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ON SMOOTH GORENSTEIN POLYTOPES

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Abstract. A Gorenstein polytope of index r is a lattice polytope whose rth dilate is a reflexive polytope. These objects are of interest in combinatorial commutative algebra and enumerative combinatorics, and play a crucial role in Batyrev's and Borisov's computation of Hodge numbers of mirror-symmetric generic Calabi-Yau complete intersections. In this paper we report on what is known about smooth Gorenstein polytopes, i.e., Gorenstein polytopes whose normal fan is unimodular. We classify d-dimensional smooth Gorenstein polytopes with index larger than $(d+3)/3$. Moreover, we use a modification of Øbro's algorithm to achieve classification results for smooth Gorenstein polytopes in low dimensions. The first application of these results is a database of all toric Fano d-folds whose anticanonical divisor is divisible by an integer r satisfying $r \geq d-7$. As a second application we verify that there are only finitely many families of Calabi-Yau complete intersections of fixed dimension that are associated to a smooth Gorenstein polytope via the Batyrev-Borisov construction.

Organization of the paper. This paper is concerned with smooth Gorenstein polytopes and their associated toric varieties. Our main motivation is their relevance in the combinatorial mirror symmetry construction due to Batyrev and Borisov.

The paper is organized as follows. Section 1 introduces smooth Gorenstein polytopes and our main computational classification result (Table 1.2). In Section 2, we present the corresponding classification results for toric Fano manifolds. Section 3 gives the combinatorial proof of the classification of smooth Gorenstein polytopes with large index (Theorem 3.2). Finally, Section 4 explains our original motivation and main application: the finiteness of generic n-dimensional Calabi-Yau complete intersections associated to smooth Gorenstein polytopes (Corollary 4.2). Figures 1 and 2 give the complete list of their stringy Hodge numbers for $n = 3$.

1. Introduction to smooth Gorenstein polytopes.

1.1. The combinatorial setting. Let us start by introducing the notions of reflexive and Gorenstein polytopes. We refer the reader to [12] and [51] for more details and the algebro-geometric background. Let us fix a dual pair of lattices M and N. We define $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and its dual vectorspace $N_{\mathbb{R}}$ analogously. A *lattice polytope* $P \subset M_{\mathbb{R}}$ is

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the convex hull of finitely many lattice points (elements in the lattice M). If such a lattice polytope contains the origin in its interior, then the *dual polytope* P^* is defined as

$$P^* = \{y \in N_{\mathbb{R}} : \langle y, x \rangle \geq -1 \text{ for all } x \in P\}.$$

We remark that P^* does not have to be a lattice polytope anymore. A *reflexive polytope* $P \subseteq M_{\mathbb{R}}$ is a lattice polytope containing the origin in its interior such that P^* is also a lattice polytope. A lattice polytope P is *Gorenstein of index* r, if rP is a reflexive polytope up to translation by a lattice point. This index is uniquely determined. In other words, Gorenstein polytopes of index r are in one-to-one correspondence to reflexive polytopes that are 'divisible' by r (i.e., if v is a vertex of such a reflexive polytope $\tilde{P}$, then $(\tilde{P} - v)/r$ is a lattice polytope). Two lattice polytopes are considered *isomorphic* (or *unimodularly equivalent*), if there is a lattice automorphism mapping their vertex sets onto each other. In each dimension there exist only finitely many reflexive polytopes up to isomorphisms. They are known up to dimension four by massive computer calculations by Kreuzer and Skarke [44, 45]. Recently, Skarke described a procedure how to possibly extend their algorithm to Gorenstein polytopes [59].

1.2. Classification results. A polytope is called d-polytope if its affine span is a d-dimensional affine subspace. Such a polytope is called *simple*, if each vertex is contained in precisely d edges. A d-dimensional lattice polytope P is called *smooth*, if P is simple and at each vertex v the primitive edge directions $v_1 - v, \ldots, v_d - v$ form a lattice basis. Equivalently, the associated normal fan is unimodular, or its associated toric variety is nonsingular. We refer to [26] for standard results in toric geometry.

Øbro described in [54] an algorithm which he used to classify all smooth reflexive polytopes for $d \leq 8$. With an improved implementation, this was extended to $d = 9$ by Andreas Paffenholz and the first author, see [47]. In this paper we apply a modified version of this algorithm to compute high-dimensional smooth reflexive polytopes that are highly divisible (for more details see Section 3.5). The outcome of these computations is summarized in Table 1.2. The database of these polytopes can be found online [46].

The reader might have noticed certain regularities in the table, if the index r is large. For instance, there is only one smooth Gorenstein polytope (the unimodular simplex, see Section 3.1) that satisfies $r > \lceil \frac{d+1}{2} \rceil$. These observations can be easily explained and follow from some well-known results in toric geometry. We refer to Theorem 3.2 in Section 3.1 for a description of all smooth Gorenstein polytopes of index $r > \frac{d+3}{3}$.

1.3. Background. In this paper we report on what is known for smooth Gorenstein polytopes of either 'small' dimension or 'large' index. Our goal is to provide the community with a database of interesting examples. Gorenstein polytopes are of relevance in combinatorial commutative algebra, enumerative combinatorics and Ehrhart theory. They can be algebraically characterized by their semigroup algebras being Gorenstein [48], and combinatorially by their Ehrhart h^*-polynomials being symmetric [36]. They have gained increased interest [20, 55, 57, 25, 56, 12, 13], initiated by the proof of Stanley's conjecture on the unimodality of the coefficients of the Ehrhart h^*-polynomial of the Birkhoff polytope [3]. It

d \ r	13	12	11	10	9	8	7	6	5	4	3	2	1	
20	0	0	1	2	**5**	11								
19	0	0	0	2	3	7								
18	0	0	0	1	2	**5**								
17	0	0	0	0	2	3	7							
16	0	0	0	0	1	2	**5**							
15	0	0	0	0	0	2	3							
14	0	0	0	0	0	1	2	**5**						
13	0	0	0	0	0	0	2	3						
12	1	0	0	0	0	0	1	2	**6**	27				
11		1	0	0	0	0	0	2	3	14	154			
10			1	0	0	0	0	1	2	**6**	64	3273		
9				1	0	0	0	0	2	4	23	896	8229721	
8					1	0	0	0	1	2	**13**	258	749892	
7						1	0	0	0	2	4	85	72256	
6							1	0	0	1	3	**28**	7622	
5								1	0	0	2	12	866	
4									1	0	1	4	**124**	**n = 3**
3										1	0	3	18	
2											1	1	5	
1												1	1	
0													1	

TABLE 1. Number of isomorphism classes of smooth Gorenstein polytopes of dimension d and index r.

remains an open question, whether unimodality holds for any integrally closed Gorenstein polytope [50], cf. Definition 4.6. So far, all smooth Gorenstein polytopes we checked confirmed this conjecture.

