characteristics of the new distribution are obtained. Plots for the cumulative distribution function, pdf and hazard function, tables with values of skewness and kurtosis are provided. As a motivation, the statistical application of the results to a problem of ball bearing data has been provided. It is observed that the new distribution is skewed to the right and bears most of the properties of skewed distribution. It is found that our newly proposed distribution fits better than size-biased Rayleigh and Maxwell distributions and many other distributions. Since many researchers have studied the procedure of the weighted distributions in the estates of forest, biomedicine and biostatistics etc., we hope in numerous fields of theoretical and applied sciences, the findings of this paper will be useful for the practitioners.

Keywords

Exponential Distribution, Moments, Moment Ratios, Estimation

1. Introduction

Weighted distributions are suitable in the situation of unequal probability sampling, such as actuarial sciences, ecology, biomedicine biostatistics and survival data analysis. These distributions are applicable, when observations are recorded without any experiment, repetition and random process. The notion of weighted distributions has been used as a device for the collection of suitable model for observed data, during last 25 years. The idea is most applicable when sampling frame is not available and random sampling is not possible. Firstly the idea of weighted distributions was introduced by

Fisher [1]. Cox [2] firstly provided the idea of length-biased sampling and after that Rao [3] established a unifying method that can be used for several sampling situations and can be displayed by means of the weighted distributions. Cox [4] estimated the mean of the original distribution built on length-biased data. Zelen [5] presented the concept of weighted distribution in studying cell kinetics and early discovery of disease. Warren [6] applied these distributions in forest product research. Patil and Rao [7] surveyed the idea and applications of weighted and size-biased sampling distributions. Patil and Rao [8] also discussed weighted binomial distribution to model the human families and estimation of the wildlife family size. Gupta and Keating [9] described the relationship between reliability measures of original and size-biased distribution. Arnold and Nagaraja [10] gave the idea of bivariate weighted distribution whereas Jain and Nanda [11] extended this idea and discussed multivariate aspect of weighted distribution.

Let $f(x;\theta)$ be the pdf of the random variable x and θ be the unknown parameter.

The weighted distribution is defined as;

$$g(x;\theta)=\frac{w(x)f(x;\theta)}{E[w(x)]},\quad x\in R,\theta>0 \tag{1}$$

where $w(x)$ is a weight function. When $w(x)=x^m$, then these distributions are termed as size-biased distribution of order m. When $m=1$ it is called size-biased of order 1 or say length biased distribution, whereas for $m=2$ it is called the area-biased distribution (Ord and Patil [12], Patil [13] and Mahfound [14]).

In forest product research, equilibrium and length biased distributions have been used as moment distributions. Kochar and Gupta [15] discussed the moment distributional properties in assessment with the actual distributions and derived the bound on the moments of moment distributions.

Oluyede [16] described inequalities for the reliability measures of size-biased and the original distributions. Navarro *et al.* [17] discussed characterization of the original and the size-biased distribution using reliability measures. Gove [18] offered the uses of size-biased distributions in forest science and ecology. Sunoj and Maya [19] established relationships among weighted and original distributions in the situation of repairable system and also characterized the sized-biased and the original distribution. Shen *et al.* [20] used semi-parametric transformations to model the length biased data. Hussain and Ahmad [21] presented misclassification in the size-biased modified power series distributions and its applications.

Mir and Ahmad [22] derived generalized forms of size-biased discrete distributions and discussed the practical applications in the field of Medical, Zoology and Accidental studies. Mir [23] derived size-biased Geeta distribution and size-biased consul distribution respectively, different properties are discussed and contrasts with original distributions are also done. Das and Roy [24] established size-biased form of generalized Rayleigh distribution and apply the consequences to the environmental data. They also applied the concept of size-biased sampling in the field of environmental studies

Dara [25] derived reliability measures for size-biased forms of several moment distributions as the special cases of moment distributions. Iqbal and Ahmad [26] found compound scale mixtures of limiting distribution of generalized log Pearson type VII distribution with different continuous and moment distributions. Hasnain [27] introduced a new family of distributions named as exponentiated moment exponential (EME) distribution and developed its properties. Iqbal *et al.* [28] found a more general class for EME distribution and built up different properties including characterization through conditional moments.

Zahida and Munir [29] worked on Weighted Weibull Distributions (WWD), Double Weibull Distributions (DWD), Weighted Double Weibull Distributions (WDWD), Double Weighted Exponential Distributions (DWED) (both in size-biased and area biased). Some basic theoretical properties of all these distributions including cumulative density function, central moments, skewness, kurtosis and moments are studied. Shannon entropy, Renyi entropy, moment generating function and information generating function of all these distributions are derived. Reliability measures including survival function, failure rates, reverse hazard rate function and Mills ratios of these distributions are also obtained. Parameters are evaluated by using method of maximum likelihood estimation along with derivation of practical examples.

