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ON SOLUBLE GROUPS OF MODULE AUTOMORPHISMS OF FINITE RANK

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Abstract. Let R be a commutative ring, M an R-module and G a group of R-automorphisms of M, usually with some sort of rank restriction on G. We study the transfer of hypotheses between $M/C_M(G)$ and $[M,G]$ such as Noetherian or having finite composition length. In this we extend recent work of Dixon, Kurdachenko and Otal and of Kurdachenko, Subbotin and Chupordia. For example, suppose $[M,G]$ is R-Noetherian. If G has finite rank, then $M/C_M(G)$ also is R-Noetherian. Further, if $[M,G]$ is R-Noetherian and if only certain abelian sections of G have finite rank, then G has finite rank and is soluble-by-finite. If $M/C_M(G)$ is R-Noetherian and G has finite rank, then $[M,G]$ need not be R-Noetherian.

Keywords: soluble group; finite rank; module automorphisms; Noetherian module over commutative ring

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Kurdachenko, Subbotin and Chupordia's paper [3] is devoted to proving the following theorem. Let R be an integral domain, M an R-module and G a subgroup of $\mathrm{Aut}_R M$ such that $M/C_M(G)$ has a composition series as R-module of finite length, l say. If p is the characteristic of some R-composition factor of $M/C_M(G)$ (so p is a prime or zero), assume that there is a (finite) bound r_p on the ranks of the elementary abelian p-sections (free abelian sections if $p=0$) of G. Then $[M,G]$ also has finite R-composition length. Moreover, this length can be bounded, for example in terms of the l, p and r_p above. (The case where R is a field had earlier been discussed by Dixon, Kurdachenko and Otal in [2].) The authors regard this as at least a superficial analogue of Schur's theorem (e.g. [9], 1.18) that if the centre of some group G has finite index in G, then the derived subgroup of G is also finite.

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Here we give a much shorter proof of this theorem. Actually we lengthen our proof a little in order to prove rather more, but we indicate below which lemmas can be omitted just to obtain a proof of this theorem from [3].

Throughout this paper R denotes a commutative ring and M an R-module. Let $\pi(M)$ denote the set of primes p such that M contains an element of additive order p together with 0 if M contains an element of infinite additive order. (If M has finite R-composition length, then $\pi(M)$ is the same as the set of primes and possibly zero considered in the theorem quoted above from [3], see below.) Note that if M is Noetherian, then the $\mathbb{Z}$-torsion submodule of M has finite exponent $e(M)$ and hence $\pi(M)$ is finite. In order to cover all cases simultaneously, by an elementary abelian 0-group we mean a free abelian group.

Proposition 1. *Suppose M is Noetherian as R-module. Let G be a subgroup of $\mathrm{Aut}_R M$ such that for every $p \in \pi(M)$ every elementary abelian p-section of G has finite rank. Then G is soluble-by-finite and has finite rank. If $0 \notin \pi(M)$, then G is even abelian-by-finite.*

Whenever G is a subgroup of $\mathrm{Aut}_R M$, let $\mathbf{g}$ denote the augmentation ideal of G in the group ring RG, so $C_M(G) = \mathrm{Ann}_M(\mathbf{g})$ and $[M, G] = M\mathbf{g}$. We need a couple of small extensions to Proposition 1.

Proposition 2. *Let G be a subgroup of $\mathrm{Aut}_R M$. If either*

(a) *$M/C_M(G)$ has finite R-composition length and every elementary abelian p-section of G has finite rank for every p in $\pi(M/C_M(G))$, or*

(b) *$M/C_M(G)$ is R-Noetherian and every elementary abelian p-section of G has finite rank for every p in $\pi(M\mathbf{g})$, or*

(c) *$M\mathbf{g}$ is R-Noetherian and every elementary abelian p-section of G has finite rank for every p in $\pi(M\mathbf{g})$,*

then G is soluble-by-finite and has finite rank.

We will see from the proofs of Propositions 1 and 2 that the rank of G and the index of its maximal soluble normal subgroup can be bounded in terms of the p-ranks for each p in $\pi(M/C_M(G))$ (or $\pi(M\mathbf{g})$) and certain structural constants of these R-modules.

Proposition 3. *Let G be a subgroup of $\mathrm{Aut}_R M$ of finite rank r, and s a positive integer.*

(a) *If $M/\mathrm{Ann}_M(\mathbf{g}^s)$ has finite R-composition length l, then $M\mathbf{g}^s$ has finite R-composition length at most lr^s.*

(b) *If $M\mathbf{g}^s$ has finite R-composition length l, then $M/\mathrm{Ann}_M(\mathbf{g}^s)$ has finite R-composition length at most lr^s.*

(c) *If $M\mathbf{g}^s$ is R-Noetherian, then $M/\mathrm{Ann}_M(\mathbf{g}^s)$ is R-Noetherian.*

Note that the theorem from [3] quoted above follows at once from Proposition 2 (a) and Proposition 3 (a) with $s = 1$ (and with R an integral domain). Further, Theorem A of [2] also follows at once from Propositions 2 and 3 (with R a field), but Proposition 2 is not strong enough to read off Theorems B and C of [2]. At the end of this paper we give some simple examples limiting possible extensions of Propositions 2 and 3. However, [2] is devoted to the case where $R = F$ is a field and in this case a much stronger version of Proposition 2 holds. The reason for this is that $\pi(M) = \{\mathrm{char} F\}$ for all nonzero vector spaces M over F. Theorems B and C of [2] are immediate from Proposition 3 and Proposition 4 below.