2. Application to toric Fano manifolds. A *Gorenstein Fano variety* X is a d-dimensional projective complex variety such that its anticanonical divisor $-K_X$ is an ample Cartier divisor. The *index* i_X of X is defined as the largest positive integer r such that there exists some Cartier divisor D with $-K_X = rD$. Fano varieties play an essential role as 'building blocks' in the Minimal Model Program. Of special importance are *Fano manifolds*, i.e., nonsingular (Gorenstein) Fano varieties. They are completely known up to dimension 3. In higher dimension, much work has been done to classify all Fano manifolds with large index. If X is smooth, then $i_X \leq d+1$ with equality only for $\mathbb{P}^d$. By now, Fano manifolds with $i_X \geq d-2$ or $i_X \geq \frac{d+1}{2}$ are completely known. We refer for these results to [61] and Section 2 in [1] with the references therein.

In the toric situation more can be shown. We refer to [26, Section 8.3], and [39] for surveys on toric Fano varieties. Isomorphism classes of toric Fano manifolds X correspond bijectively to isomorphism classes of smooth reflexive polytopes P. Here, the index i_X corresponds to the maximal r for which P is 'divisible' by r (as defined in Section 1.1). In particular, any such X is given by a smooth Gorenstein polytope of index i_X. However, note that several smooth Gorenstein polytopes may define the same toric Fano variety (e.g., the reflexive polytope associated to $\mathbb{P}^3$ is divisible by 1, 2 and 4). In other words, the reader should not confuse the index i_X of X with the index r of a Gorenstein polytope.

From the classification of smooth Gorenstein polytopes in Section 3.1 we can deduce the following observation.

PROPOSITION 2.1. *Let X be a toric Fano d-fold. Then $i_X > \frac{d+3}{3}$ if and only if $X \cong \mathbb{P}^d$, or $X \cong \mathbb{P}^{\frac{d}{2}} \times \mathbb{P}^{\frac{d}{2}}$ (for $d \geq 4$ even), or*

$$X \cong \mathbb{P}_{\mathbb{P}^{d+1-r}}(\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_t) \oplus \mathcal{O}^{r-t}),$$

where $a_1, \ldots, a_t$ is some integer partition of $d + 2 - 2r$ for $\frac{d+3}{3} < r \leq \frac{d+1}{2}$. In the last case, $i_X = r$.

Here, an *integer partition* of N is a multiset of nonnegative integers summing up to N. Let us remark that the previous result seems to be folklore to experts in toric geometry, even if we couldn't find a reference in the literature.

By now, toric Fano manifolds are known up to dimension 9 [54, 17, 47] extending previous classifications [60, 4, 6, 58, 42]. As an application of our algorithmic results (Table 1.2) combined with the previous proposition we can determine all non-isomorphic toric Fano manifolds with $i_X \geq d - 7$.

COROLLARY 2.2. *Let X be a toric Fano d-fold with index i_X.*

(1) *Let $i_X = d$* (cf. [41]). *Then $X \cong \mathbb{P}^1 \times \mathbb{P}^1$.*

(2) (Toric del Pezzo manifolds, cf. [31, 32, 11]) *Let $i_X = d - 1$. Then*

d	2	3	4
X	$\# = 3$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}(1) \oplus \mathcal{O})$	$\mathbb{P}^2 \times \mathbb{P}^2$

(3) (Toric Mukai manifolds, cf. [49, 37]) *Let $i_X = d - 2$. Then*

d	3	4	5	6
X	$\# = 15$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}_{\mathbb{P}^3}(\mathcal{O}(2) \oplus \mathcal{O})$, $\mathbb{P}^1 \times \mathbb{P}^3, \mathbb{P}^1 \times \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}(1) \oplus \mathcal{O})$	$\mathbb{P}_{\mathbb{P}^3}(\mathcal{O}(1) \oplus \mathcal{O}^2)$	$\mathbb{P}^3 \times \mathbb{P}^3$

(4) *Let $i_X = d - 3$. Then*

d	4	5	6	7	8
X	$\# = 118$	$\# = 11$	$\# = 3$	$\mathbb{P}_{\mathbb{P}^4}(\mathcal{O}(1) \oplus \mathcal{O}^3)$	$\mathbb{P}^4 \times \mathbb{P}^4$

(5) *Let $i_X = d - 4$. Then*

d	5	6	7	8	9	10
X	$\# = 853$	$\# = 27$	$\# = 4$	$\# = 2$	$\mathbb{P}_{\mathbb{P}^5}(\mathcal{O}(1) \oplus \mathcal{O}^4)$	$\mathbb{P}^5 \times \mathbb{P}^5$

(6) *Let $i_X = d - 5$. Then*

d	6	7	8	9	10	11	12
no. of X	7590	83	12	4	2	1	1

(7) *Let $i_X = d - 6$. Then*

d	7	8	9	10	11	12	13	14
no. of X	72167	256	23	6	3	2	1	1

(8) *Let* $i_X = d - 7$. *Then*

d	8	9	10	11	12	13	14	15	16
no. of X	749620	891	63	13	6	3	2	1	1

The database of the associated smooth Gorenstein polytopes can be found online [46].

3. Classification of smooth Gorenstein polytopes of large index.

3.1. Notation and description of results. For convenience, we will denote the lattice by $\mathbb{Z}^d$, as long as it is not important to differentiate between M and N.

For a positive integer d we define the *unimodular d-simplex* as

$$S_d := \operatorname{conv}(0, e_1, \dots, e_d)\,,$$

where $e_1, \dots, e_d$ is the standard lattice basis.