The exponential distribution has a fundamental role in describing a large class of phenomena, particularly in the area of reliability theory. This distribution is commonly used to model waiting times between occurrences of rare events, lifetimes of electrical or mechanical devices. It is also used to get approximate solutions to difficult distribution problems.

2. Methodology

2.1. Size-Biased Double Weighted Exponential Distribution (SDWED)

The size-biased double weighted exponential distribution is given by:

$$g_0(x;\lambda,c)=\frac{f(x)F(cx)}{\int_0^\infty f(x)F(cx)\mathrm{d}x},\ x\ge 0,\ \lambda,c>0 \tag{2}$$

where $f(x)$ is the first weight and $f(x)=\dfrac{w(x)g(x)}{\int_0^\infty xg(x)\mathrm{d}x}$

Here $w(x)=x$ and $g(x)=\lambda\mathrm{e}^{-\lambda x}$, $\lambda>0$, $x\ge 0$ is the pdf of exponential distribution.

Thus the pdf of SDWED is

$$g_0(x;\lambda,c)=\frac{\lambda^2(1+c)^3}{c^3+3c^2}\left[x\mathrm{e}^{-\lambda x}\left(1-\mathrm{e}^{-\lambda cx}-\lambda cx\mathrm{e}^{-\lambda cx}\right)\right],\ x\ge 0,\ \lambda,c>0 \tag{3}$$

where λ is shape parameter and c is scale parameter.

Graphs of Probability Density Function

Figure 1 and Figure 2 show the probability density function of SDWED.

2.2. Distribution Function of SDWED

Distribution function of a density function is defined as:

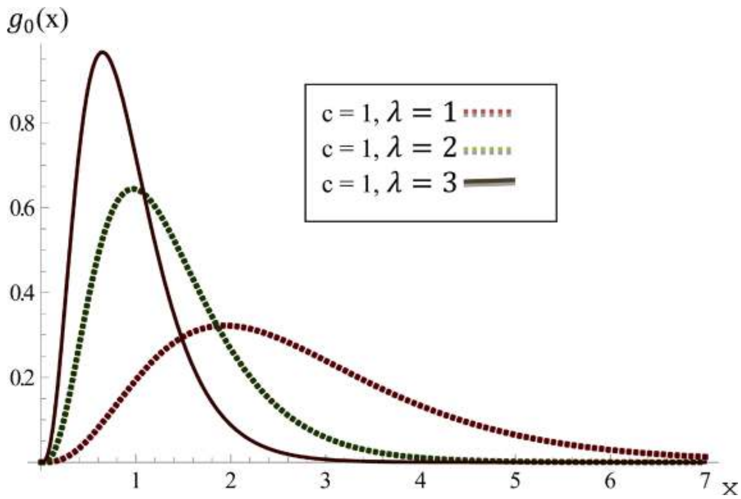

Figure 1. The probability density function of SDWED for the indicated values of *c* and *λ*.

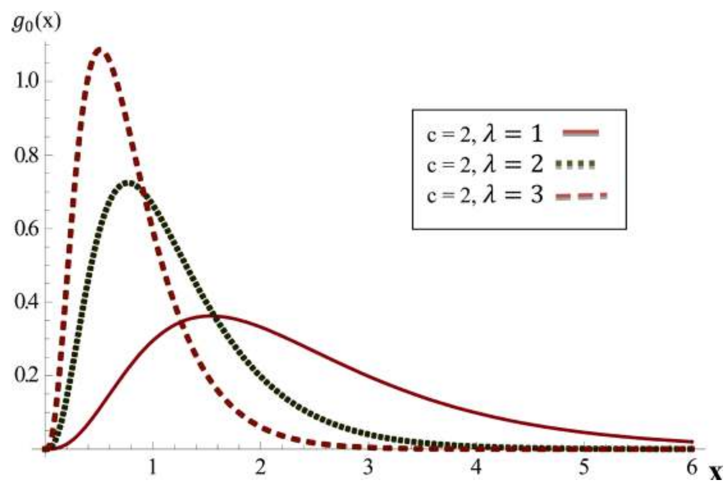

Figure 2. The probability density function of SDWED for the indicated values of *c* and *λ*.

$$F(x;\lambda,c)=\int_0^x h(t)\,\mathrm{d}t$$

$$F(x)=\frac{\lambda^2(1+c)^2}{c^3+3c^2}\int_0^x t\mathrm{e}^{-\lambda t}\left(1-\mathrm{e}^{-\lambda t}-\lambda ct\mathrm{e}^{-\lambda ct}\right)\mathrm{d}t$$

After some simplification we have:

$$F(x)=\frac{(1+c)^3\left[1-\mathrm{e}^{-\lambda x}-\lambda x\mathrm{e}^{-\lambda x}\right]}{c^3+3c^2}-\frac{(1+c)\left[1-\mathrm{e}^{-\lambda x(1+c)}-\lambda x(1+c)\mathrm{e}^{-\lambda x(1+c)}\right]}{c^3+3c^2}$$
$$-\frac{\left[-\lambda^2(1+c)^2\left(x^2\mathrm{e}^{-\lambda x(1+c)}\right)-2\lambda x(1+c)\mathrm{e}^{-\lambda x(1+c)}-2\mathrm{e}^{-\lambda x(1+c)}+2\right]}{c^2+3c} \quad (4)$$

Figure 3 shows the commutative distribution function of SDWED.

