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ON SKEW BROWNIAN MOTION

J. M. HARRISON[1] AND L. A. SHEPP

Stanford University and Bell Laboratories

We consider the stochastic equation $X(t) = W(t) + \beta l_0^X(t)$, where W is a standard Wiener process and $l_0^X(\cdot)$ is the local time at zero of the unknown process X. There is a unique solution X (and it is adapted to the fields of W) if $|\beta| \le 1$, but no solutions exist if $|\beta| > 1$. In the former case, setting $\alpha = (\beta + 1)/2$, the unique solution X is distributed as a skew Brownian motion with parameter α. This is a diffusion obtained from standard Wiener process by independently altering the signs of the excursions away from zero, each excursion being positive with probability α and negative with probability $1 - \alpha$. Finally, we show that skew Brownian motion is the weak limit (as $n \to \infty$) of $n^{-1/2}S_{[nt]}$, where S_n is a random walk with exceptional behavior at the origin.

1. Introduction. Recently Walsh (1978) has resurrected a simple but intriguing diffusion process that Itô and McKean (1965, Section 4.2, Problem 1) called skew Brownian motion. This is really a class of diffusions $X_\alpha = \{X_\alpha(t), t \ge 0\}$, indexed by $0 \le \alpha \le 1$, which Itô and McKean constructed by the following procedure. Let $Z = \{Z(t), t \le 0\}$ be a reflecting Wiener process on $[0, \infty)$ and consider the excursions of Z away from the origin. Change the sign of each excursion independently with probability $1 - \alpha$ so that a given excursion is positive with probability α and negative with probability $1 - \alpha$. Itô and McKean assert (but do not prove) that the resulting process is a diffusion, and they compute its scale and speed measures.

As an aid to intuition, it is helpful to keep in mind this construction of skew Brownian motion by random flipping of Wiener excursions. In our formal development, however, we shall define X_α directly in terms of its scale and speed mesasures. The construction of X_α from its scale and speed has been discussed by Walsh (1978) and will be reviewed in Section 2 below.

Our primary objective here is to connect skew Brownian Motion with a particular stochastic equation. Let $W = \{W(t), t \ge 0\}$ be a standard Wiener process with respect to a filtration $\{\mathscr{F}_t, t \ge 0\}$ on some probability space $(\Omega, \mathscr{F}, P)$. Given a real constant β, we seek a process $X = \{X(t), t \ge 0\}$ which is adapted to $\{\mathscr{F}_t\}$ and satisfies

$$X(t) = W(t) + \beta l_0^X(t), \qquad t \ge 0, \tag{1}$$

where $l_0^X(\cdot)$ is the local time at zero of the unknown process X, meaning that

$$l_0^X(t) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \text{measure}\{0 \le u \le t : |X(u)| \le \epsilon\}. \tag{2}$$

Such a process X will be called a solution of the stochastic equation (1)–(2). It will be shown in Section 3 that there is no solution if $|\beta| > 1$, and there is a unique solution if $|\beta| \le 1$. In the latter case, the unique solution X is adapted to the fields $\{\mathscr{F}_t^W\}$ of the Wiener process W and is distributed as the skew Brownian motion X_α with parameter $\alpha = (\beta + 1)/2$.

In Section 4 we show that X_α can be obtained as the weak limit of a normalized random

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walk which has special behavior at the origin. We consider a Markov chain $\{S_n\}$ on the integers which behaves like a symmetric ± 1 random walk except at zero. Starting from zero, the process moves up by one with probability α and down by one with probability $1 - \alpha$. It is shown that $n^{-1/2}S_{[nt]}$ converges in distribution to $X_\alpha(t)$ as $n \to \infty$ for all $t > 0$. This argument can easily be extended to show weak convergence in function space.