DEFINITION 3.1. Let $P_0, \dots, P_k \subset \mathbb{R}^s$ be lattice polytopes. Then we define the *Cayley polytope* of $P_0, \dots, P_k$ as

$$P_0 * \cdots * P_k := \operatorname{conv}(P_0 \times e_0, \dots, P_k \times e_k) \subseteq \mathbb{R}^s \oplus \mathbb{R}^{k+1}\,,$$

where $e_0, \dots, e_k$ is a lattice basis of $\mathbb{R}^{k+1}$. Here, $P_0, \dots, P_k$ are called *Cayley factors*. Note that if the dimension of the affine span $\operatorname{aff}(P_0, \dots, P_k)$ equals s, then the Cayley polytope is a lattice polytope of dimension $s + k$.

Cayley polytopes appear naturally when studying lattice polytopes that have a large 'codegree' and in the adjunction theory of polarized toric manifolds, we refer to [35, 28, 29, 30, 2]. Now, we can formulate the following result.

THEOREM 3.2. *Let P be a smooth Gorenstein polytope of dimension d and index r. Then $r > \frac{d+3}{3}$ if and only if*

(1) $P \cong S_d$ *(here,* $r = d + 1$*),*
(2) $P \cong 2S_d$ *with* $d \geq 5$ *odd (here,* $r = \frac{d+1}{2}$*),*
(3) $P \cong S_{\frac{d}{2}} \times S_{\frac{d}{2}}$ *with* $d \geq 4$ *even (here,* $r = \frac{d+2}{2}$*),*
(4) $P \cong (a_1 + 1)S_{d+1-r} * \cdots * (a_t + 1)S_{d+1-r} * S_{d+1-r} * \cdots * S_{d+1-r}$, *where there are* $\frac{d+3}{3} < r \leq \frac{d+1}{2}$ *Cayley factors, and* $a_1, \dots, a_t$ *is an integer partition of* $d + 2 - 2r$.

Different integer partitions in (4) *yield non-isomorphic Gorenstein polytopes*

We will give a combinatorial proof of Theorem 3.2 in the remainder of this section.

While strictly speaking not necessary, we will often express bounds in terms of the so-called *Calabi-Yau dimension* $n := d + 1 - 2r$, since this is closer to the point of view of Section 4. Note that

$$r > \frac{d+3}{3} \iff d > 3n + 3\,.$$

REMARK 3.3. For the algebro-geometric reader, this is how we will proceed. It follows from Mukai's conjecture [14, 22, 53] that large index implies Picard number $\rho \leq 2$. In the toric situation Kleinschmidt [40] showed that these Fano manifolds have to be projective toric bundles. It remains to use the assumption on the index to specify them and their associated reflexive polytopes. This can be done using numerical criteria. The proof presented here follows along these lines. It is slightly longer, but more elementary and combinatorial.

3.2. Bounding the number of vertices. We denote by $\mathcal{V}(Q)$ the vertex set of a polytope Q. Let us consider a d-dimensional *simplicial* reflexive polytope $Q \subset N_{\mathbb{R}}$ (i.e., dual to a simple reflexive polytope). Casagrande defined in [22] the number

$$\delta_Q := \min\{\langle v, u\rangle \ : \ v \in \mathcal{V}(Q),\ u \in \mathcal{V}(Q^*),\ v \notin F_u\} \in \mathbb{Z}_{\geq 0}\,,$$

where F_u is the facet of the polytope Q corresponding to the vertex u of Q^*. Here $\langle \cdot, \cdot \rangle$ denotes the inner pairing of N and M.

Casagrande showed the following result (Theorem 3(ii) in [22]). For readers with a background in algebraic geometry, this proves the validity of Mukai's conjecture $\rho_X(i_X - 1) \leq d$ (where ρ_X is the Picard number) for $\mathbb{Q}$-factorial Gorenstein toric Fano varieties X.

THEOREM 3.4 (Casagrande '06). *If $\delta_Q > 0$, then*

$$|\mathcal{V}(Q)| \leq d + \frac{d}{\delta_Q}\,.$$

The following observation is also well-known (Lemma 2 in [22]):

LEMMA 3.5. *Let P be a simple Gorenstein polytope of index r. Then $(rP)^*$ is a simplicial reflexive polytope with*

$$\delta_{(rP)^*} \geq r - 1\,.$$

From these results it is straightforward to deduce an upper bound on the number of vertices of a simple Gorenstein polytope of small Calabi-Yau dimension.

PROPOSITION 3.6. *Let P be a simple Gorenstein polytope of Calabi-Yau dimension n and of dimension $d > 3n + 3$. Then $(rP)^*$ is a simplex or has $d + 2$ vertices.*

PROOF. If $r = 1$, then $n = d - 1$, so $3n + 3 = 3d > d$, a contradiction. Hence, $r > 1$. Let us consider the simplicial reflexive polytope $Q := (rP)^*$. Then Theorem 3.4 and Lemma 3.5 imply

$$|\mathcal{V}(Q)| \leq d + \frac{d}{r-1} = d + \frac{2d}{d-1-n}\,.$$

The statement now follows from

$$\frac{2d}{d-1-n} < 3 \iff 3n + 3 < d\,.$$

□

REMARK 3.7. Let us note that the bound is sharp: for any $n \geq 0$, the smooth Gorenstein polytope $P = S_{n+1} \times S_{n+1} \times S_{n+1}$ has dimension $d = 3n + 3$, Calabi-Yau dimension n and $d + 3$ facets.

3.3. Proof of Theorem 3.2. By Proposition 3.6 we only have to consider two cases in order to prove Theorem 3.2: either P is a simplex or it has $d + 2$ facets.

