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# On sizes of complete caps in projective spaces PG($n, q$) and arcs in planes PG($2, q$)

Alexander A. Davydov, Giorgio Faina, Stefano Marcugini and Fernanda Pambianco

*Dedicated to the memory of Giuseppe Pellegrino (1926–2007)*

**Abstract.** More than thirty new upper bounds on the smallest size $t_2(2, q)$ of a complete arc in the plane PG$(2, q)$ are obtained for $169 \leq q \leq 839$. New upper bounds on the smallest size $t_2(n, q)$ of the complete cap in the space PG$(n, q)$ are given for $n = 3$ and $25 \leq q \leq 97$, $q$ odd; $n = 4$ and $q = 7, 8, 11, 13, 17$; $n = 5$ and $q = 5, 7, 8, 9$; $n = 6$ and $q = 4, 8$. The bounds are obtained by computer search for new small complete arcs and caps. New upper bounds on the largest size $m_2(n, q)$ of a complete cap in PG$(n, q)$ are given for $q = 4$, $n = 5, 6$, and $q = 3$, $n = 7, 8, 9$. The new lower bound $534 \leq m_2(8, 3)$ is obtained by finding a complete 534-cap in PG$(8, 3)$. Many new sizes of complete arcs and caps are obtained. The updated tables of upper bounds for $t_2(n, q)$, $n \geq 2$, and of the spectrum of known sizes for complete caps are given. Interesting complete caps in PG$(3, q)$ of large size are described. A proof of the construction of complete caps in PG$(3, 2^h)$ announced in previous papers is given; this is modified from a construction of Segre. In PG$(2, q)$, for $q = 17$, $\delta = 4$, and $q = 19, 27$, $\delta = 3$, we give complete $(\frac{1}{2}(q + 3) + \delta)$-arcs other than conics that share $\frac{1}{2}(q+3)$ points with an irreducible conic. It is shown that they are unique up to collineation. In PG$(2, q)$, $q \equiv 2 \pmod 3$ odd, we propose new constructions of $\frac{1}{2}(q+7)$-arcs and show that they are complete for $q \leq 3701$.

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**Keywords.** complete arcs in planes, complete caps, small complete caps and arcs, computer search, projective spaces.

## 1. Introduction

Let PG$(n, q)$ be the projective space of dimension $n$ over the Galois field $F_q$. A $k$-cap in PG$(n, q)$ is a set of $k$ points, no three of which are collinear. A $k$-cap in PG$(n, q)$ is called complete if it is not contained in a $(k + 1)$-cap of

$\mathrm{PG}(n,q)$. If $n=2$, then a $k$-cap is usually called a $k$-arc. A survey of results on caps and arcs can be found in [54,55]; see also [56,57,75,78].

The main questions on caps in $\mathrm{PG}(n,q)$, which are also of interest in Coding Theory, concern the size of very large caps, especially near $m_2(n,q)$, the size of the largest complete cap, and near $m_2'(n,q)$, the size of the second largest complete cap. On the other hand, very small complete caps have also been investigated, especially the problem of determining $t_2(n,q)$, the size of the smallest complete cap. Finally, the spectrum of possible sizes of complete caps is one of most important problems in study of projective spaces.

In this paper we consider all problems listed above, collect known results, including recent ones, and obtain new sizes and bounds. This leads to updated tables of upper bounds for $t_2(n,q)$, $n \geq 2$, and for the spectrum of known sizes of complete caps; see [17,23]. In particular, new upper bounds on $t_2(n,q)$ for values of $n$ and $q$ are obtained as follows:

| $n$ | $q$ |
|---|---|
| 2 | $169, 256, 263, 283, 307, 317, 331, 349, 389, 421, 433, 521, 523, 547, 557, 607,$ $619, 631, 641, 643, 653, 661, 673, 701, 739, 743, 751, 811, 823, 827, 829, 839$ |
| 3 | $25 \leq q \leq 97, \quad q$ odd |
| 4 | $7, 8, 11, 13, 17$ |
| 5 | $5, 7, 8, 9$ |
| 6 | $4, 8$ |

The new lower bound $534 \leq m_2(8,3)$ is given. The bounds mentioned above are obtained by finding corresponding new complete arcs and caps. The new upper bounds for $m_2(n,q)$ are obtained for $q=4$, $n=5,6$, and $q=3$, $n=7,8,9$.

In $\mathrm{PG}(2,q)$, $q \equiv 2 \pmod 3$ odd, we propose two news constructions of $\frac{1}{2}(q+7)$-arcs sharing $\frac{1}{2}(q+3)$ points with an irreducible conic. We show that the new arcs are complete for $q \leq 3701$.

Also a construction of complete caps in $\mathrm{PG}(3,2^h)$, modified from that of Segre, that was announced by Pambianco and Storme in 1995 and cited in [57, Table 4.8], is given. In addition, the structure and combinatorial properties of interesting complete caps in $\mathrm{PG}(3,q)$ of large size are described.

For new computer results we used the randomized greedy algorithms considered in [17, Section 2], [23, Section 2], the back-tracking algorithms [17, Section 2], the breadth-first algorithm [65, Section 3], algorithms combining orbits of groups, and other geometrical algorithms.

The points of a $k$-cap $K$ in $\mathrm{PG}(n,q)$ can be viewed as columns of the *generator matrix* $G$ of the *associated code* $\mathcal{A}(K)$ with length $k$ and dimension $n+1$. For the *dual code* $\mathcal{A}^{\perp}(K)$ with codimension $n+1$ the matrix $G$ is its *parity check matrix*. The dual code of complete cap has Hamming distance four and covering radius two [56]. For an introduction to coverings of vector spaces over finite fields and the concept of the code covering radius, see [16,74]. We use the connection with covering codes in Sect. 8 for lower bounds on $t_2(n,2)$. On the other hand, geometrical properties of caps help to obtain the weight spectrum of $\mathcal{A}(K)$; see Sect. 5.

Throughout the paper the best known values of $m_2(n,q)$, $m'_2(n,q)$, and $t_2(n,q)$ are denoted by $\overline{m}_2(n,q)$, $\overline{m}'_2(n,q)$, and $\bar{t}_2(n,q)$, respectively. In all tables new bounds and new sizes of complete caps obtained are marked by an asterisk $\star$. New bounds and sizes obtained are written in bold font. "Ref." means "References". Also, in all tables the completeness of the spectrum of possible sizes of complete caps is marked by the dot after the value of $q$ or $n$.

In Sect. 2 we give the updated table of the smallest known sizes $\bar{t}_2(2,q)$ of complete arcs in PG$(2,q)$; more than thirty sizes are new. We also give some new sizes of complete arcs, extending our knowledge on the spectrum of their sizes. Two new constructions of $\frac{1}{2}(q+7)$-arcs in PG$(2,q)$ are proposed. In addition, we consider complete $(\frac{1}{2}(q+3)+\delta)$-arcs other than conics and sharing $\frac{1}{2}(q+3)$ points with an irreducible conic. In Sect. 3 the known families of complete caps in $PG(3,q)$ are described. We give a proof of the modification of Segre's construction. In Sect. 4 the spectrum of the known sizes of complete caps in PG$(3,q)$, $3 \le q \le 23$, is given. Section 5 describes properties of interesting complete caps in PG$(3,q)$ providing the upper and lower bounds. In Sect. 6 the spectrum of complete cap sizes in PG$(n,q)$, $n \ge 4$, is considered. Small caps in PG$(n,q)$, $n \ge 3$, $q > 2$, are investigated in Sect. 7. Some updated tables are given in Sects. 6 and 7. The spectrum of sizes of binary complete caps and a few conjectures connected with it are considered in Sect. 8.

Some of the results from this work were briefly presented without proofs in [18].

## 2. Complete arcs in planes PG$(2,q)$

For $q \le 821$, the sizes of the known small complete arcs in planes PG$(2,q)$ are collected in [17, Table 3]. With the help of the randomized greedy algorithms [17,23], we obtained more than thirty small complete arcs in PG$(2,q)$ giving new upper bounds on $t_2(2,q)$. The updated table of $\bar{t}_2(2,q)$ is given as Table 1. For $q = 2^7$ a complete 34-arc is obtained from the affinely complete 32-arc of [48, Appendix, Lemma 4.3] by adding two points; see also [21, Section 2]. From Table 1, we obtain Theorem 1, improving the corresponding theorem in [17, p. 55].

**Theorem 1.** *In* PG$(2,q)$,

$$\begin{aligned}
&t_2(2,q) < 4\sqrt{q} && \textit{for} \quad 3 \le q \le 841;\\
&t_2(2,q) \le 4\sqrt{q}-8 && \textit{for} \quad 23 \le q \le 269,\ q = 281, 283, 289, 307, 317;\\
&t_2(2,q) \le 4\sqrt{q}-7 && \textit{for} \quad 19 \le q \le 353,\ q = 361, 383;\\
&t_2(2,q) \le 4\sqrt{q}-6 && \textit{for} \quad 9 \le q \le 401,\ q = 421, 431, 433;\\
&t_2(2,q) \le 4\sqrt{q}-5 && \textit{for} \quad 8 \le q \le 443,\ q = 509, 521, 523;\\
&t_2(2,q) \le 4\sqrt{q}-4 && \textit{for} \quad 7 \le q \le 557,\ q = 625, 729, 841;
\end{aligned} \tag{1}$$

TABLE 1 The smallest known sizes $\bar{t}_2 = \bar{t}_2(2,q) < 4\sqrt{q}$ of complete arcs in planes PG$(2,q)$. $A_q = \lfloor 4\sqrt{q} - \bar{t}_2(2,q) \rfloor$