2.3. Survival Function

The survival function of SDWED is defined as

$$S(x) = 1 - F(x)$$

$$S(x) = 1 - \frac{(1+c)^3\left[1 - \mathrm{e}^{-\lambda x} - \lambda x \mathrm{e}^{-\lambda x}\right]}{c^3 + 3c^2} - \frac{(1+c)\left[1 - \mathrm{e}^{-\lambda x(1+c)} - \lambda x(1+c)\mathrm{e}^{-\lambda x(1+c)}\right]}{c^3 + 3c^2} - \frac{\left[-\lambda^2(1+c)^2\left(x^2\mathrm{e}^{-\lambda x(1+c)}\right) - 2\lambda x(1+c)\mathrm{e}^{-\lambda x(1+c)} - 2\mathrm{e}^{-\lambda x(1+c)} + 2\right]}{c^2 + 3c} \quad (5)$$

Figure 4 shows the survival function of SDWED.

2.4. Hazard Rate Function of SDWED

The hazard rate function is defined as:

$$h(x) = \frac{g_0(x)}{S(x)}$$

At $c = 1$ and $\lambda = 1$, the hazard rate function will be:

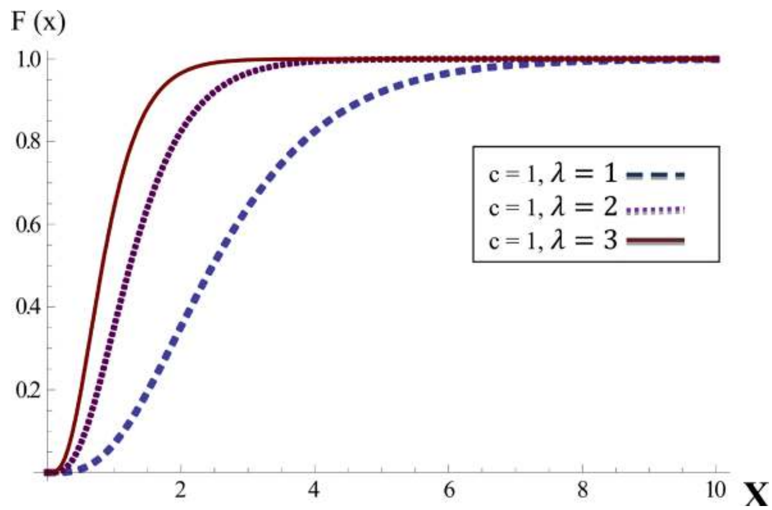

Figure 3. Cumulative Distribution Function of SDWED for the indicated values of c and λ.

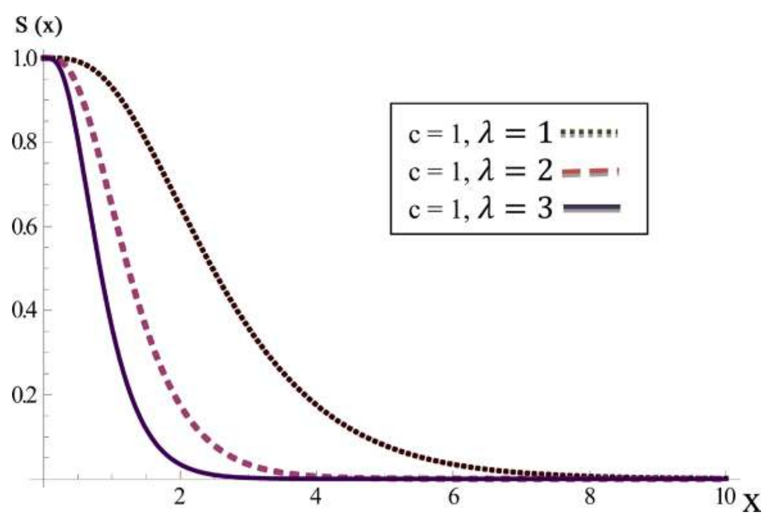

Figure 4. The Survival Function of SDWED for the indicated values of c and λ.

$$h(x)=\frac{2xe^{-x}-2xe^{-2x}-2x^2e^{-2x}}{1-2e^{-x}-2xe^{-2x}+e^{-2x}+1.25xe^{-2x}+x^2e^{-2x}} \quad (6)$$

Figure 5 shows the Hazard Rate Function of SDWED.