PROPOSITION 3.8. *Let P be a smooth Gorenstein simplex of Calabi-Yau dimension n and of dimension $d \geq 3n$. Then either $n < 0$ and $P \cong S_d$, or $n = 0$ and $P \cong 2S_d$ (with odd d).*

PROOF. In this case, $X_P \cong \mathbb{P}^d$, and $P \cong kS_d$ for some $k \in \mathbb{Z}_{\geq 1}$. Since $rP \cong (d+1)S_d$, we get $rk = d+1$. Therefore, $\frac{d+1-n}{2}k = d+1$. For $n \leq -1$, we get $k < 2$, so $k = 1$. For $n = 0$, we have $k = 2$, as desired. Finally, let $n > 0$, so $k \geq 3$. In particular, $\frac{k}{k-2} \leq 3$. This implies

$$d + 1 = \frac{k}{k-2} n \leq 3n\,,$$

a contradiction. □

The following statement can be deduced from [12, Theorem 2.6]. Recall that S_d is a smooth Gorenstein polytope of index $d+1$.

LEMMA 3.9. *If $b_1, \ldots, b_r \in \mathbb{Z}_{>0}$, $\sum_{i=1}^{r} b_i = d + 2 - r$, and $r \in \{1, \ldots, d+1\}$, then*

$$P = b_1 S_{d+1-r} * \cdots * b_r S_{d+1-r}\,,$$

is a smooth d-dimensional Gorenstein polytope of index r such that $(rP)^*$ has $d+2$ vertices.

In our situation, the converse also holds.

PROPOSITION 3.10. *Let P be a smooth Gorenstein polytope of dimension $d \geq 3n+3$ such that $(rP)^*$ has $d+2$ vertices. Then $n \geq -1$ and there exists a unique integer partition $a_1, \ldots, a_t$ of $n+1$ such that*

$$P \cong (a_1+1)S_{d+1-r} * \cdots * (a_t+1)S_{d+1-r} * S_{d+1-r} * \cdots * S_{d+1-r}\,,$$

where there are r Cayley factors.

Recall that we consider integer partitions as multisets (so uniqueness is up to permutation).

PROOF. By Kleinschmidt's classification of nonsingular toric Fano varieties whose associated fan has $d+2$ rays [40] (see also Section 7.3 in [26]), we can assume that the vertices of the reflexive polytope $(rP)^*$ are of the following form:

$$e_1, \ldots, e_d\,, \quad v_1 = -e_1 - \cdots - e_k\,, \quad v_2 = a_1 e_1 + \cdots + a_k e_k - e_{k+1} - \cdots - e_d,$$

where $1 \leq k \leq d-1$, and $a_i \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq k$ such that $m := \sum_{i=1}^{k} a_i \leq d-k$.

Combinatorially, $(rP)^*$ is a free sum (i.e., dual to the product) of a k-dimensional simplex with vertices $e_1, \ldots, e_k, v_1$ and a $(d-k)$-dimensional simplex with vertices $e_{k+1}, \ldots, e_d, v_2$. This implies that the vertices of rP are of the following types, where all entries not specified are equal to -1 (here $i \in \{1, \ldots, k\}$, $j \in \{k+1, \ldots, d\}$):

		i		j			corresponding facet(s) of $(rP)^*$
$(-1,$	$\ldots,$	$-1,$	$\ldots,$	$-1,$	$\ldots,$	$-1)$	not containing v_1 and v_2
$(-1,$	$\ldots,$	$-1,$	$\ldots,$	$d-k-m,$	$\ldots,$	$-1)$	not containing v_1 and e_j
$(-1,$	$\ldots,$	$k,$	$\ldots,$	$-1,$	$\ldots,$	$-1)$	not containing e_i and v_2
$(-1,$	$\ldots,$	$k,$	$\ldots,$	$a_i(k+1)+d-k-m,$	$\ldots,$	$-1)$	not containing e_i and e_j

Let us translate the first vertex v of rP into the origin. This yields the following vertices for the lattice polytope $(rP - v)/r$ (which is isomorphic to P):

		i		j		
(0,	...,	0,	...,	0,	...,	0)
(0,	...,	0,	...,	$\frac{d+2-m-(k+1)}{r}$,	...,	0)
(0,	...,	$\frac{k+1}{r}$,	...,	0,	...,	0)
(0,	...,	$\frac{k+1}{r}$,	...,	$\frac{(a_i-1)(k+1)+d+2-m}{r}$,	...,	0)

Note that r divides $k+1$, and thus $d+2-m$. By our assumption

$$3n+3 \leq d = n+2r-1 \quad \Longrightarrow \quad n+1 \leq r-1\,,$$

so $d+2-m \leq d+2 = n+2r+1 \leq 3r-1$. Hence, $d+2-m$ equals r or $2r$. Furthermore, by our assumption $m \leq d-k$, so $d-m+2 > k+1$, thus,

$$0 < \frac{d+2-m}{r} - \frac{k+1}{r} \leq 1\,,$$

which implies $d+2-m = 2r$ and $k+1 = r$. Therefore, P is isomorphic to the Cayley polytope

$$S_{d+1-r} * (a_1+1)S_{d+1-r} * \cdots * (a_k+1)S_{d+1-r}\,,$$

where $\sum_{i=1}^k a_i = m = n+1$. Moreover, by construction, the polytopes do not depend on the ordering of these numbers, so by choosing $t \leq k$ non-zero a_i's the result follows. Finally, different multisets of coefficients define affinely non-isomorphic Cayley polytopes, since

$$(-a_1+1)e_1 + \cdots + (-a_k+1)e_k + e_{k+1} + \cdots + e_d + v_1 + v_2 = 0$$

is the unique affine relation of the vertices of $(rP)^*$, see also [40]. □

PROOF OF THEOREM 3.2. If $r > \frac{d+3}{3}$, then Proposition 3.8 implies cases (1) and (2), while Proposition 3.10 yields (3) for $n = -1$ (all a_i's are equal to 0), and (4) for $n \geq 0$. The 'if'-statement follows for (4) from Lemma 3.9. □

It is a somewhat lucky coincidence that essentially the same bound appears quite naturally in Propositions 3.6 and 3.10. One should also compare this with the fact [59, Lemma 2] that any so-called basic IP weight system (the building blocks of Gorenstein polytopes in Skarke's classification algorithm) satisfies $d \leq 3n-1$.

3.4. Application to toric Fano manifolds. The proof of Proposition 2.1 follows directly by translating Theorem 3.2 into algebraic geometry.