| $q$ | $\bar{t}_2$ | $A_q$ | Ref. | $q$ | $\bar{t}_2$ | $A_q$ | Ref. | $q$ | $\bar{t}_2$ | $A_q$ | Ref. | $q$ | $\bar{t}_2$ | $A_q$ | Ref. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 4. | 2 | [55] | 131 | 36 | 9 | [17] | 349 | 67 | 7 | ⋆ | 601 | 96 | 2 | [17] |
| 4 | 6. | 2 | [55] | 137 | 37 | 9 | [17] | 353 | 68 | 7 | [17] | 607 | 96 | 2 | ⋆ |
| 5 | 6. | 2 | [55] | 139 | 37 | 10 | [17] | 359 | 69 | 6 | [17] | 613 | 97 | 2 | [17] |
| 7 | 6. | 4 | [55] | 149 | 39 | 9 | [17] | 361 | 69 | 7 | [17] | 617 | 97 | 2 | [17] |
| 8 | 6. | 5 | [55] | 151 | 39 | 10 | [17] | 367 | 70 | 6 | [17] | 619 | 97 | 2 | ⋆ |
| 9 | 6. | 6 | [55] | 157 | 40 | 10 | [17] | 373 | 71 | 6 | [17] | 625 | 96 | 4 | [47] |
| 11 | 7. | 6 | [55] | 163 | 41 | 10 | [17] | 379 | 71 | 6 | [17] | 631 | 98 | 2 | ⋆ |
| 13 | 8. | 6 | [68] | 167 | 42 | 9 | [17] | 383 | 71 | 7 | [17] | 641 | 99 | 2 | ⋆ |
| 16 | 9. | 7 | [68] | 169 | 42 | 10 | ⋆ | 389 | 72 | 6 | ⋆ | 643 | 99 | 2 | ⋆ |
| 17 | 10. | 6 | [68] | 173 | 44 | 8 | [17] | 397 | 73 | 6 | [17] | 647 | 99 | 2 | [17] |
| 19 | 10. | 7 | [68] | 179 | 44 | 9 | [17] | 401 | 74 | 6 | [17] | 653 | 100 | 2 | ⋆ |
| 23 | 10. | 9 | [68] | 181 | 45 | 8 | [17] | 409 | 75 | 5 | [17] | 659 | 100 | 2 | [17] |
| 25 | 12. | 8 | [64] | 191 | 46 | 9 | [17] | 419 | 76 | 5 | [17] | 661 | 100 | 2 | ⋆ |
| 27 | 12. | 8 | [64] | 193 | 47 | 8 | [17] | 421 | 76 | 6 | ⋆ | 673 | 102 | 1 | ⋆ |
| 29 | 13. | 8 | [64] | 197 | 47 | 9 | [17] | 431 | 77 | 6 | [17] | 677 | 103 | 1 | [17] |
| 31 | 14 | 8 | [68] | 199 | 47 | 9 | [17] | 433 | 77 | 6 | ⋆ | 683 | 103 | 1 | [17] |
| 32 | 14 | 8 | [68] | 211 | 49 | 9 | [17] | 439 | 78 | 5 | [17] | 691 | 104 | 1 | [17] |
| 37 | 15 | 9 | [63] | 223 | 51 | 8 | [17] | 443 | 79 | 5 | [17] | 701 | 104 | 1 | ⋆ |
| 41 | 16 | 9 | [17] | 227 | 51 | 9 | [17] | 449 | 80 | 4 | [17] | 709 | 105 | 1 | [17] |
| 43 | 16 | 10 | [17] | 229 | 52 | 8 | [17] | 457 | 81 | 4 | [17] | 719 | 106 | 1 | [17] |
| 47 | 18 | 9 | [15] | 233 | 52 | 9 | [17] | 461 | 81 | 4 | [17] | 727 | 106 | 1 | [17] |
| 49 | 18 | 10 | [17] | 239 | 53 | 8 | [17] | 463 | 82 | 4 | [17] | 729 | 104 | 4 | [47] |
| 53 | 18 | 11 | [17] | 241 | 53 | 9 | [17] | 467 | 82 | 4 | [17] | 733 | 107 | 1 | [17] |
| 59 | 20 | 10 | [17] | 243 | 54 | 8 | [17] | 479 | 83 | 4 | [17] | 739 | 107 | 1 | ⋆ |
| 61 | 20 | 11 | [63] | 251 | 55 | 8 | [17] | 487 | 84 | 4 | [17] | 743 | 108 | 1 | ⋆ |
| 64 | 22 | 10 | [15] | 256 | 55 | 9 | ⋆ | 491 | 84 | 4 | [17] | 751 | 108 | 1 | ⋆ |
| 67 | 23 | 9 | [66] | 257 | 56 | 8 | [17] | 499 | 85 | 4 | [17] | 757 | 109 | 1 | [17] |
| 71 | 22 | 11 | [63] | 263 | 56 | 8 | ⋆ | 503 | 85 | 4 | [17] | 761 | 109 | 1 | [17] |
| 73 | 24 | 10 | [17] | 269 | 57 | 8 | [17] | 509 | 85 | 5 | [17] | 769 | 110 | 0 | [17] |
| 79 | 26 | 9 | [17] | 271 | 58 | 7 | [17] | 512 | 86 | 4 | [17] | 773 | 111 | 0 | [17] |
| 81 | 26 | 10 | [17] | 277 | 59 | 7 | [17] | 521 | 86 | 5 | ⋆ | 787 | 112 | 0 | [17] |
| 83 | 27 | 9 | [17] | 281 | 59 | 8 | [17] | 523 | 86 | 5 | ⋆ | 797 | 112 | 0 | [17] |
| 89 | 28 | 9 | [17] | 283 | 59 | 8 | ⋆ | 529 | 88 | 4 | [47] | 809 | 113 | 0 | [17] |
| 97 | 30 | 9 | [17] | 289 | 60 | 8 | [17] | 541 | 89 | 4 | [17] | 811 | 113 | 0 | ⋆ |
| 101 | 30 | 10 | [17] | 293 | 61 | 7 | [17] | 547 | 89 | 4 | ⋆ | 821 | 114 | 0 | [17] |
| 103 | 31 | 9 | [17] | 307 | 62 | 8 | ⋆ | 557 | 90 | 4 | ⋆ | 823 | 114 | 0 | ⋆ |
| 107 | 32 | 9 | [17] | 311 | 63 | 7 | [17] | 563 | 92 | 2 | [17] | 827 | 115 | 0 | ⋆ |
| 109 | 32 | 9 | [17] | 313 | 63 | 7 | [17] | 569 | 93 | 2 | [17] | 829 | 115 | 0 | ⋆ |
| 113 | 33 | 9 | [17] | 317 | 63 | 8 | ⋆ | 571 | 93 | 2 | [17] | 839 | 115 | 0 | ⋆ |
| 121 | 34 | 10 | [15] | 331 | 65 | 7 | ⋆ | 577 | 93 | 3 | [17] | 841 | 112 | 4 | [47] |

Table 1 continued

| $q$ | $\bar{t}_2$ | $A_q$ | Ref. | $q$ | $\bar{t}_2$ | $A_q$ | Ref. | $q$ | $\bar{t}_2$ | $A_q$ | Ref. | $q$ | $\bar{t}_2$ | $A_q$ | Ref. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 125 | 35 | 9 | [17] | 337 | 66 | 7 | [17] | 587 | 94 | 2 | [17] | | | | |
| 127 | 35 | 10 | [17] | 343 | 67 | 7 | [17] | 593 | 95 | 2 | [17] | | | | |
| 128 | 34 | 11 | [48] | 347 | 67 | 7 | [17] | 599 | 95 | 2 | [17] | | | | |

$$t_2(2,q) \le 4\sqrt{q} - 2 \quad \textit{for} \quad 3 \le q \le 661;$$
$$t_2(2,q) \le 4\sqrt{q} - 1 \quad \textit{for} \quad 3 \le q \le 761.$$

For even $q = 2^h$, $10 \le h \le 15$, the smallest known sizes of complete $k$-arcs in $\mathrm{PG}(2,q)$ are obtained in [21]; they are as follows: $k = 124, 201, 307, 461, 665, 993$, for $h = 10, 11, 12, 13, 14, 15$, respectively. Also, $6(\sqrt{q}-1)$-arcs in $\mathrm{PG}(2,q)$, $q = 4^{2h+1}$, are constructed in [22]; for $h \le 4$ it is proved that they are complete. It gives the complete 3066-arc in $\mathrm{PG}(2, 2^{18})$.

Note that in [43, Table 2.4] and [17, Table 2], some entries on the existence of complete $\frac{1}{2}(q+7)$-arcs in $\mathrm{PG}(2,q)$ are based on [70,71]. The papers [70,71] contain inaccuracies indicated in the recent work [61]. Below we give a few results confirming the correctness of all entries on the existence of complete $\frac{1}{2}(q+7)$-arcs in [43, Table 2.4] and [17, Table 2] and, moreover, extending these tables.

In the first, we note that for $q \le 125$, the validity of the entries in [43, Table 2.4] and [17, Table 2] have checked by computer, see [61, Introduction]. In addition, in [46], for $q \equiv 1 \pmod 4$, a construction of $\frac{1}{2}(q+7)$-arcs in $\mathrm{PG}(2,q)$ is proposed and it is showed by computer that the arcs are complete if $q \le 337, q \ne 17$. Also, in [61] it is proved that if $q = ct - 1 \ge 17$, $t$ odd prime, $c \in \{2,4\}$, then in $\mathrm{PG}(2,q)$ there is a complete $\frac{1}{2}(q+7)$-arc. So, for $q = 19, 25, 27, 37, 61, 67, 73, 121, 157, 163$, the complete arcs needed exist. Finally, by the method of [61], we obtained $\frac{1}{2}(q+7)$-arcs for $q = 97, 109, 139, 151$, and checked by computer that they are complete.

Now we give two explicit constructions of $\frac{1}{2}(q+7)$-arcs in $\mathrm{PG}(2,q)$.

**Construction S.** Let $q \equiv 2 \pmod 3$ be an odd prime, $q \ge 11$. Then $-3$ is a non square element in $F_q$ [55, Section 1.5]. Let $(x_0, x_1, x_2, x_3)$ be a point of $\mathrm{PG}(3,q)$. We denote $A_\infty = (0,0,0,1)$, $P = (0,1,3,0)$, $P_1 = (0,1,0,0), P_2 = (0,1,-3,-12)$. Let $\mathcal{Q}$ be the quadric of the equation $3x_1^2 + x_2^2 = x_0x_3$. In [25,26] the complete $\frac{1}{2}(q^2+q+8)$-cap $\mathcal{K}$ in $\mathrm{PG}(3,q)$ is obtained such that $\mathcal{K} = \mathcal{K}_1 \cup \{P_1, P_2\}$ where

$$\mathcal{K}_1 = \big\{(1, v, d, 3v^2 + d^2) | v \in F_q, d = v - 2i, i \in \{v, v+1, \dots, q-1\}\big\} \cup \{A_\infty, P\}$$

and $\mathcal{K}_1 \backslash P \subset \mathcal{Q}$, see [26, Lemma 2]. The $q$ points of the form $(1, v, -v, 4v^2)$ and the point $A_\infty$ lie on tangents to $\mathcal{Q}$ through $P$ [26, Lemma 1], while the rest of points lies on bisecants of $\mathcal{Q}$ through $P$ so that on every bisecant exactly one point of $\mathcal{Q}$ is included to $\mathcal{K}_1$, cf. Segre's Construction A in Section 3. Let $s$ be a nonzero square in $F_q$. The plane $\pi(s)$ of the equation $x_3 = sx_0$ meets the quadric $\mathcal{Q}$ in the conic $\mathcal{C}$ of the equation $3x_1^2 + x_2^2 = sx_0^2$. At that end, $\pi(s)$ meets the cap $\mathcal{K}$ in the $\frac{1}{2}(q+7)$-arc $\mathcal{K}(s)$, points $(x_0, x_1, x_2)$ of which can be represented in $\pi(s)$ as follows:

$$\mathcal{K}(s) = \{(1, v, d) | v \in F_q, d = v - 2i, i \in \{v, v+1, \ldots, q-1\},\\ 3v^2 + d^2 = s\} \cup \{P, P_1\}$$

where $P = (0, 1, 3), P_1 = (0, 1, 0)$. By above, $\mathcal{K}'(s) = \mathcal{K}(s) \backslash \{P, P_1\} \subset \mathcal{C}$. Putting $v = \pm\frac{1}{2}\sqrt{s}$ and $i = v$, we obtain two points $(1, v, -v) \in \mathcal{K}'(s)$ lying on tangents to $\mathcal{C}$ through $P$, while other $\frac{1}{2}(q-1)$ points of $\mathcal{K}'(s)$ lie on bisecants of $\mathcal{C}$ through $P$ so that on every bisecant exactly one point of $\mathcal{C}$ belongs to $\mathcal{K}'(s)$.