2. Skew Brownian motion. In this section, to avoid trivial complications, we treat only the case $|\alpha| < 1$. Our terminology and notation for one-dimensional diffusions follow Freedman (1971). Let

$$\sigma_\alpha^2(x) = \begin{cases} (1-\alpha)^2 & \text{if } x \geq 0, \\ \alpha^2 & \text{if } x < 0. \end{cases} \tag{3}$$

Let $W = \{W(t), t \geq 0\}$ be a standard Brownian motion on some probability space and set

$$Y_\alpha(t) = W(T_\alpha(t)), \qquad t \geq 0, \tag{4}$$

where the time change T_α is defined by

$$t = \int_0^{T_\alpha(t)} du/\sigma_\alpha^2(W(u)), \qquad t \geq 0. \tag{5}$$

Thus Y_α is a diffusion in natural scale with state space $\mathbb{R}$ (the whole real line) and infinitesimal variance function $\sigma_\alpha^2(x)$. That is, the speed measure of Y_α (in Freedman's terminological system) is

$$m_\alpha(dx) = 2\, dx/\sigma_\alpha^2(x), \qquad x \in \mathbb{R} \tag{6}$$

Next let

$$r_\alpha(x) = \begin{cases} x/(1-\alpha), & \text{if } x \geq 0 \\ x/\alpha, & \text{if } x < 0. \end{cases} \tag{7}$$

and define X_α by

$$X_\alpha(t) = r_\alpha(Y_\alpha(t)), \qquad t \geq 0. \tag{8}$$

Thus the scale function of the diffusion X_α is

$$s_\alpha(x) = \begin{cases} (1-\alpha)x, & \text{if } x \geq 0 \\ \alpha x, & \text{if } x < 0, \end{cases} \tag{9}$$

(the inverse of r_α). The construction, given by (4) and (8), of the diffusion X_α from its scale function and speed measure is of course standard in the theory of diffusions, cf., Freedman (1971), pages 102–106. In specifying the scale and speed for skew Brownian motion, Walsh (1978) seems to have made some minor computational errors, and we have corrected them. Walsh gives several other interesting characterizations of this process, and he calculates the generator and transition density of X_α. (We believe that these calculations are correct with our construction of the process.)

Skew Brownian motion appears as a weak limit in a theorem proved by Rosenkrantz (1975) and by Portenko (1976), although these authors do not attach any name to the process. Note that Rosenkrantz (1975) uses the terminological system of Mandl (1968), rather than the Freedman (1971) system employed here. (The scale and speed measures are defined differently in the two systems.)

3. The stochastic equation. Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\{\mathscr{F}_t, t \geq 0\}$ an increasing family of sub-σ-fields. Let $W = \{W(t), t \geq 0\}$ be a standard Brownian motion adapted to $\{\mathscr{F}_t\}$ and such that $W(t+u) - W(t)$ is independent of $\mathscr{F}_t$ if $0 \leq t \leq t + u < \infty$. (This is the usual set-up for the study of stochastic differential equations.) For concreteness, assume $W(0) = 0$. In this section we prove existence and uniqueness results for the stochastic equation (1)–(2).

Suppose first that X is adapted to $\{\mathscr{F}_t\}$ and satisfies (1)–(2) for a constant β with $|\beta| < 1$. Let $\alpha = (\beta + 1)/2$ and define the strictly increasing and continuous function $s_\alpha(\cdot)$ by (9). Next let

$$f(x) = \begin{cases} (1-\alpha) & \text{if } x > 0 \\ 1/2 & \text{if } x = 0, \\ \alpha & \text{if } x < 0 \end{cases} \tag{10}$$

so that $f(\cdot)$ is the average of the left and right derivatives of $s_\alpha(\cdot)$. Observe that $s_\alpha(\cdot)$ is twice continuously differentiable (with second derivative equal to zero) except possibly at zero, where its first derivative jumps by $\gamma = (1 - 2\alpha)$. Let $Y(t) = s_\alpha(X(t))$ for $t \geq 0$, and note that $Y(0) = X(0) = 0$. From (1) and (2) we have that X is a semimartingale, since W is a martingale and l_0^X is adapted and nondecreasing. Directly applying the generalized Itô formula that appears as Theorem 5.52 of Jacod (1979), we then have that Y is a semimartingale with differential

$$\begin{aligned} dY(t) &= f(X(t))\, dX(t) + \frac{1}{2}\gamma\, dl_0^X(t) \\ &= f(X(t))(dW(t) + \beta\, dl_0^X(t)) + \frac{1}{2}\gamma\, dl_0^X(t) \\ &= f(Y(t))\, dW(t) + f(0)\beta\, dl_0^X(t) + \frac{1}{2}\gamma\, dl_0^X(t) \\ &= f(Y(t))\, dW(t). \end{aligned} \tag{11}$$