2.5. Reverse Hazard Rate Function

The reverse Hazard rate function of SDWED is given by

$$r(x)=\frac{g_0(x)}{F(x)}$$

At $c=1$ and $\lambda=1$, the reverse hazard rate function will be:

$$r(x)=\frac{2xe^{-x}-2xe^{-2x}-2x^2e^{-2x}}{2e^{-x}+2xe^{-2x}-e^{-2x}-1.25xe^{-2x}-x^2e^{-2x}} \quad (7)$$

Figure 6 shows the Reverse Hazard Rate Function of SDWED.

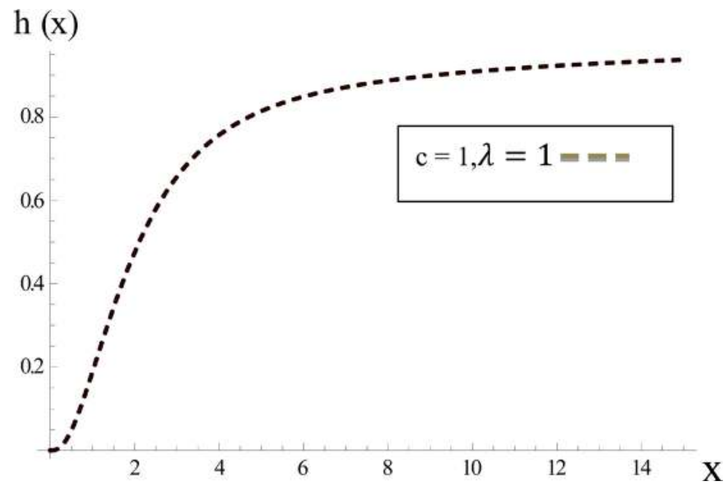

Figure 5. The Hazard Function of SDWED for the indicated values of c and λ.

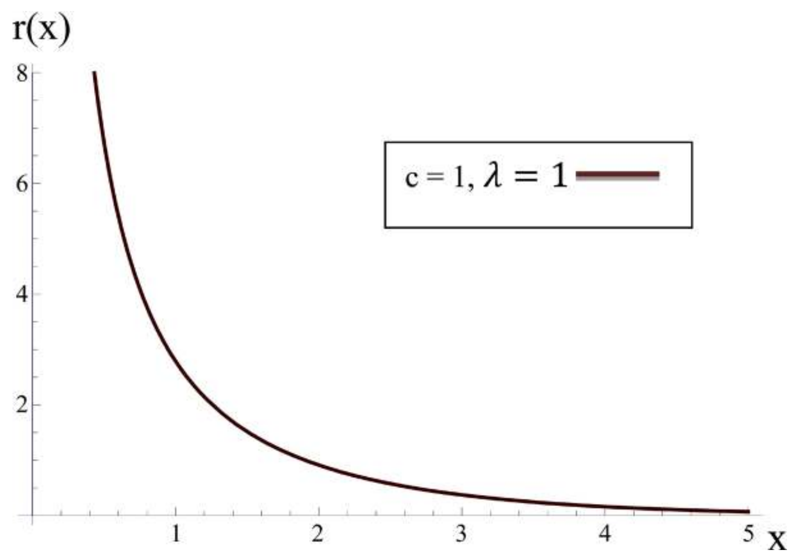

Figure 6. Reverse Hazard Function of SDWED for the indicated values of c and λ.

2.6. Mills Ratio

The Mills Ratio is given by:

$$m(x) = \frac{1}{h(x)}$$

At $c = 1$ and $\lambda = 1$, the Mills Ratio will be:

$$m(x) = \frac{1 - 2e^{-x} - 2xe^{-2x} + e^{-2x} + 1.25xe^{-2x} + x^2e^{-2x}}{2xe^{-x} - 2xe^{-2x} - 2x^2e^{-2x}} \tag{8}$$

Figure 7 shows the mills ratio of SDWED.

2.7. Moment Generating Function of SDWED

The moment generating function of SDWED is:

$$M_X(t) = \int_0^\infty e^{tx} g_0(x) dx \tag{9}$$

Using Equation (3)

$$M_X(t) = \int_0^\infty e^{tx} \cdot \frac{\lambda^2(1+c)^3}{c^3 + 3c^2}\left[xe^{-\lambda x}\left(1 - e^{-\lambda cx} - \lambda cxe^{-\lambda cx}\right)\right]dx$$

$$= \frac{\lambda^2(1+c)^3}{c^3 + 3c^2}\int_0^\infty\left[xe^{-x(\lambda - t)} - xe^{-x(\lambda(1+c)-t)} - x^2e^{-x(\lambda(1+c)-t)}\right]dx$$