**Theorem 2.** *Let $q \equiv 2 \pmod 3$ be an odd prime, $q \geq 11$. Let $s$ be a non-zero square in $F_q$. Then in* $\mathrm{PG}(2, q)$, *the point set $\mathcal{K}(s)$ of Construction S is a $\frac{1}{2}(q+7)$-arc other than a conic that shares $\frac{1}{2}(q+3)$ points with an irreducible conic. For $q \leq 3701$, $q \neq 17$, the $\frac{1}{2}(q+7)$-arc $\mathcal{K}(1)$ is complete. For $q = 17$, the arc $\mathcal{K}(s)$ is an incomplete 12-arc embedded in the complete 14-arc.*

*Proof.* The $\frac{1}{2}(q+7)$-set $\mathcal{K}(s)$ is an arc as it is an intersection of a cap in $\mathrm{PG}(3, q)$ and a plane. The completeness of the arcs $\mathcal{K}(1)$ is checked by computer. □

**Construction C.** Let $q \equiv 2 \pmod 3$ be a power of an odd prime, $q \geq 11$. In $\mathrm{PG}(2, q)$ we denote the points $A_i = (\frac{1}{2}(i^2+1), i, 1)$, $i \in F_q$, $T = (1, 0, 0)$, $P = (0, 1, 0)$, $Q = (-1, -1, 1)$. The conic $\mathcal{C}$ of the equation $x_1^2 + x_2^2 = 2x_0x_2$ can be represented as $\mathcal{C} = \{A_i | i \in F_q\} \cup \{T\}$. For $i \neq \pm 1$, we define a 3-cycle $C_i = \{A_i, A_{\frac{i-3}{1+i}}, A_{\frac{i+3}{1-i}}\}$ of three points. It holds that $C_i = C_{\frac{i-3}{1+i}} = C_{\frac{i+3}{1-i}}$ and $C_{-i} = \{A_{-i}, A_{-\frac{i-3}{1+i}}, A_{-\frac{i+3}{1-i}}\}$. The $(q-5)$-set $\mathcal{C} \backslash (C_0 \cup \{T, A_1, A_{-1}\})$ is partitioned to $\frac{1}{6}(q-5)$ pairs of "opposite" 3-cycles $\{C_i, C_{-i}\}$, $i \neq 0$. We form a $\frac{1}{2}(q+7)$-set $K = K_1 \cup \{P, Q, T, A_0, A_1, A_{-3}\}$ where a $\frac{1}{2}(q-5)$-set $K_1$ consists of $\frac{1}{6}(q-5)$ 3-cycles $C_i$, $i \neq 0$. From every pair $\{C_i, C_{-i}\}$, exactly one 3-cycle is included in $K_1$. At that end, any of two 3-cycles can be taken. So, there are $2^{(q-5)/6}$ formally distinct variants of the set $K_1$. The points $T, A_0$ and $A_1, A_{-3}$ lie on the tangents to $\mathcal{C}$ through $P$ and $Q$.

**Theorem 3.** *Let $q \equiv 2 \pmod 3$ be a power of an odd prime, $q \geq 11$. Then in* $\mathrm{PG}(2, q)$, *for the all $2^{(q-5)/6}$ variants of the set $K_1$, the set $K$ of Construction C is a $\frac{1}{2}(q+7)$-arc other than a conic that shares $\frac{1}{2}(q+3)$ points with an irreducible conic. For $q \leq 3701$, there is a variant of $K_1$ such that the $\frac{1}{2}(q+7)$-arc $K$ is complete. Also, for $q \leq 131$, $q \neq 17, 59$, all the variants of $K_1$ give complete arcs $K$. For $q = 17$, there are two equivalent variants of $K_1$ providing incomplete $\frac{1}{2}(q+7)$-arcs $K$ embedded in complete $\frac{1}{2}(q+11)$-arcs; for $q = 59$, there are four equivalent variants of $K_1$ providing incomplete $\frac{1}{2}(q+7)$-arcs $K$ embedded in complete $\frac{1}{2}(q+9)$-arcs.*

The properties of the set $K$ of Construction C and Theorem 3 are proved in [27]. We do not give here these proofs to save the space. For $q \leq 3701$,

TABLE 2 The sizes of the known complete $k$-arcs in $\mathrm{PG}(2,q)$

| $q$ | $t_2(2,q)$ | Sizes $k$ of the known complete arcs with $t_2(2,q) \le k \le m_2'(2,q)$ | $m_2'(2,q)$ | $m_2(2,q)$ | References |
|---|---|---|---|---|---|
| 43 | $12 \le$ | $16 \le k \le 26 = (q+9)/2$, $k = 28 = (q+13)/2$ | $\le 42$ | 44 | [17,61] |
| 59 | $14 \le$ | $20 \le k \le 34 = (q+9)/2$ | $\le 57$ | 60 | [17,61] |
| 64 | $13 \le$ | $22 \le k \le 35 = (q+6)/2$, $k = 42 = 6\sqrt{q}-6, k = 57$ | 57 | 66 | [17,22] |
| 89 | $17 \le$ | $28 \le k \le 48 = (q+7)/2$ | $\le 87$ | 90 | [17,46],⋆ |
| 97 | $18 \le$ | $30 \le k \le 52 = (q+7)/2$ | $\le 94$ | 98 | [17,46],⋆ |
| 101 | $18 \le$ | $30 \le k \le 54 = (q+7)/2$ | $\le 98$ | 102 | [17,46],⋆ |
| 109 | $19 \le$ | $32 \le k \le 58 = (q+7)/2$ | $\le 106$ | 110 | [17,46],⋆ |
| 113 | $19 \le$ | $33 \le k \le 60 = (q+7)/2$ | $\le 110$ | 114 | [17,46],⋆ |
| 128 | $18 \le$ | $34 \le k \le 67 = (q+6)/2$ | $\le 114$ | 130 | [17,21,48],⋆ |
| 137 | $21 \le$ | $37 \le k \le 72 = (q+7)/2$ | $\le 134$ | 138 | [17,46],⋆ |
| 149 | $22 \le$ | $39 \le k \le 78 = (q+7)/2$ | $\le 145$ | 150 | [17,46],⋆ |
| 163 | $23 \le$ | $41 \le k \le 85 = (q+7)/2$ | $\le 160$ | 164 | [17],⋆ |
| 167 | $23 \le$ | $42 \le k \le 87 = (q+7)/2$ | $\le 164$ | 168 | [17],⋆ |

the completeness of the arcs is checked by computer. We state the following conjecture.

**Conjecture 1.** *Let $q \equiv 2 \pmod 3$ be a power of an odd prime, $q \ge 11$. Then, in Construction C, there exists at least one variant of the set $K_1$ providing a complete $\frac{1}{2}(q+7)$-arc $K$.*

By Theorems 2 and 3, in $\mathrm{PG}(2,q)$ there is a complete $\frac{1}{2}(q+7)$-arc, in particular, for $q = 11, 23, 29, 41, 47, 53, 59, 71, 83, 89, 101, 113, 125, 131, 137, 149, 167$.

So, we have confirmed that all entries in [43, Table 2.4] and [17, Table 2] on the existence of complete $\frac{1}{2}(q+7)$-arcs are correct.

Now we summarize new data extending the tables of [43, Table 2.4] and [17, Table 2]. Using the greedy algorithms we obtained complete $k$-arcs in $\mathrm{PG}(2,q)$ with $k = 35$ for $q = 128$, $k = 80, 82, 83$ for $q = 163$, and $k = 83, 84$ for $q = 167$. Also, in [22] a complete 42-arc in $\mathrm{PG}(2,64)$ is obtained. In [61] a complete 26-arc in $\mathrm{PG}(2,43)$ and a complete 34-arc in $\mathrm{PG}(2,59)$ are constructed. By the method of [61], we obtained complete $\frac{1}{2}(q+7)$-arcs for $q = 97, 109$. Finally, by Theorems 2 and 3, for $q = 89, 101, 113, 137, 149$, complete $\frac{1}{2}(q+7)$-arcs in $\mathrm{PG}(2,q)$ exist. In Table 2 we give the updated list of the known sizes of complete arcs in $\mathrm{PG}(2,q)$. Note also that in [59] it is shown that $m_2'(2,31) = 22$, $m_2'(2,32) = 24$.

**Theorem 4.** *In $\mathrm{PG}(2,q)$, $q$ odd, let $K_q(\delta)$ be a complete $(\frac{1}{2}(q+3)+\delta)$-arc other than a conic but sharing $\frac{1}{2}(q+3)$ points with an irreducible conic. If $\Delta_q$ is the maximum value of $\delta$, then*

(i) $\Delta_{17} = 4,\ \Delta_{19} = \Delta_{27} = 3,\ \Delta_{11} = \Delta_{23} = \Delta_{29} = \Delta_{31} = 2$;
(ii) *there is no any arc* $K_{17}(3)$;
(iii) *for* $\delta = 3, 4$ *and* $q \leq 27$, *the arcs* $K_q(\delta)$ *are unique up to collineations.*

*Proof.* The assertion is proved by an exhaustive computer search. □

*Remark* 1. Let $q = 2p - 1$ where $p$ is an odd prime. In [60] the following is proved: $\Delta_{2p-1} \leq 4$; if, in addition, $(2p-1)^2 \equiv 1 \pmod{16}$, then $\Delta_{2p-1} \leq 2$. An example of the arc with $\Delta_{2p-1} = 4$ is given in [8] for $p = 7$, $q = 13$. So, $\Delta_{25} = 2$ as $25^2 \equiv 1 \pmod{16}$, and $\Delta_{13} = 4$.

The results of [70,71] can be compared with Theorem 4, Remark 1, and the arcs $K_q(\Delta_q)$ below. The unique 14-arc $K_{17}(4)$ is a counterexample to [71]. The 14-arc $K_{19}(3)$ is obtained in [71] but here is shown to be unique. Finally, the unique 18-arc $K_{27}(3)$ is new.

Points of the unique 14-arc $K_{17}(4)$ are given in [26]:

$$\{(1,10,12),(1,6,8);(1,0,6),(1,0,11),(1,1,4),(1,1,13),(1,6,9),$$
$$(1,10,5),(1,14,3),(1,3,14);(0,1,3),(0,1,0),(1,5,1),(1,14,10)\}.$$

The first ten points lie on the conic $3x_1^2 + x_2^2 = 2x_0^2$ and the last four are placed outside it. In addition, the first twelve points are the arc $\mathcal{K}(2)$ of Construction S. The semicolons separate the points into orbits of the stabilizer group. For $K_{17}(4)$ the stabilizer is the dihedral group $\mathbf{D}_4$ of order eight.

The unique 14-arc $K_{19}(3)$ may be represented as follows:

$$\{(1,5,6),(1,2,4);(1,0,0),(1,7,11),(1,13,17);(1,1,1),(1,3,9),$$
$$(1,4,16),(1,6,17),(1,9,5),(1,17,4);(1,13,6),(1,1,11),(1,6,8)\}.$$

The first 11 points belong to the conic $x_1^2 = x_0x_2$.

The unique 18-arc $K_{27}(3)$ may be represented as follows:

$$\{(1,14,1),(1,12,23),(1,10,19);(1,0,0),(0,0,1),(1,2,3),$$
$$(1,22,17),(1,13,25),(1,11,21);(1,8,15),(1,20,13),(1,19,11),$$
$$(1,16,5),(1,4,7),(1,5,9);(0,1,0),(1,6,8),(1,21,12)\}.$$

The field $F_{27}$ is generated by the polynomial $x^3 - x^2 - 2$. The elements of $F_{27}$ are represented as follows: $0 = 0$, $\alpha^i = i + 1$, where $\alpha$ is a primitive element of the field. The first 15 points of $K_{27}(3)$ belong to the conic $x_1^2 = x_0x_2$.

The stabilizer group of the arcs $K_{19}(3)$ and $K_{27}(3)$ is the symmetric group $\mathbf{S}_3$ of order six. In $K_{17}(4), K_{19}(3)$, and $K_{27}(3)$ the points outside the conic form an orbit of the stabilizer group.

It should be noted that, independently of this work, in the recent paper [61], the complete arcs $K_{17}(4), K_{19}(3), K_{27}(3), K_{43}(3)$, and $K_{59}(3)$ are obtained with the help of an interesting theoretical approach supported by computer search. Moreover, in [61, Theorem 6.1] infinite families of $(\frac{1}{2}(q+3)+\delta)$-arcs are constructed for $q \equiv 3 \pmod 4$. The arc $K_{17}(4)$ is obtained also in [46].