LEMMA 3.11. *Let $s, l \in \mathbb{Z}_{>0}$, and $c_1, \ldots, c_l \in \mathbb{N}$. Then*

$$S_s * (c_1+1)S_s * \cdots * (c_l+1)S_s \subset M_{\mathbb{R}}$$

defines a toric projective bundle

$$\mathbb{P}_{\mathbb{P}^s}(\mathcal{O} \oplus \mathcal{O}(c_1) \oplus \cdots \oplus \mathcal{O}(c_l))\,.$$

For a proof of this fact see [40] or [26, Section 7.3].

PROOF OF PROPOSITION 2.1. Let X be a toric Fano d-fold with $r := i_X > \frac{d+3}{3}$. Then there exists a smooth Gorenstein polytope $P \subset M_{\mathbb{R}}$ such that the reflexive polytope rP is associated to the anticanonical divisor $-K_X$. Now, we apply Theorem 3.2 and the previous Lemma. The converse statement follows from combining the previous Lemma with Lemma 3.9. □

Let us note that we recover a result of Wiśniewski [61] in this toric setting.

COROLLARY 3.12. *Let X be a toric Fano d-fold with $i_X \geq \frac{d+1}{2}$. Then X is isomorphic to one of the following cases:*

- $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, *$d = 3$, and $i_X = 2$,*
- $\mathbb{P}_{\mathbb{P}^{\frac{d+1}{2}}}(\mathcal{O}(1) \oplus \mathcal{O}^{\frac{d-1}{2}})$, *$d$ is odd, and $i_X = \frac{d+1}{2}$,*
- $\mathbb{P}^{\frac{d}{2}} \times \mathbb{P}^{\frac{d}{2}}$, *$d \geq 4$ is even, and $i_X = \frac{d+2}{2}$,*
- $\mathbb{P}^d$, *and $i_X = d + 1$.*

PROOF. The case $d = 3$ is well-known, it can also be easily checked using the database. Since $d > 3$ is equivalent to $\frac{d+1}{2} > \frac{d+3}{3}$, the statements follow immediately from Proposition 2.1. □

3.5. The classification algorithm for smooth Gorenstein polytopes of given index.

Let us briefly describe the modified version of the algorithm to classify smooth Fano polytopes by Mikkel Øbro [54]. The key ingredient for his algorithm is the notion of a special facet.

DEFINITION 3.13. Let P be a reflexive polytope, a facet F of P is called *special*, if the sum of all vertices $v_1, \dots, v_k$ of P is a point within the cone over the facet F, i.e., $\sum_{i=1}^k v_i \in \text{cone}(F)$.

Clearly, every reflexive polytope has at least one special facet. Now, let P be a smooth reflexive d-polytope so that P is divisible by r, as in Section 1.1. Then the dual polytope P^* is a simplicial reflexive polytope where the vertices of each facet form a lattice basis of $\mathbb{Z}^d$. Let us denote here such reflexive polytopes as *dual-smooth*. By applying a unimodular transformation, we can assume that $\text{conv}(e_1, \dots, e_d)$ is a special facet of P^*. Øbro has shown that all remaining vertices of such a dual-smooth reflexive polytope must lie within some explicit finite set. The original algorithm enumerates this set in a clever way to efficiently generate all dual-smooth reflexive polytopes. Note that the vector $-\mathbf{1} = (-1, \dots, -1)$ defines the special facet used above. Hence, by Lemma 3.5, we know that all vertices $v \in \mathcal{V}(P^*)$ that are not contained in this facet must evaluate to $\langle v, -\mathbf{1} \rangle \geq r - 1$. Using this property we can further restrict the set of possible vertices. By adding this condition to the original algorithm we were able to generate a superset of all simplicial reflexive polytopes whose dual polytopes are r-multiples of smooth Gorenstein polytopes for given dimension d and index r larger than some r_0. For $d = 10$ and $r \geq 3$ this algorithm took about 1 hour. This should be compared to two weeks for the original (parallelized) algorithm without additional conditions for $d = 9$. Table 1.2 contains all isomorphism classes for values of d and r that we could compute so far.

4. Smooth Gorenstein polytopes in combinatorial mirror symmetry.

4.1. The Batyrev-Borisov construction. Over the last three decades mirror symmetry has spurred the interest in finding all possible Hodge numbers of Calabi-Yau n-folds, in particular, for $n = 3$. The 'Hodge diamond' of a Calabi-Yau threefold is completely described by the pair of Hodge numbers $(h^{1,1}, h^{1,2})$. So far ten thousands of these pairs have been found [38, 21], all of them in the range of $h^{1,1} + h^{1,2} \leq 502$. This can be regarded as striking evidence for an affirmative answer to the following question attributed to Yau: are there only finitely many families of irreducible Calabi-Yau n-folds? For related results in these directions we refer to [34, 27, 24, 23, 59].

The vast amount of examples of Calabi-Yau manifolds are so-called *CICY's*: (resolutions of) generic complete intersection Calabi-Yau varieties in Gorenstein toric Fano varieties. Here, reflexive polytopes play a key role and were introduced for this purpose by Batyrev [5]. To avoid confusion we adopt the notation in [5]. We will omit technical details, the interested reader is invited to look at the survey paper [12].

An s-dimensional reflexive polytope $\Delta \subset M_{\mathbb{R}}$ defines a Gorenstein toric Fano variety X given by the fan over the faces of Δ^*. A generic anticanonical hypersurface Y in X is a (possibly singular) Calabi-Yau variety of dimension $s - 1$. For $s \leq 4$ it can be (crepantly) resolved by a Calabi-Yau manifold $\widehat{Y}$. Exploiting the duality of reflexive polytopes, Batyrev showed that for $s = 4$ the Calabi-Yau 3-folds $\widehat{Y}$, $\widehat{Y^*}$ constructed by Δ and Δ^* in this way have mirror-symmetric Hodge numbers: $h^{1,1}(\widehat{Y}) = h^{1,2}(\widehat{Y^*})$. For $s > 4$, the possibly singular Y may not be resolvable in this way. Therefore, one considers *stringy Hodge numbers* $h^{p,q}_{\mathrm{st}}(Y)$, see [10, 7]. In the case that Y can be crepantly resolved by a Calabi-Yau manifold $\widehat{Y}$, the stringy Hodge numbers of Y equal the usual Hodge numbers of $\widehat{Y}$.