Applying the transformations and after simplifying:

$$M_X(t) = \frac{\lambda^2(1+c)^3\left[\left[\left(\lambda(1+c)-t\right)^3\right] - (\lambda - t)^2\left[\lambda(1+c)-t\right] - 2\lambda(\lambda - t)^2 c\right]}{\left[\lambda(1+c)-t\right]^3\left(c^3 + 3c^2\right)(\lambda - t)^2} \tag{10}$$

2.8. Information Generating Function of SDWED

The information generating function is defined as:

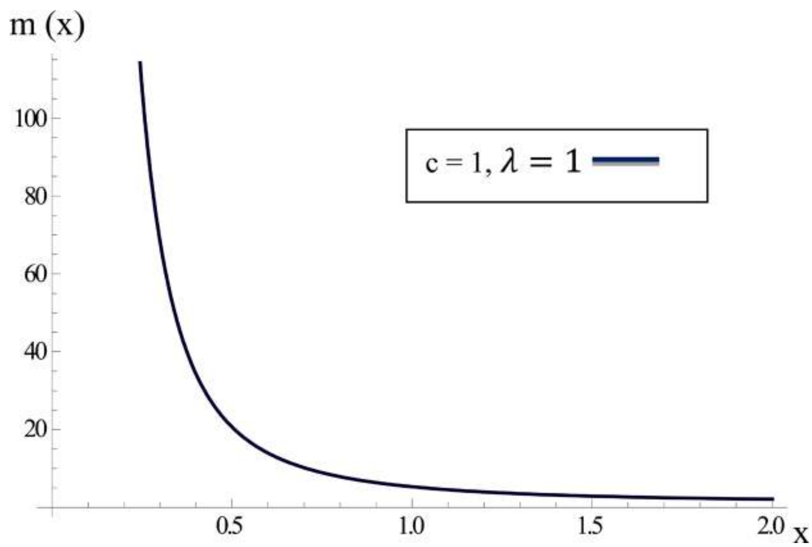

Figure 7. Mills ratio of SDWED for the indicated values of c and λ.

$$T(s)=E\left[g_0(x)\right]^s=\int_0^\infty\left(g_0(x)\right)^s g_0(x)\,\mathrm{d}x$$

Using Equation (3)

$$=\frac{\lambda^{2(s+1)}(1+c)^{3(s+1)}}{\left(c^3+3c^2\right)^{s+1}}\int_0^\infty x^{s+1}\mathrm{e}^{-\lambda x(1+s)}\left(1-\mathrm{e}^{-\lambda xc}-\lambda xc\mathrm{e}^{-\lambda xc}\right)^{1+s}\mathrm{d}x$$

Putting

$$\left(1-\mathrm{e}^{-\lambda xc}-\lambda xc\mathrm{e}^{-\lambda xc}\right)^{1+s}=\sum_{i=0}^{s+1}(s+1)_i(-1)^i\left[\left((1+\lambda xc)\mathrm{e}^{-\lambda xc}\right)^i\right]$$

and after a long simplification, the information generating function will be:

$$T(s)=\sum_{i=0}^{s+1}(s+1)_i(-1)^i\frac{\lambda^{2s}(1+c)^{3(s+1)}}{\lambda^{s+1}[ci+s+1]^{s+2}}\sum_{j=0}^{i}(i)_j\left(\frac{c}{ci+s+1}\right)^j\Gamma(s+j+2) \tag{11}$$

2.9. Limit and Mode of SDWED

Note that the limit of the density function given in Equation (3) is as follows:

$$x\to 0,\ g_0(x;\lambda,c)=x\to 0,\ \frac{\lambda^2(1+c)^3}{c^3+3c^2}\left[x\mathrm{e}^{-\lambda x}\left(1-\mathrm{e}^{-\lambda x}-\lambda cx\mathrm{e}^{-\lambda cx}\right)\right]=0 \tag{12}$$

$$x\to\infty,\ g_0(x;\lambda,c)=\frac{\lambda^2(1+c)^3}{c^3+3c^2}x\to\infty,\ x\mathrm{e}^{-\lambda x}\left(1-\mathrm{e}^{-\lambda x}-\lambda cx\mathrm{e}^{-\lambda cx}\right)=0 \tag{13}$$

since

$$x\to\infty,\ \mathrm{e}^{-\lambda x}=0\quad\text{and}\quad x\to\infty,\left(1-\mathrm{e}^{-\lambda x}-\lambda cx\mathrm{e}^{-\lambda cx}\right)=1 \tag{14}$$

2.10. Mode of SDWED

Taking log of Equation (3) on both sides:

$$\log\left(g_0(x;\lambda,c)\right)=\log\left(\frac{\lambda^2(1+c)^3}{c^3+3c^2}\right)+\log x-\lambda x+\log\left(1-\mathrm{e}^{-\lambda x}-\lambda cx\mathrm{e}^{-\lambda cx}\right) \tag{15}$$