## 3. The families of complete caps in PG$(3, q)$

Surveys of the known families of complete caps in PG$(3, q)$ can be found, e.g., in [40, 42, 43, 57, 78]. However, these surveys are insufficiently complete for our goals and do not contain some details, refinements, corrections, developments and recent results, important for our presentation; see, e.g., Theorems 5 and 6, formulas (4),(7)–(9),(14),(19).

It is well known that $m_2(3, q) = q^2 + 1$, provided by an elliptic quadric [54], [57, Theorem 4.1], [75].

In PG$(3, q)$ we consider complete $k$-caps $\mathcal{K}$ having $t$ points common with an elliptic quadric $\mathcal{Q}$ where $t < k$. So, $\mathcal{K} \nsubseteq \mathcal{Q}$. The known constructions of caps often start with a given cap of size $T_q = 1 + (q+1) + \frac{1}{2}q(q-1) = \frac{1}{2}(q^2 + q + 4)$ containing all but just one point from an elliptic quadric of PG$(3, q)$. As a matter of fact, $T_q$ is an important number in the study of complete caps since any cap containing at least $T_q$ points from an elliptic quadric is entirely contained in it [2, 54, 72, 75]. Therefore we consider the situation $t \le T_q - 1$.

In the beginning we consider the families of complete $k$-caps in PG$(3, q)$ with $k \ge T_q$. For such families the following construction plays an important role.

**Segre's Construction A.** [75, Theorem V, p. 73]. Given an elliptic quadric $\mathcal{Q} \subset$ PG$(3, q)$, the $T_q$-cap comprises a point $P \notin \mathcal{Q}$, the $q + 1$ common points of $\mathcal{Q}$ with the polar plane of $P$ together with one point from each bisecant through $P$, choosing one of its two common points with $\mathcal{Q}$. The resulting cap is complete or it can be completed by adding at most $q + 1$ points.

By [75, Theorem V, p. 73], [54, Section 18.2], in PG$(3, q)$ there are complete $k$-caps with

$$T_q \le k \le T_q + q + 1, \quad t = T_q - 1, \quad q \ge 2. \tag{2}$$

In [75, Theorem V, p. 73] complete $k$-caps in PG$(3, q)$ are constructed with

$$\begin{aligned} T_q + 1 \le k \le T_q + q + 1, \quad t = T_q - 1,\ q \text{ is odd}, \\ q \equiv 2 \pmod 3 \text{ or } q = 3^h \ge 9. \end{aligned} \tag{3}$$

By Remark 1, the results of [2], where it is assumed that always $\Delta_{2p-1} \le 2$, should be corrected. We give these results in the following form, see [6, 25, 26] for $q = 5$:

In PG$(3, q)$ there are complete $k$-caps with parameters

$$\begin{aligned} T_q \le k \le T_q + 3, \quad t = T_q - 1, \quad (q+1)/2 \text{ odd prime}, \\ q^2 \equiv 1 \pmod{16} \text{ or } q = 5. \end{aligned} \tag{4}$$

In [72, Theorems IV,V] the results of (3),(4) are developed and it is proved that in PG$(3, q)$ there are complete $k$-caps with parameters

$$k = T_q + 1, \quad t = T_q - 1, \quad q > 13, \quad q = 2p - 1, \quad p \text{ is odd prime.} \tag{5}$$

$$k = T_q + 1, \quad t = T_q - 1, \quad q = 4v + 1. \tag{6}$$

$$T_q + 1 \leq k, \quad t = T_q - 1, \quad q \text{ is odd}, \quad q \neq 2^r - 1. \tag{7}$$

The condition $q \neq 2^r - 1$ of (7) includes the conditions $q \equiv 2 \pmod 3$ and $q = 3^h \geq 9$, of (3) [72, p. 285 ]. Note that in [72, Table, p. 271] there is a misprint in the entry for the situation of (7).

In [25,26] complete $k$-caps in $\mathrm{PG}(3, q)$ are constructed with parameters

$$k = T_q + 2, \quad t = T_q - 1, \quad q \text{ odd prime, } q \equiv 2 \pmod 3, \quad q \geq 11. \tag{8}$$

The construction of caps in (8), slightly modified, furnishes also one of two inequivalent complete 20-caps in $\mathrm{PG}(3, 5)$ considered in [6].

In [7] the family of complete $k$-caps in $\mathrm{PG}(3, q)$ is obtained with parameters

$$k = T_q, \quad t = T_q - 1, \quad q \equiv 3 \pmod 4, \ q \geq 7. \tag{9}$$

The caps of (9) are invariant under a linear collineation group of $\mathrm{PG}(3, q)$ acting transitively on the common points of the cap and the elliptic quadric.

In [2,7,25,26,54,72,75] to obtain the results of (2)–(9) Segre's Construction A is used with distinct points $P$ and distinct rules for the choice of one point of the quadric on every bisecant through $P$.

**Theorem 5.** *In* $\mathrm{PG}(3, q)$, $q \geq 3$, *there is a complete* $T_q$*-cap sharing* $T_q - 1$ *points with an elliptic quadric.*

*Proof.* The existence of complete $T_q$-caps in $\mathrm{PG}(3, q)$ is proved in the following works: for $q \equiv 3 \pmod 4$ in [75, Theorem V, p. 73], [54, Theorem 18.2.5], [7]; for $q \equiv 1 \pmod 4$, $q \neq 5, 9$, in [72]; for $q = 2^h$ with odd $h > 3$ in [31]; for all even $q > 8$ in [1]; for $q = 8$ in [3]; for $q = 9$ in [43, Table 3.2]; for $q = 4$ in [76]. Also, the situation for $q = 4$ together with $q = 3, 5$ is considered in [6,40], [43, Table 3.1]. See also the references in [54, Section 18.5]. The $T_q$-caps of [1,3,7,54,72,75] are constructed by Segre's Construction A, therefore they have $t = T_q - 1$. For $q = 4, 5, 9$, the known caps do not have $t = T_q - 1$. We obtained by computer $T_q$-caps with $t = T_q - 1$, $q = 4, 5, 9$, using Segre's Construction A with a random choice of one point on a bisecant. □

Now we consider the families of complete $k$-caps in $\mathrm{PG}(3, q)$ with $k < T_q$.

**Segre's Construction B.** [76]. Let $q = 2^h$, $h \geq 2$, and let $\pi_1, \pi_2$ be planes of $\mathrm{PG}(3, q)$ with the intersection line $l$. We consider irreducible conics $C_1 \subset \pi_1$ and $C_2 \subset \pi_2$ such that the both conics touch the line $l$ at the same point $T$ and have the same nucleus $O \in l$ with $O \neq T$. We denote by $A_j$ a point of $C_j \backslash \{T\}$, $j = 1, 2$. In [76] it is proved that every line $A_1A_2$ intersects $q - 1$ other such lines in the same point, say $A_3$. There are $q$ points of the form $A_3$. All lie in the same plane, say $\pi_3$, containing the line $l$. The points $A_3$ together with the point $T$ form an irreducible conic $C_3 \subset \pi_3$ with the same nucleus $O$. Moreover, the points of $C_1 \cup C_2 \cup \{O\}$ form a $(2q + 2)$-cap $\mathcal{K}^*$ covering the whole space $\mathrm{PG}(3, q)$ excepting $\pi_3 \backslash \{C_3 \cup l\}$. To obtain a complete cap we can add to $\mathcal{K}^*$ any points of $\pi_3 \backslash \{C_3 \cup l\}$ *chosen freely with the only condition that* *the set of points obtained by aggregating $T$ and $O$ to them is a $k_2$-arc $\mathcal{A} \subset \pi_3$* [76, Theorem II, Proof]. In particular, as the arc $\mathcal{A}$ one can take a hyperoval. This gives the family of complete $k$-caps with

$$k = 3q + 2, \quad q = 2^h, \quad h \geq 2. \tag{10}$$

In 1995, Pambianco and Storme announced the following result, connected with the paper [68] and cited in [57, Table 4.8]:

$$t_2(3, q) \leq 2q + t_2(2, q), \quad q = 2^h, \quad h \geq 2. \tag{11}$$

As a development of the idea of (11), from [76, Theorem II, Proof] and [68, Remark 2.2(3)] we have the following result.

**Theorem 6.** *Let $q \geq 4$ be even. For every complete $k_2$-arc in the plane $\mathrm{PG}(2, q)$ there is a complete $(2q + k_2)$-cap in the space $\mathrm{PG}(3, q)$.*

*Proof.* Let $(x_0, x_1, x_2, x_3)$ be a point of $\mathrm{PG}(3, q)$, $x_i \in F_q$. We take the planes $x_3 = 0$ as $\pi_1$ and $x_2 = 0$ as $\pi_2$. We take also the conics

$$C_1 = \{(1, g, g^2, 0) \mid g \in F_q\} \cup \{A_1^\infty\}, \quad C_2 = \{(1, g, 0, g^2) \mid g \in F_q\} \cup \{A_2^\infty\},$$

where $A_1^\infty = (0, 0, 1, 0)$, $A_2^\infty = (0, 0, 0, 1)$. Then $T = (1, 0, 0, 0)$, $O = (0, 1, 0, 0)$, and $l$ is the line $x_2 = x_3 = 0$. Similarly to [68, Remark 2.2(3)] it can be shown that $\pi_3$ is the plane $x_2 = x_3$ and $C_3 = \{(1, g, g^2, g^2) | g \in F_q\} \cup \{A_3^\infty\}$, $A_3^\infty = (0, 0, 1, 1)$.

Take in $\mathrm{PG}(2, q)$ a complete $k_2$-arc $\mathcal{A}^*$ that is not a hyperoval. Without loss of generality, let

$$\mathcal{A}^* = \{(1, 0, 0), (0, 1, 0), (0, 1, f), (1, t_4, u_4), \ldots, (1, t_{k_2}, u_{k_2})\}.$$

Let $\mathcal{A}_s^* = \phi_s(\mathcal{A}^*)$ where $\phi_s(a, b, c) = (a, b + sc, c)$ is a projectivity with $s \in F_q$. We replace $q$ arcs $\mathcal{A}_s^*$ in $\pi_3$. Now the points of $\mathcal{A}_s^*$ have the form

$$(1, 0, 0, 0) = T, \quad (0, 1, 0, 0) = O, \quad (0, 1 + sf, f, f), \quad (1, t_j + su_j, u_j, u_j).$$

Every point of the form $(1, t_j + su_j, u_j, u_j)$ coincides with a point $(1, v, v^2, v^2) \in C_3 \backslash \{T, A_3^\infty\}$ for one and only one value of $s$. If $f = 1$ the point $(0, 1 + sf, f, f)$ coincides with $A_3^\infty$ for $s = 1$. Hence we have at most $k_2 - 2$ arcs $\mathcal{A}_s^*$ not convenient for our goal. But $k_2 - 2 < q$ and at least one of the arcs $\mathcal{A}_s^*$ can be taken as the needed arc $\mathcal{A}$ in Segre's Construction B. □

In [75] it is shown that in $\mathrm{PG}(n, q)$ there are complete $k$-caps with

$$k = 2^n, \quad q = 3, \quad n \geq 3. \tag{12}$$

$$k = 2^{n+1} - 2, \quad q = 4, \quad n \geq 3. \tag{13}$$

On the families of (10)–(13) see also [54,68], [78, Theorems 6.9–6.11].

In [4] a complete $k$-cap in $\mathrm{PG}(3, 2^h)$ with $k = (n-3)(q+1)+2$ is constructed from a complete $n$-arc of $\mathrm{PG}(2, 2^h)$ having special properties. An example of such arc with $n = (q + 8)/3$, $q = 2^{2h}$, $h \geq 4$, is described. It implies the existence of a family with

$$k = \frac{q^2 + 5}{3}, \quad q = 2^{2h}, \quad h \geq 4. \tag{14}$$

In [39] families with $k < T_q$ are considered for odd $q = p^h$ with $p, q$ sufficiently large. The following is shown:

$$A = \left\{k/q^2 \mid \text{there is a complete } k\text{-cap in PG}(3,q)\right\}$$
$$\text{is dense in the interval } \left[\frac{1}{3}, \frac{1}{2}\right].$$

In [41], a family is described with

$$k = \frac{q^2 + rq + 6}{3}, \quad r \in \{1, 2\},$$
$$t = k - 2, \quad q \geq 5, \ q \text{ is an odd prime or } q = 9, \tag{15}$$

where $r$ is the remainder of the division $[(q-3)/2 : 3]$.