Batyrev and Borisov generalized Batyrev's results to complete intersections [15, 7, 9] using the framework of Gorenstein polytopes. A generic CICY Y of dimension $n = s - r$ in an s-dimensional Gorenstein toric Fano variety X is given by a Minkowski decomposition of the s-dimensional reflexive polytope $\Delta = \Delta^{(1)} + \cdots + \Delta^{(r)}$ into r lattice polytopes $\Delta^{(1)}, \ldots, \Delta^{(r)}$. (Here, we use the notation $\Delta^{(i)}$ for these lattice polytopes, since Δ_i is sometimes used to denote the unimodular i-simplex.) More precisely, Y is defined as the compactification in X of the intersection of the hypersurfaces given by the generic Laurent polynomials

$$F_{\Delta^{(i)}}(z) := \sum_{m \in \Delta^{(i)} \cap M} c_m z^m \quad (\text{for } c_m \in \mathbb{C}^*),$$

for $i = 1, \ldots, r$. In this combinatorially described setting, one defines the Cayley polytope

$$P := \Delta^{(1)} * \cdots * \Delta^{(r)},$$

which is a Gorenstein polytope of dimension $d := s + r - 1$ and of index r. Note that P has Calabi-Yau dimension $d + 1 - 2r = n$. Using such a combinatorial datum, Batyrev and Borisov showed (in the notation of [12]) that the *stringy E-polynomial* of Y

$$E_{\mathrm{st}}(Y) := \sum_{p,q} (-1)^{p+q} h^{p,q}_{\mathrm{st}}(Y)\, u^p v^q \tag{1}$$

equals $E_{\text{st}}(P)$, a rather complicated combinatorial expression called the *stringy E-polynomial* of P (as defined in [12] based on [16], see also [52]).

It is important to remark that not every Gorenstein polytope is given by such a Minkowski decomposition, still, $E_{\text{st}}(P)$ is always of the form (1), so stringy Hodge numbers of Gorenstein polytopes are well-defined, see [52]. In particular, this motivates why it makes sense to define the Calabi-Yau dimension of *any* Gorenstein polytope P of dimension d and index r to be $n := d + 1 - 2r$. Gorenstein polytopes also satisfy a beautiful duality, and under additional hypotheses (the existence of a so-called *nef-partition*) it is possible to show that a CICY has an analogously constructed mirror partner (on the level of stringy Hodge numbers), for more on this see [15, 8, 7, 9, 12].

4.2. The main result. After these preparations we can state the result which originally motivated our investigations. (We remark that $E_{\text{st}}(P) = 0$, if $n < 0$, see [52].)

THEOREM 4.1. *Let P be a smooth Gorenstein polytope of Calabi-Yau dimension $n \geq 0$ and of dimension $d > 3n + 3$. Then $E_{\text{st}}(P) = E_{\text{st}}(P')$ for P' a smooth Gorenstein polytope of Calabi-Yau dimension n and of dimension at most $3n+1$. More precisely, $E_{\text{st}}(P)$ equals the E-polynomial of an n-dimensional Calabi-Yau manifold Y given as the complete intersection in projective space $\mathbb{P}^{\tilde{s}}$ of generic hypersurfaces of degrees $d_1, \ldots, d_{\tilde{r}} \in \mathbb{Z}_{\geq 2}$ with $d_1 + \cdots + d_{\tilde{r}} = \tilde{s} + 1$, where $\tilde{s} \leq 2n + 1$. Any n-dimensional generic Calabi-Yau complete intersection associated to the Gorenstein polytope P (in the sense of Batyrev-Borisov) is isomorphic to Y.*

Since there are only finitely many Gorenstein polytopes in fixed dimension, this answers affirmatively for the special class of *smooth* Gorenstein polytopes Question 4.21 in [12] which asks whether there should be (up to multiples) only finitely many stringy E-polynomials of Gorenstein polytopes of given Calabi-Yau dimension. In particular, the validity of this purely combinatorial conjecture for any Gorenstein polytope would imply the finiteness of stringy Hodge numbers of all irreducible CICY's given by the Batyrev-Borisov construction. In our present situation the previous theorem shows even more:

COROLLARY 4.2. *There are only finitely many families of (possibly singular) generic Calabi-Yau complete intersections of dimension n that are associated to* smooth *Gorenstein polytopes.*

Table 1.2 gives a complete list of all smooth Gorenstein polytopes of Calabi-Yau dimension $n \leq 3$ up to large d. Note that by Theorem 4.1 it is enough to consider $d \leq 12$ for $n = 3$. Of course, the assumption of smoothness is very strong and yields only very few Gorenstein polytopes. The subtle issue which of the stringy Hodge numbers of these Gorenstein polytopes are realized by CICY's is addressed in Section 4.4. In particular, we found 7 Hodge numbers of Calabi-Yau 3-folds that were not yet contained in the database [38] by Benjamin Jurke.

REMARK 4.3. Let us remark that in [59] the Kreuzer-Skarke algorithm was extended to a potential classification procedure of Gorenstein polytopes of given d and n, however, it is yet unclear how computationally feasible this will be. Lists of Gorenstein polytopes with $n = 3$ coming from so-called basic IP weight systems can be found on the webpage [43].

4.3. Proof of Theorem 4.1. The proof will be a direct consequence of the following observation. It is straightforward from an algebro-geometric viewpoint, however it seems to be quite a challenge to prove it using combinatorics only!

LEMMA 4.4. *Let $b_2 + \cdots + b_r = s$ (all these numbers being positive integers). Then*

$$S_s * b_2 S_s * \cdots * b_r S_s \quad \text{and} \quad b_2 S_{s-1} * \cdots * b_r S_{s-1}$$

are smooth Gorenstein polytopes with the same Calabi-Yau dimension and the same stringy E-polynomial. More precisely, the CICY's given by $S_s + b_2 S_s + \cdots + b_r S_s$, respectively by $b_2 S_{s-1} + \cdots + b_r S_{s-1}$, are isomorphic.