(In writing the third equality of (11), we have used the fact that $l_0^X(\cdot)$ increases only when $X(\cdot) = 0$.) Moreover, if Y is any process adapted to $\{\mathscr{F}_t\}$ and satisfying (11) plus $Y(0) = 0$, one can apply the generalized Itô formula to conclude that the process X defined by $X(t) = r_\alpha(Y(t))$, $t \geq 0$, satisfies (1)–(2). Thus we have that X satisfies (1)–(2) if and only if $Y = r_\alpha(X)$ satisfies

$$Y(t) = \int_0^t f(Y(u))\, dW(u), \qquad t \geq 0. \tag{12}$$

Now the Theorem of Nakao (1972) says that this stochastic differential equation (12), with its strictly positive and discontinuous coefficient function f, has a unique solution Y, and that solution is adapted to the fields $\{\mathscr{F}_t^W\}$ of the Wiener process. (In the usual language of stochastic differential equations, (12) has a strong solution, and there are no additional weak solutions.) Combining this with our earlier observations, it follows that (1)–(2) has a unique solution X, and this solution is adapted to $\{\mathscr{F}_t^W\}$ as well.

Finally, if Y is the solution of (12), any of several standard arguments can be used to show that Y is a diffusion in natural scale with infinitesimal variance function $\sigma_\alpha^2(\cdot)$. One can, for example, use the theorem on page 112 of Gihman and Skorokhod (1972) to show that Y satisfies (4) and (5) with another standard Wiener process W^* in place of W. Thus Y has the same distribution as the process Y_α discussed in Section 2, implying that the unique solution X of (1)–(2) is distributed as the skew Brownian motion X_α.

Next suppose that X is adapted to $\{\mathscr{F}_t\}$ and satisfies (1)–(2) for $\beta = 1$. (The case $\beta = -1$ is treated in a symmetric fashion.) Set $\alpha = 1$ and define

$$s(x) = \begin{cases} x & \text{if } x \leq 0 \\ 0 & \text{if } x > 0, \end{cases} \tag{13}$$

$$f(s) = \begin{cases} 1 & \text{if } x < 0 \\ ½ & \text{if } x = 0 \\ 0 & \text{if } x > 0. \end{cases} \tag{14}$$

Note that these definitions extend (9) and (10) to the case $\alpha = 1$. Let $Y(t) = s(X(t))$ for t

≥ 0. Exactly as in (11), we now apply the generalized Itô formula to obtain

$$Y(t) = \int_0^t f(X(t))\, dW(t), \qquad t \geq 0. \tag{15}$$

From (13) we see that $Y(\cdot)$ is nonpositive and continuous with $Y(0) = 0$, while (15) says that Y is a martingale. Thus $Y(\cdot) \equiv 0$ almost surely, meaning that

$$X(t) \geq 0 \quad \text{for all} \quad t \geq 0 \quad \text{almost surely.} \tag{16}$$

Now let $M(t) = -\inf\{W(u): 0 \leq u \leq t\}$ for $t \geq 0$. From (1), (2) and (16) it follows that

$$l_0^X(t) = M(t), \qquad t \geq 0. \tag{17}$$

Here is the argument. From (1) and (16) we have $l_0^X(t) \geq M(t)$ for all t. If strict equality holds for some t, then it must hold for some t that is a point of increase for $l_0^X(\cdot)$. But this means that $l_0^X(\cdot)$ increases at a time t where $X(t) > 0$, contradicting (2).

It is well known that the process $X(t) = W(t) + M(t)$ satisfies (1)–(2), meaning that $l_0^X = M$ when $X = W + M$, and we have shown above that it is the only possible solution. Finally, it is well known that $X = W + M$ is one representation for a reflecting Wiener process on the upper half-line, and this is how one defines the skew Brownian motion X_α when $\alpha = 1$.