In [67], complete $k$-caps in PG$(3, q)$ are obtained with

$$k = \frac{q^2 + 6}{3}, \quad t = k - 2, \quad q = 3^n, \quad n \geq 2. \tag{16}$$

In [69] a family is constructed with parameters

$$k = \frac{q^2 + 7}{2}, \quad t = k - 2, \quad q \text{ is odd prime}, \ q = 7 \text{ and } 13 \leq q \leq 931. \tag{17}$$

In the caps of (15) and (17) two points not belonging to a quadric lie on a line external to it. In the caps of (16) two points not belonging to a quadric lie on a tangent line to it.

In [73] two families are constructed with parameters

$$k = \frac{q^2 - 2q + 9}{2}, \quad q \geq 5, \quad q \text{ odd}. \tag{18}$$
$$k = (m + 1)(q + 1) + w, \quad w = 0, 1, 2, \quad q \geq 7, \quad q \text{ odd}, \tag{19}$$

where $m$ is the greatest integer such that $\binom{m}{2} \leq (q + 1)/4$. In the previous formula we have $q \geq 7$ as in PG$(3, 5)$ a complete 19-cap does not exist. Asymptotically the caps of (19) are the smallest known ones for odd $q$ as the order of their size is approximately $q\sqrt{q/2}$.

Caps entirely lying in only one orbit of the stabilizer group are considered in [5, 6, 77].

## 4. On the spectrum of sizes of complete caps in PG$(3, q)$

Table 3 gives the known sizes of complete caps in PG$(3, q)$. We used the sizes of complete caps and bounds from the works [1–7, 13], [23, Table 1], [25, 26, 30, 31, 37–44, 54, 56, 57, 66–69, 72, 73, 75–78], see also the references therein. We applied the relations (2)–(19) and Theorem 6.

The complete $k$-caps with $k = 17$ in PG$(3, 7)$, $k = 20$ in PG$(3, 8)$, $k = 24$ in PG$(3, 9)$, and $k = 30$ in PG$(3, 11)$, are obtained in [66]. In [13] it is proved that $t_2(3, 7) = 17$. The complete 41-cap in PG$(3, 16)$ is noted in [68, Table I]; see

TABLE 3 The sizes of the known complete $k$-caps in $\mathrm{PG}(3,q)$. $T_q = (q^2+q+4)/2$

| $q$ | $t_2(3,q)$ | Sizes $k$ of the known complete caps with $t_2(3,q) \le k \le m_2'(3,q)$ | $m_2'(3,q)$ | $m_2(3,q)$ | References |
|---|---|---|---|---|---|
| 3. | 8 | $8 = T_q$ | 8 | 10 | [40,43,75] |
| 4. | 10 | $10, 12, 13, 14 = T_q + 2$ | 14 | 17 | [40,42,43,75,76] |
| 5. | 12 | $12 \le k \le 18$ and $k = 20 = T_q + 3$ | 20 | 26 | [5,6,25,26,30,40,41, 43,73] |
| 7 | 17 | $17 \le k \le 30$ and $k = 32 = T_q + 2$ | 32 | 50 | [7,13,23,37,38,41–43, 53–57,66,69,73,75] |
| 8 | $14 \le$ | $20 \le k \le 41 = T_q + 3$ | $\le 60$ | 65 | [3,23,40,43,54,66,68, 76],Th. 6 |
| 9 | $15 \le$ | $24 \le k \le 48 = T_q + 1$ | $\le 78$ | 82 | [23,30,40,41,43,66,67, 72,73,75] |
| 11 | $18 \le$ | $30 \le k \le 70 = T_q + 2$ | $\le 116$ | 122 | [7,23,25,26,40–43, 54,66,72,73,75] |
| 13 | $21 \le$ | $36 \le k \le 93 = T_q$ | $\le 162$ | 170 | [23,25,26,40,41,43,66, 69,72,73,75] |
| 16 | $25 \le$ | $41 \le k \le 138 = T_q$ | $\le 242$ | 257 | [1,23,31,40,43,44,54, 57,68,76],Th. 6 |
| 17 | $26 \le$ | $51 \le k \le 157 = T_q + 2$ | $\le 278$ | 290 | [23,25,26,40–43, 66,69,72,73,75] |
| 19 | $29 \le$ | $58 \le k \le 192 = T_q$ | $\le 348$ | 362 | [7,23,25,26,40,41,43, 54,69,72,73,75] |
| 23 | $35 \le$ | $72 \le k \le 280 = T_q + 2$ | $\le 512$ | 530 | [23,25,26,40,41,43,54, 69,72,73,75] |

also (11). The complete 51-cap in $\mathrm{PG}(3,17)$ is described in [23]. The complete $k$-caps with $k = 36$ in $\mathrm{PG}(3,13)$, $k = 58$ in $\mathrm{PG}(3,19)$, and $k = 72$ in $\mathrm{PG}(3,23)$, are obtained in [25,26]. A number of relatively small complete caps in $\mathrm{PG}(3,q)$ are given in [41,42]. The complete 20-cap in $\mathrm{PG}(3,5)$ and 36-cap in $\mathrm{PG}(3,9)$ are obtained in [30]. The equality $m_2'(3,5) = T_5 + 3 = 20$ is proved in [6].

The result $m_2'(3,7) = T_7 + 2 = 32$ is obtained in [37]. In [3] a $(T_8+1)$-cap with $t = T_8 - 1$ in $\mathrm{PG}(3,8)$ is given. The complete $k$-caps with $k = T_8 + 3 = 41$ in $\mathrm{PG}(3,8)$ and $k = T_9 + 1 = 48$ in $\mathrm{PG}(3,9)$ are obtained in [23]. The complete $k$-caps with $k = T_q$, $q = 13, 16, 19$, follow from Theorem 5. The complete $k$-caps with $k = T_q + 2$, $q = 11, 17, 23$, follow from (8).

In the planes $\mathrm{PG}(2,8)$ and $\mathrm{PG}(2,16)$ there are complete $k_2$-arcs with $k_2 = 6, 10$ and $k_2 = 9, \ldots, 13, 18$, respectively; see [43]. Hence, by Theorem 6, in the spaces $\mathrm{PG}(3,8)$ and $\mathrm{PG}(3,16)$ there are complete $k$-caps with $k = 22, 26$ and $k = 41, 42, 43, 44, 45, 50$, respectively.

We use the following estimates: $m_2'(3,q) \leq q^2 - q + 6$, $q \geq 7$ odd [54, Theorem 18.4.1]; $m_2'(3,q) \leq q^2 - q/2 - \sqrt{q}/2 + 2$, $q = 2^h \geq 4$ [54, Theorem 18.5.1], [43, Theorem 3.3(a)] and the references therein; $m_2'(3,q) \leq q^2 - q + 2$, $q = 2^h \geq 16$ [44];

$$\frac{t_2(n,q)(t_2(n,q)-1)}{2}(q-1) + t_2(n,q) \geq |\mathrm{PG}(n,q)|. \tag{20}$$

The estimate of (20) is also used in Tables 6, 7 and 8. See also the lower bounds on $t_2(n,q)$ in [40, Lemma 2.5], [43, Theorem 3.4(d), Table 3.2].

The completeness of the spectrum of possible sizes of complete caps for $q = 3,4,5$ is shown in [6, 40], [43, Table 3.1].

**Open problem.** To complete the spectrum of sizes of complete caps in PG(3, 7), it only remains to solve the problem: "Is there a complete 31-cap in PG(3, 7)?"

The bounds which follow are obtained from the relations (2)–(9), Theorem 5 and Table 3, and from [1–7, 23, 25, 26, 31, 37, 40, 43], [54, Sections 18.2, 18.5], [72, 73], [75, Theorem V, p. 73]; see also the references therein:

$$T_q \leq m_2'(3,q) \quad \text{for all } q \geq 3; \tag{21}$$

$$\begin{aligned} &T_q + 2 = m_2'(3,q) \quad \text{for } q = 4,7; \quad 20 = T_5 + 3 = m_2'(3,5);\\ &41 = T_8 + 3 \leq m_2'(3,8); \end{aligned} \tag{22}$$

$$\begin{aligned} &T_q + 1 \leq m_2'(3,q) \quad \text{for } q \text{ odd with } q \neq 2^r - 1,\\ &\qquad \text{or } q = 4v+1, \quad \text{or } (q+1)/2 \text{ an odd prime.} \end{aligned} \tag{23}$$

$$T_q + 2 \leq m_2'(3,q) \text{ for } q \text{ an odd prime, } q \equiv 2 \pmod 3,\ q \geq 11. \tag{24}$$

*Remark* 2. Complete caps sharing $T_q - 1$ points with an elliptic quadric in PG$(3,q)$ in the first were constructed as examples of relatively small sizes. However, at present time they give the lower bounds of $m_2'(3,q)$ while the smallest known sizes are provided by quite other constructions; see (10)–(19). The known upper bounds on $m_2'(3,q)$ are approximately $cq^2$ with $c = 1$ for $q > 2$ even and for $q$ odd but not prime, while $c = 2641/2700$ for $q$ an odd prime. At the same time, the lower bounds are of order $T_q = (q^2+q+4)/2$. For references on bounds; see [2, 23, 43, 44, 51, 54], [57, Theorem 4.2,Table 4.2(ii)], [72, 73, 75, 76, 78] and (21)–(24). So, our knowledge on $m_2'(3,q)$ seems to be insufficient.

## 5. Some complete caps in PG(3, $q$) with boundary sizes

There exist two nonequivalent complete 20-caps in PG(3, 5) where $20 = T_5 + 3 = m_2'(3,5)$ [6]. These caps, called $K_1$ and $K_2$, are studied in [5, 6] where it is noted that each of them is preserved by a collineation group acting sharply transitively on it. Orbits and structures of the stabilizer group and its subgroups, list of points, the number of points on the secant planes, and other

TABLE 4 Relations of points of the 20-cap $K_1 \subset \mathrm{PG}(3,5)$ and the quadric $\mathcal{Q}$

| $\star$ | [6] | Cap points off $\mathcal{Q}$ | $\pi$ | $\star$ | [6] | Cap points on $\mathcal{Q}$ | $\pi$ |
|---|---|---|---|---|---|---|---|
| $P_1$ | **D** | 1111 | All | $T_1$ | $\mathbf{B}_1$ | 1144 | $\tau_1$ |
| $P_2$ | $\mathbf{B}_3$ | 1441 | All | $T_2$ | $\mathbf{B}_2$ | 1414 | $\tau_2$ |
| $P_3$ | $\mathbf{A}_{11}$ | 0141 | $\Sigma_3$ | $T_3$ | $\mathbf{C}_4$ | 0111 | $\Sigma_1$ |
| $P_3^*$ | $\mathbf{A}_2$ | 1410 | $\Sigma_3$ | $T_3^*$ | $\mathbf{C}_1$ | 1110 | $\Sigma_1$ |
| $P_4$ | $\mathbf{A}_{10}$ | 0144 | $\Sigma_4$ | $T_4$ | $\mathbf{A}_8$ | 1041 | $\Sigma_2$ |
| $P_4^*$ | $\mathbf{A}_3$ | 1140 | $\Sigma_4$ | $T_4^*$ | $\mathbf{A}_5$ | 1401 | $\Sigma_2$ |
| $\star$ | [6] | Cap points on $\mathcal{Q}$ | $\pi$ | Quadric points off $K_1$ | $\pi$ | | |
| $S_1$ | $\mathbf{A}_7$ | 1044 | $\Sigma_3$ | 0122 | $\Sigma_3$ | | |
| $S_1^*$ | $\mathbf{A}_6$ | 1104 | $\Sigma_3$ | 1130 | $\Sigma_3$ | | |
| $S_2$ | $\mathbf{A}_{12}$ | 0114 | $\Sigma_1$ | 1002 | $\Sigma_1$ | | |
| $S_2^*$ | $\mathbf{A}_1$ | 1440 | $\Sigma_1$ | 1003 | $\Sigma_1$ | | |
| $S_3$ | $\mathbf{C}_3$ | 1011 | $\Sigma_2$ | 1411 | $\Sigma_2$ | | |
| $S_3^*$ | $\mathbf{C}_2$ | 1101 | $\Sigma_2$ | 1141 | $\Sigma_2$ | | |
| $S_4$ | $\mathbf{A}_9$ | 1014 | $\Sigma_4$ | 1310 | $\Sigma_4$ | | |
| $S_4^*$ | $\mathbf{A}_4$ | 1404 | $\Sigma_4$ | 0131 | $\Sigma_4$ | | |

properties of $K_1$ and $K_2$ are described in [5,6], although no geometric constructions are given. The construction of $K_2$ is given in [25,26]. Here we describe a variant of the construction of the cap $K_1$ using computer results.