PROOF. In this case, the Minkowski sum $S_s + b_2 S_s + \cdots + b_r S_s$ defines a generic Calabi-Yau complete intersection Y in $\mathbb{P}^s$ given by generic hypersurfaces of degrees $1, b_2, \ldots, b_s$. Identifying the generic hypersurface of degree 1 with P^{s-1}, we note that Y can be regarded as a generic complete intersection of hypersurfaces of degrees $b_2, \ldots, b_s$ in $\mathbb{P}^{s-1}$. This finishes the proof. □

PROOF OF THEOREM 4.1. Note that $n \geq 0$ if and only if $r \leq \frac{d+1}{2}$, so only cases (2) and (4) may occur in Theorem 3.2. In case (2), we have $n = 0$, and $E_{\text{st}}(2S_d) = 2 = E_{\text{st}}([-1, 1])$, see e.g. Example 4.12 in [12]. Hence, it suffices to consider case (4). We may assume that $a_1, \ldots, a_t$ are all positive integers (since their sum equals $n + 1 \geq 1$). Now, Lemma 4.4 shows that it suffices to consider $t = r$ (note again that $a_i = 0$ for $i > t$). Therefore, $n+1 = \sum_{i=1}^r a_i \geq r$, thus $d = (d+1-2r)+2r-1 = n+2r-1 \leq n+2(n+1)-1 = 3n+1$. This proves the first statement. The second claim follows from $s = d+1-r = n+r \leq 2n+1$ and the proof of Lemma 4.4.

For the last statement of the theorem, let P be a Gorenstein polytope associated to a Minkowski decomposition $\Delta = \Delta^{(1)} + \cdots + \Delta^{(r)}$ as above, i.e.,

$$P = \Delta^{(1)} * \cdots * \Delta^{(r)}. \tag{2}$$

Let $d > 3n + 3$. Again, we have two cases. If $n = 0$, then $P \cong 2S_d$, hence $r = 1$ (since $2S_d$ is not a Cayley polytope). Therefore, $d = 2$, a contradiction to $d > 3n + 3$. Therefore, we are in case (4) of Theorem 3.2, so by Definition 3.1

$$P \cong \operatorname{conv}((a_1 + 1)S_s \times e_1, \ldots, (a_t + 1)S_s \times e_t, S_s \times e_{t+1}, \ldots, S_s \times e_r), \tag{3}$$

where $a_1, \ldots, a_t$ is an integer partition of $n + 1$. Equation (2) implies the existence of a surjective, affine lattice homomorphism $\phi : P \to S_{r-1}$. We may assume $a_1 > 0$. Since for every vertex of $(a_1 + 1)S_s \times e_1$ every adjacent edge contains a lattice point in its relative interior, $\phi((a_1 + 1)S_s \times e_1)$ has to be a vertex of S_{r-1}. Hence, $S_s \times e_1$ lies in the fiber space of ϕ. Therefore, every factor in expression (3) maps onto a vertex via ϕ. Since the map is surjective and there are r factors, it follows that these two Cayley decompositions are the same (up to a permutation of the factors). Hence, the Minkowski decomposition of Δ equals up to permutation and translations by lattice points the Minkowski decomposition

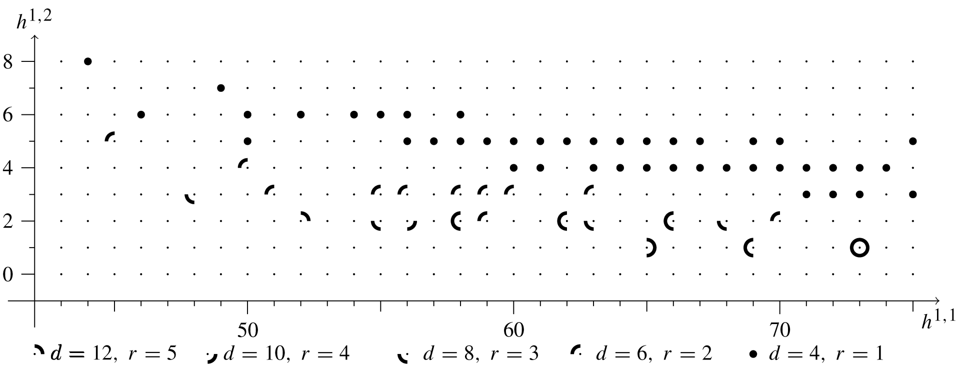

FIGURE 1. Stringy Hodge numbers of the duals of smooth Gorenstein polytopes with Calabi-Yau dimension 3 and $h^{1,1} \leq 75$.

$(s+1)S_s = (a_1+1)S_s + \cdots + (a_t+1)S_s + S_s + \cdots + S_s$. This implies the statement (e.g., as in the proof of Lemma 4.4). □

In the notation of [9, 12, 59], this shows that for $d > 3n+3$ the 'reflexive Gorenstein cone' over P is 'completely split'.

4.4. Realization of stringy Hodge numbers by Calabi-Yau complete intersections. Table 1.2 combined with Theorem 3.2 gives a complete enumeration of all smooth Gorenstein polytopes P with Calabi-Yau dimension $0 \leq n \leq 3$. In this section, we would like to discuss which stringy Hodge numbers of these smooth Gorenstein polytopes are possibly realized by Calabi-Yau complete intersections.

First, let us recall that every Gorenstein polytope P has a *dual Gorenstein polytope* $P^\times$ of the same dimension, the same index, and the same Calabi-Yau dimension n, see [9] or [12], satisfying the combinatorial mirror symmetry

$$E_{\text{st}}(P; u, v) = (-u)^n E_{\text{st}}(P^\times; u^{-1}, v)\,.$$

In particular, for $n = 3$, the stringy Hodge numbers $(h^{1,1}, h^{1,2})$ are interchanged between P and $P^\times$, see [7, 12, 52]. Figures 1 and 2 show the complete list of pairs $(h^{1,1}, h^{1,2})$ of stringy Hodge numbers of $P^\times$ for $n = 3$ and $d \leq 12$. Note that by Theorem 4.1 this is the complete list for $n = 3$.

To compute the stringy Hodge numbers of these polytopes we implemented the formula for the stringy E-polynomial described in [12, Definition 4.8] in the `polymake` framework [33], using an interface to Normaliz [18, 19] for the Ehrhart h^*-polynomials.