For the negative result, suppose that X is adapted to $\{\mathscr{F}_t\}$ and satisfies (1)–(2) with $\beta > 1$. (The case $\beta < -1$ is treated in a symmetric fashion.) Let

$$s(x) = \begin{cases} (1-\beta)x & \text{if} \quad x < 0 \\ (1+\beta)x & \text{if} \quad x > 0 \end{cases} \tag{18}$$

and

$$f(x) = \begin{cases} 1-\beta & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x = 0 \\ 1+\beta & \text{if} \quad x > 0. \end{cases} \tag{19}$$

Let $Y(t) = s(X(t))$ for $t \geq 0$. Applying the generalized Itô formula as in (11), we again have $dY(t) = f(X(t))\, dW(t)$, and hence Y is a martingale. But Y is nonpositive by (18) and $Y(0) = X(0) = 0$, so it must be that $Y(\cdot) \equiv 0$ almost surely. This implies $X(\cdot) \equiv 0$, which contradicts (1)–(2). We conclude that there can exist no solution X of (1)–(2) with $|\beta| > 1$.

4. A random walk result. Let $\{S_0, S_1, \ldots\}$ be a Markov chain on the integers with $S_0 = 0$ and transition probabilities

$$P\{S_{k+1} = S_k + 1 \mid S_0, \ldots, S_k\} = \begin{cases} 1-\alpha & \text{if} \quad S_k = 0 \\ 1/2 & \text{otherwise,} \end{cases}$$

$$P\{S_{k+1} = S_k - 1 \mid S_0, \ldots, S_k\} = \begin{cases} \alpha & \text{if} \quad S_k = 0 \\ 1/2 & \text{otherwise.} \end{cases}$$

Fix $t > 0$. We seek to show that

$$n^{-1/2} S_{[nt]} \to_{\mathscr{D}} X_\alpha(t) \quad \text{as} \quad n \to \infty, \tag{20}$$

for which it suffices to show

$$E\{\exp(i\rho n^{-1/2} S_{[nt]})\} \to E\{\exp(i\rho X_\alpha(t))\} \tag{21}$$

as $n \to \infty$ for all real ρ. This can be done without calculation in the following way. It is easy to verify that

$$P(S_k = m) = \begin{cases} \alpha P(|S_k| = m), & \text{if} \quad m > 0 \\ (1-\alpha) P(|S_k| = |m|), & \text{if} \quad m < 0 \end{cases}$$

which implies that

$$E\{\exp(i\rho n^{-1/2}S_{[nt]})\} = \alpha\psi_n(\rho) + (1-\alpha)\psi_n(-\rho), \tag{22}$$

where

$$\psi_n(\rho) = E\{\exp(i\rho n^{-1/2}\,|\,S_{[nt]}\,|)\}. \tag{23}$$

From the explicit results of Walsh (1978), (3)) we obtain the analogous formula

$$E\{\exp(i\rho X_\alpha(t))\} = \alpha\psi(\rho) + (1-\alpha)\psi(-\rho), \tag{24}$$

where

$$\psi(\rho) = E\{\exp(i\rho\,|\,X_\alpha(t)\,|)\}. \tag{25}$$

(This is of course obvious from the Itô-McKean construction of X_α.) Walsh (1978) shows that $|X_\alpha(t)|$ is a reflected Wiener process on $[0, \infty)$. Similarly, $|S_n|$ is an ordinary reflected walk on the nonnegative integers. Thus using standard results on random walks, we have that $n^{-1/2}|S_{[nt]}|$ converges weakly to $|X_\alpha(t)|$, which implies

$$\psi_n(\rho) \to \psi(\rho) \quad \text{as} \quad n \to \infty. \tag{26}$$

Combining (22)–(26), we have (21) and hence (20). This argument can easily be extended to show convergence of all finite-dimensional distributions (not just one-dimensional distributions), and weak convergence in the function space $D[0, \infty)$ can then be proved using the moment criterion for tightness on page 128 of Billingsley (1968).

The result (20) can also be extended to a Markov chain $\{S_n\}$ having a more general type of behavior at the origin. Suppose that, starting from zero, the chain $\{S_n\}$ moves to a new state distributed as an integer-valued random variable Z. It can be shown that (20) holds with $\alpha = EZ^+/E|Z|$.

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Graduate School of Business
Stanford University
Stanford, California 94305

Bell Laboratories
600 Mountain Avenue
Murray Hill, New Jersey 07974