The maximum number of points of $K_1$ on an elliptic quadric $\mathcal{Q}$ is 14. There exist many quadrics with this property. For example, consider $\mathcal{Q}$ with equation

$$2x_0^2 + x_1^2 + x_1x_2 + x_2^2 + 2x_3^2 = 0.$$

The points of $K_1$ have a "symmetry property": if the point $A = (x, y, z, t)$ belongs to the cap then so does the point $A^* = (t, z, y, x)$. Four points of the cap with no zero coordinates have $A = A^*$; see Table 4. In the table the points of $K_1$ are noted in the column "$\star$" in compliance with the construction below and in the column "[6]" in compliance with [6, p. 10] via the bold font.

The points of $K_1$ are partitioned into subsets

$$P = \{P_1, P_2, P_3, P_3^*, P_4, P_4^*\},$$
$$T = \{T_1, T_2, T_3, T_3^*, T_4, T_4^*\},$$
$$S = \{S_1, S_1^*, S_2, S_2^*, S_3, S_3^*, S_4, S_4^*\}.$$

Points of $P$ are off the quadric $\mathcal{Q}$. Points of $T$ and $S$ lie, respectively, on tangents and bisecants of $\mathcal{Q}$ through $P_1$. Points of $\mathcal{Q}$ on these bisecants not contained in $K_1$ are written in Table 4. Two bisecants of the quadric through

Table 5 Properties of some complete caps in PG$(3,q)$ with boundary sizes

| $q$ | Cap | Stab. group (order) | Stab. group orbits | Intersecting planes | Points on intersecting planes |
|---|---|---|---|---|---|
| 5 | $K_1$ | $N_{16}\times\mathbf{S}_5(1920)$ | $20_1$ | $0_{16}, 3_{20}, 4_{80}, 6_{40}$ | $20^3_3, 20^4_{16}, 20^6_{12}$ |
| 7 | $K_7^{32}$ | (192) | $32_1$ | $0_{40}, 2_{60}, 4_{32}, 5_{96}, 6_{160}, 8_{12}$ | $32^4_4, 32^5_{15}, 32^6_{30}, 32^8_3$ |
| 8 | $K_8^{41}$ | $\mathbf{A_4}\times\mathbf{Z_2}(24)$ | $1_1, 4_2, 8_1, 12_2$ | $0_{28}, 1_{28}, 2_{20}, 3_{24}, 4_{72}, 5_{72}, 6_{232}, 7_{72}, 8_{28}, 9_5, 10_4$ | $12^3_2, 8^3_6, 12^4_8, 8^4_9, 12^4_{10}, 12^5_{10}, 20^5_{12}, 8^6_{29}, 12^6_{30}, 12^6_{32}, 8^6_{44}, 1^6_{64}, 12^7_{10}, 20^7_{12}, 8^7_{18}, 12^8_6, 8^8_7, 12^8_8, 40^9_1, 1^9_5, 16^{10}_1, 12^{10}_2$ |
| 8 | $K_{8,1}^{20}$ | $\mathbf{D}_5(10)$ | $10_2$ | $0_{35}, 1_{160}, 2_{90}, 3_{140}, 4_{100}, 5_{60}$ | $20^3_{21}, 20^4_{20}, 20^5_{15}$ |

$P_1$ have no points in $K_1$. It is an *important distinction* between the construction of the cap $K_1$ and Segre's Construction A. The corresponding points of $\mathcal{Q}$ are (0123), (1420) on the first bisecant and (0134), (1240) on the second.

The points $P_1, P_2, P_3, P_3^*$, are coplanar as are $P_1, P_2, P_4, P_4^*$. The line $P_1P_2$ is external to $\mathcal{Q}$. The sheaf of planes through $P_1P_2$ consists of planes $\tau_1$ and $\tau_2$ tangent to $\mathcal{Q}$ and secant planes $\Sigma_1, \Sigma_2, \Sigma_3$, and $\Sigma_4$. The column "$\pi$" of Table 4 gives the plane containing the corresponding point. Every secant plane meets the cap in a conic and the quadric in another conic. Finally, $\{(0123),(1420)\} \subset \Sigma_4$, $\{(0134),(1240)\} \subset \Sigma_3$.

Some properties of the 20-cap $K_1$, described in [6], are given in Table 5 where the column "stab. group (order)" gives the structure of the stabilizer group. The order of the group is written in brackets. The column "stab. group orbits" notes the lengths of orbits of the stabilizer group with a subscript that is the number of orbits of such length. In the column "intersecting planes" an entry $s_n$ remarks that the number of $s$-secant planes is $n$. Finally, in the column "points on intersecting planes" an entry $p_n^s$ says that the number of points, every of which lie on $n$ $s$-secant planes, is equal to $p$. For the structure of groups we use the notation of [55, Table 2.3]. By [6, Proposition 7], the stabilizer group of $K_1$ is the semidirect product of an elementary abelian normal subgroup $N_{16}$ and a subgroup isomorphic to $\mathbf{S}_5$.

For any $k$-cap $K$ in PG$(n,q)$, let $w_i$ be the number of codewords with weight $i$ in the associated code $\mathcal{A}(K)$ and let $h_j$ denote the number of hyperplanes in PG$(n,q)$ meeting $K$ in $j \geq 0$ points. The following result is given in [52, Theorem 4.1], [13, 26]:

$$w_0 = 1, \quad w_{k-j} = (q-1)h_j \quad \text{for } 0 \leq j < k. \tag{25}$$

Using (25) and Table 5, the weight spectrum of the associated code $\mathcal{A}(K)$ can be deduced.

In [37] a complete $(T_q + 2)$-cap in $\mathrm{PG}(3, 7)$ has been obtained by computer; denote it $K_7^{32}$. The list of points of $K_7^{32}$ and the weight spectrum of the associated code $\mathcal{A}(K_7^{32})$ can be found online, in Yves Edel's homepage. However, no geometric construction is given in [37]. Here we describe features of the structure of the cap $K_7^{32}$ using computer results. The stabilizer group of $K_7^{32}$ has order 192, it acts transitively on the cap and is a subgroup of the stabilizer group of the complete 20-cap $K_1$ of $\mathrm{PG}(3, 5)$. The maximum number of points of $K_7^{32}$ on an elliptic quadric $\mathcal{Q}$ is equal to 24. An example of a quadric with this property is $\mathcal{Q} : x_0^2 + 3x_0x_1 + x_1^2 + 6x_2x_3 = 0$. Let $S_{24} = K_7^{32} \cap \mathcal{Q}$. The stabilizer of the 24-set $S_{24}$ has order 48, it has two orbits on the points of the cap: $S_{24}$ and the remaining 8 points of the cap outside the quadric. The secant lines of the set $S_{24}$ cover all the points of $\mathrm{PG}(3, 7)$ except the remaining 26 points on the quadric and the remaining 8 points of the cap outside $\mathcal{Q}$.

The cap $K_1$ in $\mathrm{PG}(3, 5)$ has a similar structure. It has 14 points on an elliptic quadric, whose stabilizer is the same group of order 48 as the subset $S_{24}$ of $K_7^{32}$. The secants lines of the 14 points on the quadric cover all the points of $\mathrm{PG}(3, 5)$ except the remaining 12 points on the quadric and the remaining 6 points of the cap.

This motivated a computer search in $\mathrm{PG}(3, q)$, $q = 9, 11, 13$, for complete caps with a similar structure: a set of points on the elliptic quadric constituting the orbit of a certain group and $q + 1$ points outside the quadric. In particular all the subgroups of order 48 and some subgroups of order 24 of the stabilizer group of the elliptic quadric have been considered. However, no example with the required properties was found.

The following important property of $K_7^{32}$ should be noted. Let $S_8 = K_7^{32} \setminus S_{24}$ be the set of points $K_7^{32}$ outside the quadric $\mathcal{Q}$. Through every point of $S_8$ there are 8 tangents to $\mathcal{Q}$ and 21 bisecants of $\mathcal{Q}$ [54,75]. For any point of $S_8$, the cap $K_7^{32}$ contains 21 points on the bisecants and only three points on tangents. So, five tangents through a point of $S_8$ have no points in $K_7^{32}$. Recall that, for the cap $K_1$, two bisecants have no points in it.

By the above, it seems useful to consider a variant of Segre's Construction A when some tangents and/or bisecants through the point $P$ off the elliptic quadric $\mathcal{Q}$ do not contain any points of the cap constructed. Using this variant one can try to obtain complete caps with new sizes.

For $q \neq 5$ the only known example of a complete $(T_q + 3)$-cap is the 41-cap in $\mathrm{PG}(3, 8)$ obtained in [23]. We denote it $K_8^{41}$. Similarly to [23,66] we write $F_8 = \{0, 1 = \alpha^0, 2 = \alpha^1, \ldots, 7 = \alpha^6\}$, where $\alpha$ is a root of the polynomial $x^3 + x + 1$ generating the field $F_8$.

The cap $K_8^{41}$ contains four hyperovals; see Table 5. Let $g_i^{(j)}$ be the $j$-th $i$-orbit of the stabilizer group. Then $K_8^{41} = \{g_1^{(1)}, g_4^{(1)}, g_4^{(2)}, g_8^{(1)}, g_{12}^{(1)}, g_{12}^{(2)}\}$, where

$$g_1^{(1)} = \{(1, 3, 3, 3)\}, g_4^{(1)} = \{(1, 1, 6, 2)^1, (1, 3, 6, 3)^2, (0, 1, 1, 6)^3, (1, 7, 0, 4)^4\},$$
$$g_4^{(2)} = \{(1, 2, 0, 0)^1, (1, 2, 7, 0)^2, (1, 1, 7, 2)^3, (1, 7, 3, 4)^4\},$$

$$\begin{aligned} g_8^{(1)} &= \{(1,3,4,4)^1,(1,7,4,1)^2,(1,1,0,1)^3,(0,1,3,5)^4,(1,0,0,4)^1,\\ &\quad (1,7,3,5)^2,(1,0,6,3)^3,(1,1,6,5)^4\},\\ g_{12}^{(1)} &= \{(1,2,4,5)^{1,2},(1,0,7,2)^{1,2},(1,3,6,0)^{1,3},(0,1,5,4)^{1,3},(0,1,1,5)^{1,4},\\ &\quad (1,1,7,5)^{1,4},(1,3,1,2)^{2,3},(1,6,1,1)^{2,3},(1,0,6,0)^{2,4},(1,6,3,3)^{2,4},\\ &\quad (1,0,0,0)^{3,4},(1,6,7,3)^{3,4}\},\\ g_{12}^{(2)} &= \{(0,1,3,0),(1,6,7,0),(1,7,7,1),(1,3,1,5),(0,1,4,3),(1,1,2,7),\\ &\quad (1,4,3,4),(1,7,0,5),(1,1,0,7),(1,0,7,1),(1,6,3,0),(1,2,1,7)\}. \end{aligned}$$

The superscript of a point indicates the order numbers of the hyperovals containing it. Any two hyperovals meet in two points. Intersecting points of the hyperovals form the 12-orbit $g_{12}^{(1)}$. One can see the symmetric connections of the hyperovals and the orbits.