Differentiating Equation (15) with respect to x, we obtain:

$$\frac{\partial}{\partial x}\log\left(g_0(x;\lambda,c)\right)=\frac{1}{x}-\lambda+\frac{c^2\lambda^2x\mathrm{e}^{-\lambda cx}}{1-\mathrm{e}^{-\lambda cx}-\lambda cx\mathrm{e}^{-\lambda cx}}$$

The mode of the SDWED is obtained by solving the nonlinear equation with respect to x:

$$\frac{1}{x}-\lambda+\frac{c^2\lambda^2x\mathrm{e}^{-\lambda cx}}{1-\mathrm{e}^{-\lambda cx}-\lambda cx\mathrm{e}^{-\lambda cx}}=0 \tag{16}$$

The mode of SDWED is given in Table 1.

2.11. Mean of SDWED

$$\mu(x)=\frac{2c^2+8c+12}{\lambda\left(c^2+4c+3\right)} \tag{17}$$

Table 1. Mode of SDWED.

c	λ	mode
2	1	1.543
2	2	0.578
2	3	0.336
2	4	0.250
2	5	0.200

Table 2. Mean, variance and standard deviation of SDWED.

c	λ	Mean	Variance	Standard Deviation
2	1	2.4	1.98	1.41
2	2	1.2	0.50	0.70
2	3	0.8	0.04	0.22
2	4	0.6	0.01	0.12
2	5	0.4	0.01	0.08

2.12. Variance of SDWED

$$\sigma^2(x) = \frac{3c^7 + 26c^6 + 132c^5 + 356c^4 + 590c^3 + 630c^2 + 396c + 108}{\lambda^2\left(c^7 + 13c^6 + 69c^5 + 193c^4 + 307c^3 + 279c^2 + 135c + 27\right)} \tag{18}$$

Table 2 shows the Mean, Variance and Standard Deviation with some values of the parameters λ and c.

2.13. Moments of SDWED

The r^{th} moment of SDWED is given by

$$\mu_r' = E\left(x^r\right) = \frac{\lambda^{-r}}{c^3 + 3c^2}\left[(1+c)^3\Gamma(r+2) - (1+c)^{1-r}\Gamma(r+2) - c(1+c)^{-r}\Gamma(r+3)\right]$$

for r = 1, 2, 3, 4, the first four moments about the mean are

$$\mu_1 = 0$$

$$\mu_2 = \left(\frac{6c^3 + 30c^2 + 60c + 60}{\lambda^2\left(c^3 + 5c^2 + 7c + 3\right)}\right) - \left(\frac{2c^2 + 8c + 12}{\lambda\left(c^2 + 4c + 3\right)}\right)^2$$

$$\begin{aligned}\mu_3 = {} & \frac{24c^4 + 144c^3 + 360c^2 + 480c + 360}{\lambda^3\left(c^4 + 6c^3 + 12c^2 + 4c + 3\right)} \\ & -3\left(\frac{2c^2 + 8c + 12}{\lambda\left(c^2 + 4c + 3\right)}\right)\left(\frac{6c^3 + 30c^2 + 60c + 60}{\lambda^2\left(c^3 + 5c^2 + 7c + 3\right)}\right) \\ & +2\left(\frac{2c^2 + 8c + 12}{\lambda\left(c^2 + 4c + 3\right)}\right)^3\end{aligned}$$

$$\mu_4=\left(\frac{120c^5+840c^4+2520c^3+4200c^2+4200c+2520}{\lambda^4\left(c^5+7c^4+18c^3+22c^2+13c+3\right)}\right)$$
$$-4\left(\frac{2c^2+8c+12}{\lambda\left(c^2+4c+3\right)}\right)\left(\frac{24c^4+144c^3+360c^2+480c+360}{\lambda^3\left(c^4+6c^3+12c^2+4c+3\right)}\right)$$
$$+6\left(\frac{2c^2+8c+12}{\lambda\left(c^2+4c+3\right)}\right)^2\left(\frac{6c^3+30c^2+60c+60}{\lambda^2\left(c^3+5c^2+7c+3\right)}\right)-3\left(\frac{2c^2+8c+12}{\lambda\left(c^2+4c+3\right)}\right)^4$$

2.14. Moment Ratios

Table 3 shows the coefficients of skewness and kurtosis.