REMARK 4.5. We note that for $n = 3$ the 12-dimensional smooth Gorenstein polytope P from Remark 3.7 has Hodge-pair (2, 52) which did not appear for any of the polytopes of lower dimension, indicating that the bound $3n + 3$ in Theorem 4.1 could indeed be sharp. However, for general n we cannot rule out that the stringy E-polynomial of this polytope might appear for some smooth Gorenstein polytope of lower dimension. In fact, for $n =$

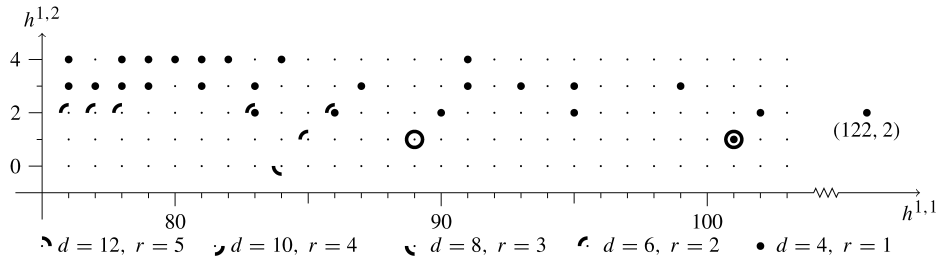

FIGURE 2. Stringy Hodge numbers of the duals of smooth Gorenstein polytopes with Calabi-Yau dimension 3 and $h^{1,1} > 75$.

3 there exists a 6-dimensional (non-smooth) Gorenstein polytope with this pair of Hodge numbers [43].

As it turns out, all of these 'virtual' stringy Hodge numbers indeed equal the Hodge numbers of Calabi-Yau manifolds. To prove this, let us recall the definition of being integrally closed.

DEFINITION 4.6. A d-dimensional lattice polytope $P \subseteq \mathbb{R}^d$ is *integrally closed*, if the semigroup of lattice points in the cone $C_P := \mathbb{R}_{\geq 0}(P \times \{1\}) \subset \mathbb{R}^{d+1}$ is generated by lattice points in $C_P \cap \mathbb{R}^d \times \{1\}$.

PROPOSITION 4.7. *Let P be a Gorenstein polytope of Calabi-Yau dimension n. If P is integrally closed, then there exists a Calabi-Yau variety Y^* of dimension n such that $E_{\text{st}}(P^\times)$ equals the stringy E-polynomial of Y^*. Moreover, if $n \leq 3$, then we can assume that Y^* is smooth.*

PROOF. As follows from [12, Cor.2.12], $P^\times$ is a Cayley polytope of length r, say, $P^\times = \nabla_1 * \cdots * \nabla_r$ in the notation of [12]. Therefore, e.g. by [12, Thm.2.6], $\nabla_1 + \cdots + \nabla_r =: Q \subset N_\mathbb{R}$ is a reflexive polytope of dimension $s := d + 1 - r$. In particular, there exists an associated generic n-dimensional CICY Y^* in the Gorenstein Fano toric variety X^* associated to Q. By the very definition of the stringy E-polynomial of Gorenstein polytopes, $E_{\text{st}}(Y^*) = E_{\text{st}}(P^\times)$.

Let $n \leq 3$. Let us recall the argument given in [5]. We choose a maximal projective crepant partial desingularization (MPCP) $\widehat{X^*}$ of X^* in the sense of [5]. This induces an MPCP from a CICY $\widehat{Y^*}$ in $\widehat{X^*}$ to Y^*, where $\widehat{Y^*}$ is again a Calabi-Yau variety whose stringy E-polynomial equals $E_{\text{st}}(Y^*)$. Note that the cones of dimension ≤ 3 of the fan corresponding to $\widehat{X^*}$ are unimodular, so the toric strata of $\widehat{X^*}$ of dimension $\geq s - 3$ are smooth (e.g., [5, Thm.2.2.9]). Since $\widehat{Y^*}$ is generic of dimension ≤ 3, it avoids toric strata of dimension $< s - 3$, hence $\widehat{Y^*}$ is smooth. □

There is a famous open conjecture: *Smooth polytopes are integrally closed*. We verified it for all smooth Gorenstein polytopes we could compute. In particular, by Theorem 4.1 this observation implies:

COROLLARY 4.8. *All stringy E-polynomials of duals of smooth Gorenstein polytopes with Calabi-Yau dimension $n \leq 3$ equal the E-polynomial of some Calabi-Yau manifold of dimension n in a Gorenstein toric Fano variety of dimension $s \leq 2n + 2$.*

PROOF. It remains to consider the case $s \geq 2n + 3$, hence $r = s - n \geq n + 3$, thus $d = s + r - 1 \geq 3n + 5$, in which case Theorem 4.1 implies the statement. □

REMARK 4.9. Among the Hodge numbers listed in Figures 1 and 2 the pairs

$$(84, 0), (85, 1), (52, 2), (69, 1), (65, 1), (55, 2), (63, 2)$$

are not yet contained in the online database 'Calabi-Yau 3-fold explorer' by Benjamin Jurke [38]. The associated smooth Gorenstein polytopes P can be found on the webpage [46]. We note that for the first five of these pairs Gorenstein polytopes with these stringy Hodge numbers can also be found in [43].

REMARK 4.10. Among the stringy Hodge-numbers of the smooth Gorenstein polytopes themselves (i.e., not their duals), only the pairs $(1, 69)$ and $(1, 85)$ are not contained in the database of known Hodge numbers of Calabi-Yau 3-folds [38]. However, all three corresponding dual Gorenstein polytopes $P^\times$ are neither integrally closed, nor do they contain a special $(r - 1)$-simplex (in the sense of [12]). Hence, we cannot deduce whether there are CICY's associated to these smooth Gorenstein polytopes P.

REMARK 4.11. The complete intersection Calabi-Yau manifolds described in Theorem 4.1 are given by a so-called *nef-partition*, see [15, 12]. In particular, there exist Calabi-Yau manifolds with mirror-symmetric Hodge numbers. This does not have to be the case in general. For instance, there is a generic CICY with Hodge numbers $(84, 0)$ (see Remark 4.9), however, there is no CICY with Hodge numbers $(0, 84)$. We refer to [9] for more on this rigidity-phenomenon.

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