The smallest known size of a complete cap in PG$(3,8)$ is 20 [66]. We obtained three inequivalent complete 20-caps in PG$(3,8)$ with the stabilizer groups $\mathbf{D}_5, \mathbf{Z}_2, \mathbf{S}_3$ of orders $10, 2, 6$. The first cap, see Table 5, is $K_{8,1}^{20} = \{g_{10}^{(1)}, g_{10}^{(2)}\}$ where

$$\begin{aligned} g_{10}^{(1)} &= \{(1,6,6,7),(1,6,6,5),(1,2,0,5),(1,2,3,2),(1,2,7,0),\\ &\quad (1,7,6,4),(1,2,2,2),(1,4,6,7),(1,0,2,0),(1,6,5,6)\},\\ g_{10}^{(2)} &= \{(1,3,5,5),(1,4,5,0),(0,1,4,5),(1,0,0,1),(1,5,0,6),\\ &\quad (1,3,2,2),(1,4,0,7),(1,0,1,2),(1,1,4,7),(1,6,5,4)\}. \end{aligned}$$

The cap $K_{8,1}^{20}$ has the dihedral stabilizer group. It has been obtained using the algorithm that joins orbits of the subgroup $\mathbf{Z}_5$ of $P\Gamma L(4,8)$.

From (25) and Table 5, the associated codes $\mathcal{A}(K_1)$ and $\mathcal{A}(K_{8,1}^{20})$ are *optimal* as they have the greatest possible minimum distance [50].

## 6. On the spectrum of complete cap sizes in PG$(n,q)$, $n \geq 4$

In [79] a few families of complete $k$-caps in PG$(4,q)$ are described with the following parameters:

$$k = 2q^2 + q, \quad k = 2q^2 + q + 5, \quad n = 4, \quad q > 2 \text{ even}; \tag{26}$$

$$k = 2q^2 + 1, \quad n = 4, \quad q \geq 3 \text{ odd}; \tag{27}$$

$$k = 2q^2 + 2q + 1, \quad n = 4, \quad q > 3 \text{ odd}. \tag{28}$$

Table 6 gives sizes of the known complete caps in PG$(n,q)$, $n \geq 4$, $q \geq 3$. We used sizes and bounds from [9,10], [23, Table 2], [28,32–37,42], [43, Table 4.3], [49–52], [57, Table 4.5], [66], [68, Table I], [75,79]; see also the relations (12), (13), (26)–(28).

Table 6 The sizes of the known complete $k$-caps in $\mathrm{PG}(n, q)$, $n \geq 4$, $q \geq 3$

| $n$ | $q$ | $t_2(n,q)$ | Sizes $k$ of the known complete caps with $t_2(n,q) \leq k \leq m_2'(n,q)$ | $m_2'(n,q)$ | $m_2(n,q)$ | References |
|---|---|---|---|---|---|---|
| 4 | 3. | 11 | $k = 11$ and $16 \leq k \leq 19$ | 19 | 20 | [40,43,75,79] |
| 4 | 4. | 20 | $20 \leq k \leq 40$ | 40 | 41 | [10,11,23,33,36,42,43,68, 75,79] |
| 4 | 5 | $21 \leq$ | $31 \leq k \leq 66$ | | $\leq 88$ | $\star$, [23,35–37,40,43,57,66,79] |
| 4 | 7 | $29 \leq$ | $\mathbf{56} \leq k \leq 124$ and $k = 126, 132$ | | $\leq 238$ | $\star$, [23,35–37,40,43,57,66,79] |
| 5 | 3 | $20 \leq$ | $k = 22$ and $26 \leq k \leq 48$ | 48 | 56 | $\star$, [9,23,35,43,52,57,68,75] |
| 5 | 4 | $31 \leq$ | $50 \leq k \leq 108$ and $k = 112, 126$ | | $\leq \mathbf{153}$ | $\star$, [23,35,43,49,50,57,66,75] |
| 6 | 3 | $34 \leq$ | $k = 44$ and $46 \leq k \leq 103$, and $k = 112$ | | $\leq 136$ | $\star$, [9,23,28,43,57,75] |
| 6 | 4 | $61 \leq$ | $\mathbf{114} \leq k \leq 288$ | | $\leq \mathbf{607}$ | $\star$, [23,34,35,43,52,57] |
| 7 | 3 | $58 \leq$ | $88 \leq k \leq 238$ and $243 \leq k \leq 248$ | | $\leq \mathbf{404}$ | $\star$, [23,28,35,43,52,57,75] |
| 8 | 3 | $100 \leq$ | $176 \leq k \leq 532$ and $k = \mathbf{534}$ | | $\leq \mathbf{1208}$ | $\star$, [23,28,35,43,52,57,75] |
| 9 | 3 | $172 \leq$ | $352 \leq k \leq 1214$ and $k = 1216$ | | $\leq \mathbf{3247}$ | $\star$, [12,23,28,32,35,43,57,75] |

In Table 6 we use the fact that in the ternary projective spaces the *doubling construction*, leaving from a complete cap, gives rise to a new one:

**Theorem 7.** [28, Theorem 1] *Let $K'$ be a complete cap in* $\mathrm{PG}(n, 3)$ *and let* $K_1 = \{(0, a) \mid a \in K'\} \cup \{(1, a) \mid a \in K'\}$. *Then $K_1$ is a complete cap in* $\mathrm{PG}(n + 1, 3)$.

**Corollary 1.** *The upper bound on the smallest size $t_2(n, 3)$ of a complete cap in the ternary projective space* $\mathrm{PG}(n, 3)$ *satisfies*

$$t_2(n, 3) \leq 11 \cdot 2^{n-4}, \quad n \geq 4. \tag{29}$$

*Proof.* Use $(n - 4)$-fold doubling of an 11-cap in $\mathrm{PG}(4, 3)$. □

At the present time no caps in $\mathrm{PG}(n, 3)$ of size smaller than the bound in (29) are known. The 11-cap in $\mathrm{PG}(4, 3)$ corresponds to the perfect ternary Golay code [74].

The complete spectrum of possible sizes for complete caps in $\mathrm{PG}(4, 3)$ is given in [40]; see also [43]. The result $t_2(4, 4) = 20$ and a complete 21-cap in $\mathrm{PG}(4, 4)$ are obtained in [10,11]. Together with the results of [23, Table 2] this provides the completeness of the spectrum of sizes for complete caps in $\mathrm{PG}(4, 4)$.

For lower bounds on $t_2(n, q)$ we use (20) and tables in [23,43].

A complete 20-cap in PG(4, 4) was obtained by T. Penttila and G. F. Royle in 1995; see also [43,68].

The complete $k$-caps with $k = 31$ in PG(4, 5) and $k = 50$ in PG(5, 4) are obtained in [66]. The complete 22-cap in PG(5, 3) is noted in [68, Table I]. The complete $k$-caps with $k = 44$ in PG(6, 3), $k = 88$ in PG(7, 3), $k = 176$ in PG(8, 3), and $k = 352$ in PG(9, 3) are written in [23, Table 2]. They come from Corollary 1.

The exact values $m_2(4, 4) = 41$ and $m'_2(4, 4) = 40$ are given by [33] and [36]. The relation $m_2(5, 3) = 56$ is provided by the Hill 56-cap in PG(5, 3) [35,52], [57, Table 4.2(i)]. The results $m'_2(5, 3) = 48$ and $m_2(6, 3) \le 136$ are described in [9]. Complete $k$-caps with $k = 248$ in PG(7, 3), $k = 524, 532$ in PG(8, 3), $k = 1120$ in PG(9, 3), $k = 66$ in PG(4, 5), and $k = 132$ in PG(4, 7), are constructed in [35]. The Glynn complete 126-cap in PG(5, 4) is obtained in [49]; another description is given in [35]. The complete 288-cap in PG(6, 4) is described in [34], where the estimate $m_2(6, q) \ge q^4 + 2q^2$ is obtained; see also [35, Table 4]. The complete 1216-cap in PG(9, 3) is constructed in [32].

The estimates $m_2(4, 5) \le 88$ and $m_2(4, 7) \le 238$ are obtained in [37]. The following are also used:

$$m_2(n, q) \le q m_2(n-1, q) - (q+1), \quad q > 2, \quad n \ge 4; \tag{30}$$

$$m_2(n, q) \le q^{n+1} \frac{n+2}{2(n+1)^2}, \quad q > 2, \quad n \ge 3; \tag{31}$$

see [52, Theorem 5.5], [57, Table 4.5] for (30) and [12] for (31).

**Proposition 1.** *The following are upper bounds for* $m_2(n, q)$*:*

(i) $m_2(5, 4) \le 153$;
(ii) $m_2(6, 4) \le 607$;
(iii) $m_2(7, 3) \le 404$;
(iv) $m_2(8, 3) \le 1208$;
(v) $m_2(9, 3) \le 3247$.

*Proof.* By [50], a $[154, 154 - 6, 4]_4$ code does not exist. So, $m_2(5, 4) \le 153$. By (30), from $m_2(5, 4) \le 153$ we obtain $m_2(6, 4) \le 607$ and from $m_2(6, 3) \le 136$ we get $m_2(7, 3) \le 404$, $m_2(8, 3) \le 1208$. Finally, by (31) we obtain $m_2(9, 3) \le 3247$. □