3. Maximum Likelihood Estimation

The maximum likelihood estimation of SDWED distribution may be defined as:

$$L\left(\lambda,c;x_1,x_2,\cdots,x_n\right)=\prod_{i=1}^{n}g_0\left(x_i\right).$$

Here the independent observations are $x_1,x_2,\cdots,x_n$, then the likelihood function of the DWED is:

$$L\left(\lambda,c;x_1,x_2,\cdots,x_n\right)=\sum_{i=1}^{n}\log\left(g_0\left(x_i;\lambda,c\right)\right)$$

$$\begin{aligned}L\left(\lambda,c\right)=2n\log\lambda+3n\log\left(1+c\right)-9n\log\left(c\right)+\sum_{i=1}^{n}\log x_i\\-\lambda\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\log\left(1-\mathrm{e}^{-\lambda cx_i}-\lambda cx\mathrm{e}^{-\lambda cx_i}\right)\end{aligned}\tag{19}$$

This admits the partial derivatives:

$$\frac{\partial L\left(\lambda,c\right)}{\partial\lambda}=\frac{2n}{\lambda}-\sum_{i=1}^{n}x_i+\frac{nc^2x_i^{\,2}\lambda\mathrm{e}^{-\lambda cx_i}}{1-\mathrm{e}^{-\lambda cx_i}-\lambda cx_i\mathrm{e}^{-\lambda cx_i}}\tag{20}$$

and

$$\frac{\partial L\left(\lambda,c\right)}{\partial c}=\frac{3n}{1+c}-\frac{9n}{c}+\frac{n\lambda^2x_i^{\,2}c\mathrm{e}^{-\lambda cx_i}}{1-\mathrm{e}^{-\lambda cx_i}-\lambda cx_i\mathrm{e}^{-\lambda cx_i}}\tag{21}$$

Equating these equations to zero, then we get:

$$\frac{2n}{\lambda}-\sum_{i=1}^{n}x_i+\frac{nc^2x_i^{\,2}\lambda\mathrm{e}^{-\lambda cx_i}}{1-\mathrm{e}^{-\lambda cx_i}-\lambda cx_i\mathrm{e}^{-\lambda cx_i}}=0\tag{22}$$

Table 3. Coefficients of skewness and kurtosis of SDWED.

c	λ	$\sqrt{\beta_1}$	β_2
2.481	1	0.0005	1.067
2.484	1	0.0049	1.076
2.480	2	0.0003	1.066
3.000	2	1.3450	2.778
3.001	3	1.0060	2.999

$$\frac{3n}{1+c}-\frac{9n}{c}+\frac{n\lambda^2 x_i^2 c\mathrm{e}^{-\lambda c x_i}}{x_i-\lambda c x_i \mathrm{e}^{-\lambda c x_i}}=0 \tag{23}$$

which can be solved simultaneously for $\hat{\lambda}$ and $\hat{c}$.

The asymptotic variance-covariance matrix is the inverse of $I(\xi,k,\theta)=-E(H(x))$

$$H(x)=\begin{pmatrix} \dfrac{\partial^2\left(\log\left(g_0(x;\lambda,c)\right)\right)}{(\partial\lambda)^2} & \dfrac{\partial^2\left(\log\left(g_0(x;\lambda,c)\right)\right)}{(\partial\lambda\partial c)} \\ \dfrac{\partial^2\left(\log\left(g_0(x;\lambda,c)\right)\right)}{(\partial c\partial\lambda)} & \dfrac{\partial^2\left(\log\left(g_0(x;\lambda,c)\right)\right)}{(\partial c)^2} \end{pmatrix}$$

The inverse of the asymptotic covariance matrix is $I(\lambda,c)=-E(H(x))$ with

$$\frac{\partial^2\left(\log\left(g_0(x;\lambda,c)\right)\right)}{(\partial\lambda)^2}=\frac{-2n}{\lambda^2}+\frac{nc^2x^2\mathrm{e}^{-\lambda cx}(1-cx)\left(1-\mathrm{e}^{-\lambda cx}\right)-\lambda cx\left(nc^2x^2\mathrm{e}^{-2\lambda cx}\right)}{\left(1-\mathrm{e}^{-\lambda cx}-\lambda cx\mathrm{e}^{-\lambda cx}\right)^2} \tag{24}$$

$$\frac{\partial^2\left(\log\left(g_0(x;\lambda,c)\right)\right)}{(\partial c)^2}=\frac{-3n}{(1+c)^2}+\frac{9n}{c^2}+\frac{\lambda^2x^2\mathrm{e}^{-\lambda cx}\left(1-\lambda cx-\mathrm{e}^{-\lambda cx}\right)}{\left(1-\mathrm{e}^{-\lambda cx}-\lambda cx\mathrm{e}^{-\lambda cx}\right)^2} \tag{25}$$

$$\begin{aligned}\frac{\partial^2\left(\log\left(g_0(x;\lambda,c)\right)\right)}{(\partial\lambda\partial c)}&=\frac{\partial^2\left(\log\left(g_0(x;\lambda,c)\right)\right)}{(\partial c\partial\lambda)}\\&=\frac{2nc\lambda x^2\mathrm{e}^{-\lambda cx}\left(1-\mathrm{e}^{-\lambda cx}\right)-nc^2\lambda^2x^3\mathrm{e}^{-\lambda cx}\left(1+\mathrm{e}^{-\lambda cx}\right)}{\left(1-\mathrm{e}^{-\lambda cx}-\lambda cx\mathrm{e}^{-\lambda cx}\right)^2}\end{aligned} \tag{26}$$

4. Numerical Example

The Ball Bearing Data Records

See for data set published in Lawless [30]

In Table 4, the approximations of the parameters are specified. For goodness-of-fit statistics Anderson-Darling and Cramer-von Mises tests have been used, SDWED model proposals the best fitting (see Figure 8 and Figure 9):

5. Conclusion

In this paper, Size-biased Double Weighted Exponential Distribution (SDWED) has

Table 4. Parameters' estimates and goodness-of-fit statistics.