The complete $k$-caps with $k = 56$ in PG(4, 7), $k = 114$ in PG(6, 4), and $k = 534$ in PG(8, 3) noted by the bold font in Table 6 are obtained by computer in this work with the help of the randomized greedy algorithms [17,23]. So,

$$t_2(4, 7) \le 56, \quad t_2(6, 4) \le 114, \quad 534 \le m_2(8, 3).$$

Also, by the greedy algorithms, many new sizes $k$ for complete $k$-caps in the region $t_2(n, q) \le k \le m'_2(n, q)$ were obtained:

$$k = 62 - 65 \quad \text{in PG}(4, 5);$$

$$k = 114 - 124,\ 126 \quad \text{in PG}(4, 7);$$

TABLE 7 The sizes $\bar{t}_2(3,q)$ of the known small complete caps in $PG(3,q)$

| $q$ | $t_2(3,q)$ | $\bar{t}_2(3,q)$ | Ref. | $q$ | $t_2(3,q)$ | $\bar{t}_2(3,q)$ | Ref. |
|---|---|---|---|---|---|---|---|
| 7 | 17 | $3q-4=17$ | [13,66] | 43 | $63 \leq$ | $3q+25=153$ | $\star$ |
| 8 | $14 \leq$ | $3q-4=20$ | [66] | 47 | $69 \leq$ | $3q+28=169$ | $\star$ |
| 9 | $15 \leq$ | $3q-3=24$ | [66] | 49 | $72 \leq$ | $3q+33=180$ | $\star$ |
| 11 | $18 \leq$ | $3q-3=30$ | [66] | 53 | $77 \leq$ | $3q+36=195$ | $\star$ |
| 13 | $21 \leq$ | $3q-3=36$ | [25,26] | 59 | $86 \leq$ | $3q+43=220$ | $\star$ |
| 16 | $25 \leq$ | $2q+9=41$ | [68],Th. 6 | 61 | $89 \leq$ | $3q+47=230$ | $\star$ |
| 17 | $26 \leq$ | $3q=51$ | [23] | 64 | $93 \leq$ | $2q+22=150$ | Th. 6 |
| 19 | $29 \leq$ | $3q+1=58$ | [25,26] | 67 | $97 \leq$ | $3q+56=257$ | $\star$ |
| 23 | $35 \leq$ | $3q+3=72$ | [25,26] | 71 | $103 \leq$ | $3q+62=275$ | $\star$ |
| 25 | $38 \leq$ | $3q+6=81$ | $\star$ | 73 | $106 \leq$ | $3q+68=287$ | $\star$ |
| 27 | $41 \leq$ | $3q+8=89$ | $\star$ | 79 | $114 \leq$ | $4q-4=312$ | $\star$ |
| 29 | $43 \leq$ | $3q+9=96$ | $\star$ | 81 | $117 \leq$ | $4q-3=321$ | $\star$ |
| 31 | $46 \leq$ | $3q+11=104$ | $\star$ | 83 | $120 \leq$ | $4q-2=330$ | $\star$ |
| 32 | $48 \leq$ | $2q+14=78$ | Th. 6 | 89 | $128 \leq$ | $4q-1=355$ | $\star$ |
| 37 | $55 \leq$ | $3q+17=128$ | $\star$ | 97 | $140 \leq$ | $4q+6=394$ | $\star$ |
| 41 | $60 \leq$ | $3q+22=145$ | $\star$ | 128 | $\leq$ | $2q+34=290$ | [48],Th. 6 |

$$k = 45 - 47 \quad \text{in } PG(5,3);$$
$$k = 47, 49, 51,\ 95 - 103 \quad \text{in } PG(6,3);$$
$$k = 255 - 287 \quad \text{in } PG(6,4);$$
$$k = 189 - 223,\ k = 225 - 238,\ k = 243 - 247 \quad \text{in } PG(7,3);$$
$$k = 381 - 447,\ k = 449 - 523,\ k = 525 - 531 \quad \text{in } PG(8,3);$$
$$k = 785 - 895,\ k = 897 - 1119,\ k = 1121 - 1214 \quad \text{in } PG(9,3).$$

See also [23, Table 2].

To obtain new computer results for relatively large caps in the spaces $PG(4,5)$, $PG(4,7), PG(7,3), PG(8,3), PG(9,3)$, we used the matrix representations of the complete $k$-caps with $k = 66, 132, 248, 532, 1216$ given online, in Yves Edel's homepage. The shortened matrices were applied as starting points for the greedy algorithms. Similarly, for $PG(5,3)$ we used the matrix of the Hill 56-cap written in [35]; for $PG(6,3)$ we applied the matrix of the 112-cap obtained by the doubling construction of Theorem 7; for $PG(6,4)$ we took the matrix of the 288-cap described in [34]. Finally, in $PG(6,3)$ for new computer results for small caps we used the twofold doubling 11-cap in $PG(4,3)$ written in [40].

## 7. Small caps in $PG(n,q)$, $n \geq 3$, $q > 2$

Tables 7 and 8 give sizes of the known small complete caps in $PG(3,q)$ and $PG(n,q)$, $n \geq 4$.

Table 8 The sizes $\bar{t}_2(n,q)$ of the known small complete caps in $\mathrm{PG}(n,q)$

| $n$ | $q$ | $t_2(n,q)$ | $\bar{t}_2(n,q)$ | Ref. | $n$ | $q$ | $t_2(n,q)$ | $\bar{t}_2(n,q)$ | Ref. |
|---|---|---|---|---|---|---|---|---|---|
| 4 | 4 | 20 | 20 | [10,11,68] | 5 | 4 | $31 \leq$ | 50 | [66] |
| 4 | 5 | $21 \leq$ | 31 | [66] | 5 | 5 | $36 \leq$ | 82 | $\star$ |
| 4 | 7 | $29 \leq$ | 56 | $\star$ | 5 | 7 | $70 \leq$ | 174 | $\star$ |
| 4 | 8 | $33 \leq$ | 53 | $\star$ | 5 | 8 | $91 \leq$ | 181 | $\star$ |
| 4 | 9 | $39 \leq$ | 87 | [23] | 5 | 9 | $115 \leq$ | 302 | $\star$ |
| 4 | 11 | $52 \leq$ | 121 | $\star$ | 6 | 3 | $34 \leq$ | 44 | [28] |
| 4 | 13 | $67 \leq$ | 162 | $\star$ | 6 | 4 | $61 \leq$ | 114 | $\star$ |
| 4 | 16 | $91 \leq$ | 153 | [20,21], (32) | 6 | 5 | $80 \leq$ | 131 | [28] |
| 4 | 17 | $100 \leq$ | 255 | $\star$ | 6 | 7 | $121 \leq$ | 349 | [28] |
| 5 | 3 | $20 \leq$ | 22 | [68] | 6 | 8 | $256 \leq$ | 437 | $\star$ |

From Tables 3, 7 and [43, Table 3.1], the following result is obtained.

**Theorem 8.** *In spaces* $\mathrm{PG}(3,q)$,

$$t_2(3,q) \leq 3q \quad \text{for } 2 \leq q \leq 17,$$
$$t_2(3,q) < 4q \quad \text{for } 2 \leq q \leq 89.$$

In Table 8 for $q = 16$ we use the complete $k$-caps in $\mathrm{PG}(n,q)$, $q = 2^h \leq 2^{15}$, constructed in [20,21]. Their parameters are as follows:

$$\begin{aligned} k &= b_q\left(q^{\frac{n-2}{2}} + q^{\frac{n-4}{2}} + q^{\frac{n-6}{2}} + \cdots + q + 1\right), \quad n \geq 4 \text{ even.} \\ k &= 2q^{\frac{n-1}{2}} + b_q\left(q^{\frac{n-3}{2}} + q^{\frac{n-5}{2}} + q^{\frac{n-7}{2}} + \cdots + q + 1\right), \quad n \geq 5 \text{ odd.} \end{aligned} \tag{32}$$

Here $b_q$ is as in the following table, cf. Sect. 2:

| $\log_2 q$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $b_q$ | 6 | 9 | 14 | 22 | 34 | 55 | 86 | 124 | 201 | 307 | 461 | 665 | 993 |

For $\mathrm{PG}(n,8)$, $n = 4,5,6$, we use the shortened matrices of the caps of (32) as starting points for the greedy algorithms.

## 8. On the spectrum of sizes of binary complete caps

In the binary space $\mathrm{PG}(n,2)$ we consider only $k$-caps with $k \leq 2^{n-1}$ since all possible sizes of binary complete caps with $k \geq 2^{n-1}+1$ are known [29].

**Theorem 9.** [29, Theorem 1] *In the space* $\mathrm{PG}(n,2)$, $n \geq 3$, *a complete* $k$*-cap, with* $k \geq 2^{n-1}+1$, *has size* $k = 2^{n-1} + 2^{n-1-g}$ *for some* $g = 0,2,3,\ldots,n-1$. *For each* $g = 0,2,3,\ldots,n-1$, *there exists a complete* $(2^{n-1}+2^{n-1-g})$*-cap in* $\mathrm{PG}(n,2)$. *Also, every such complete cap can be formed by repeated application of the doubling construction to a complete* $(2^{m-1}+1)$*-cap of* $\mathrm{PG}(m,2)$, $m < n$.

TABLE 9 The sizes of the known complete $k$-caps in $PG(n, 2)$, $k \leq 2^{n-1}$

| $n$ | $t_2(n, 2)$ | Sizes $k$ of the known small complete caps with $t_2(n, 2) \leq k \leq 2^{n-1}$ | References |
|---|---|---|---|
| 5. | 13 | 13 | [40,45] |
| 6. | 21 | $f(6) = 21 \leq k \leq 31,\ k \neq 23, 30$ | [19,23,24,45, 58,80] |
| 7 | $25 \leq$ | $f(7) = 28 \leq k \leq 63$ | [16,19,23,24, 45,80] |
| 8 | $34 \leq$ | $f(8) = 43 \leq k \leq 127$ | [16,19,23,24, 45,80] |
| 9 | $47 \leq$ | $k = f(9) = 57$ and $59 \leq k \leq 255$ | [16,19,23,24, 45,80] |
| 10 | $65 \leq$ | $k = f(10) = 89$ and $91 \leq k \leq 511$ | [16,19,23,24, 45,80] |
| 11 | $92 \leq$ | $k = f(11) = 117,\ k = 121$ and $123 \leq k \leq 1023$ | [16,19,23,24, 45,80] |
| 12 | $129 \leq$ | $k = f(12) = 181,\ k = 185$ and $187 \leq k \leq 2047$ | [16,19,23,24, 45,80] |

The structure and properties of complete $(2^{m-1} + 1)$-caps are studied in [14]; see also the references there and [29]. Numerous constructions of binary complete caps of distinct sizes are given in [14,19,24,29,45,58,62,80]; see also the references there. The smallest known complete $k$-caps in $PG(n, 2)$, $n \geq 6$, with $k = f(n)$ are constructed in [45]. Here

$$\begin{aligned} f(6) &= 21, \quad f(7) = 28, \\ f(2m) &= 23 \times 2^{m-3} - 3, \quad m \geq 4, \\ f(2m-1) &= 15 \times 2^{m-3} - 3, \quad m \geq 5. \end{aligned} \tag{33}$$

In $PG(n, 2)$, $n = 2, 3, 4$, complete $k$-caps with $k \leq 2^{n-1}$ do not exist. For $n = 5$, there is only one such cap with $k = 13$; see [40,43,54] and the references there.

Table 9 updating [19, Tables 1, 3] gives the sizes of the known complete $k$-caps in $PG(n, 2)$, $k \leq 2^{n-1}$, $n \leq 12$. For $n = 6$ completeness of the spectrum is proved in [58]. A 59-cap in $PG(9, 2)$ is obtained in [24].

In Table 9 the lower bounds on the length of linear codes with covering radius $R = 2$ and codimension $n + 1$ [16] are taken as the lower bounds on $t_2(n, 2)$ for $7 \leq n \leq 12$. We use the fact that the dual code of a complete cap has $R = 2$ [56].

The following are conjectures on sizes of binary complete caps.

**Conjecture 2.** [29, Remark 4], [19]*: In the space* $PG(n, 2)$ *a complete* $2^{n-1}$*-cap does not exist.*

**Conjecture 3.** [19,23]*: For $n \geq 7$ in the space* PG$(n, 2)$ *there exist complete caps of all sizes $k$ with $f(n) \leq k \leq 2^{n-1} - 1$, where $f(n)$ is defined in* (33).

**Conjecture 4.** *For $n \geq 6$ in the space* PG$(n, 2)$ *the bound is $t_2(n, 2) = f(n)$.*

By Table 9, Conjecture 2 is proved for $n = 5, 6$, Conjecture 3 holds for $n = 7, 8$, and Conjecture 4 is proved for $n = 6$.

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Alexander A. Davydov
Institute for Information Transmission Problems
Russian Academy of Science
Bol'shoi Karetnyi per. 19, GSP-4
Moscow 127994, Russian Federation
e-mail: adav@iitp.ru

Giorgio Faina, Stefano Marcugini, Fernanda Pambianco
Dipartimento di Matematica e Informatica
Università degli Studi di Perugia
Via Vanvitelli 1
Perugia 06123
Italy
e-mail: faina@dipmat.unipg.it;
gino@dipmat.unipg.it;
fernanda@dipmat.unipg.it

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