Distributions	$\hat{\theta}$	$\hat{k}$	$\hat{\xi}$	$\hat{\alpha}$	$\hat{c}$	$\hat{\lambda}$	$\hat{\beta}$	A_0^2	W_0^2
Size-biased Rayleigh	-	-	-	-	-	-	46.764	0.708	0.134
Size-biased Maxwell	-	-	-	40.50	-		-	1.693	0.278
Weighted Weibull (size-biased)	0.8151	0.604	4.759	-			-	0.1909	0.0332
Size-Biased Double weighted exponential distribution(SDWED)					16.64	0.027	-	0.133	0.134

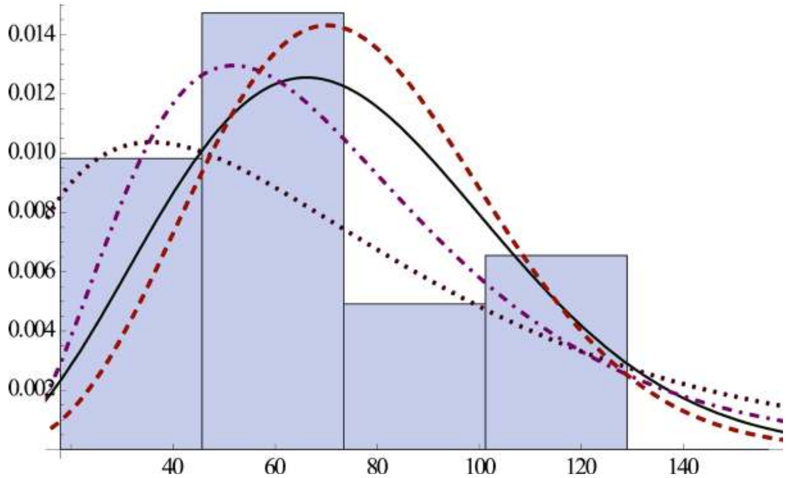

Figure 8. Size-Biased Double Weighted Exponential (Dotted Line), weighted Weibull (Solid Line), Rayleigh (Dashes Line) and Maxwell (dotted dashed Line) on the Histogram.

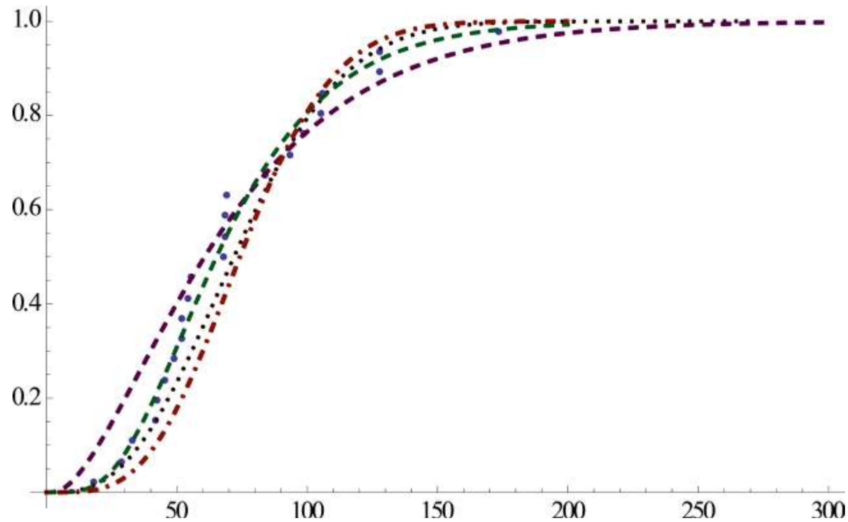

Figure 9. SWWD (dotted Line), SDWED (dott dashed), Maxwell (Solid Line), Rayleigh (Dashed Line), Density Estimates–cdf Estimates and Empirical cdf.

been introduced. The pdf of the SDWED has been studied as well as different reliability measures such as survival function, failure rate function or hazard function. The moments, mode, the coeff. of skewness and the coeff. of kurtosis of SDWED have been derived. For estimating the parameters of SDWED, MLE method has been used. The SDWED has been fitted to Ball Bearing data set. SDWED suggested a good fit of the data as comparing to other distributions.

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