Proposition 4. *Let $R = F$ be a field of characteristic $p \geqslant 0$, s a positive integer, M a vector space over F and G a subgroup of $\mathrm{Aut}_F M$ such that either $M/\mathrm{Ann}_M(\mathbf{g}^s)$ or $M\mathbf{g}^s$ is finite dimensional. If every elementary abelian p-section of G has finite rank, then G is soluble-by-finite and of finite rank.*

For brevity, if in some situation involving integers a, b, c etc. there is an integer-valued function f only of the variables b, c etc. and of none of the other information in the situation such that $a \leqslant f(b, c, \ldots)$, we shall often just say that a is $(b, c, \ldots)$-bounded.

Lemma 1. *Let G be a subgroup of $\mathrm{GL}(n, F)$, where n is a positive integer and F is a field of characteristic 0. Suppose that every free abelian section of G has finite rank. Then G has finite rank, r say, and G is soluble-by-finite and n-bounded. If r_0 is the upper bound of the ranks of the free abelian sections of G, then $r_0 \leqslant r \leqslant f_0(n, r_0)$ for f_0 being some integer-valued function of n and r_0 only.*

P r o o f. Clearly G can contain no non-cyclic free subgroups. Hence by Tits's theorem ([5], 10.17, but see also [5], 10.11) the group G has a soluble normal subgroup S whose index $(G : S)$ in G is finite and n-bounded. Now S contains a triangularizable (over the algebraic closure $\hat{F}$ of F) normal subgroup T of G with S/T finite and n-bounded (see Proposition 1 of [7]), so $(G : T)$ is n-bounded.

If U is the unipotent radical of T, then U is nilpotent of class less than n and its upper central factors are torsion-free. Hence U has finite rank (at most $r_0(n-1)$ once we know that r is finite). Also $A = T/U$ embeds into the diagonal group $\mathrm{D}(n, \hat{F})$. Thus, its torsion subgroup $\tau(A)$ has rank at most n while $A/\tau(A)$ has finite rank (at most r_0). Hence, T has finite rank and consequently so does G. Moreover the above then shows that

$$r_0 \leqslant r \leqslant (r_0 + 1)n + (G : T)$$

and $(G : T)$ is n-bounded. Hence, r is (n, r_0)-bounded. □

Lemma 2. *Let G be a subgroup of $\mathrm{GL}(n, F)$, where n is a positive integer and F is a field of characteristic $p > 0$. Suppose that every elementary abelian p-section of G has finite rank. Then G has finite rank, r say. If r_p is the upper bound of the ranks of the elementary abelian p-sections of G, then G is abelian-by-finite and (p, n, r_p)-bounded and $r_p \leqslant r \leqslant f_p(n, r_p)$ for f_p being some integer-valued function of n and r_p only.*

P r o o f. Here Tits's theorem ([5], 10.17) only yields a soluble normal subgroup S of G with G/S locally finite, but [7], Proposition 1, does at least yield a triangularizable (over $\hat{F}$) normal subgroup $T \leqslant S$ of G with $(S : T)$ n-bounded.

If U again denotes the unipotent radical of T, then U has a central series of length less than n with its factors being elementary abelian p-groups. Thus, U is finite, say of order q, where $\log_p q \leqslant r_p(n-1)$. Now $C = C_T(U)$ is nilpotent of class at most 2 (for T/U is abelian). Hence C^q is abelian. Set $A = C_G C_G(C^q)$. Then A is an abelian normal subgroup of G containing C^q (it is the centre of $C_G(C^q)$) and G/A is isomorphic to a subgroup of $\mathrm{GL}(n^2, F)$, see [5], 6.2.

Now $(S : T)$ is finite and n-bounded, $(T : C)$ divides $q!$ and $(C : C^q)$ is a finite power of p with $\log_p(C : C^q) \leqslant (r_p+1)r_p(n-1)$. Thus, $(S : A)$ is finite and (p, n, r_p)-bounded. In particular, G/A is locally finite and embeddable into $\mathrm{GL}(n^2, F)$. Further, a Sylow p-subgroup of G/A is finite, say of order p^α, where $\alpha \leqslant r_p(n^2-1)$. By the Brauer-Feit theorem (see [1] or [5], 9.6 and 9.7 for summary) there is an integer-valued function $f(m, n, p)$ of the exhibited variables only such that G/A has an abelian normal subgroup B/A of finite index with $(G : B) \leqslant f(r_p(n^2-1), n^2, p)$. In particular, B is soluble, so we may choose $S = B$. Consequently $(B : A)$ is finite and (p, n, r_p)-bounded. Finally, the torsion subgroup $\tau(A)$ has finite rank at most $\max\{n, r_p\}$ and $A/\tau(A)$ has finite rank at most r_p. Thus, A has finite rank and hence so does G. Also

$$r_p \leqslant r \leqslant 2r_p + n + (B : A) + f(r_p(n^2-1), n^2, p),$$

which is (p, n, r_p)-bounded. □

P r o o f of Proposition 1. There exists a positive integer n (depending only on M) and for each p in $\pi(M)$ a field F_p of characteristic p such that G (indeed $\mathrm{Aut}_R M$) embeds into the direct product over $p \in \pi(M)$ of the $\mathrm{GL}(n, F_p)$, see [8], 6.1 and 6.2, or less explicitly [6]. By Lemmas 1 and 2 for each p in $\pi(M)$ there exists a normal subgroup N_p of G such that G/N_p is soluble-by-finite (even abelian-by-finite if $p < 0$) of finite rank and with $I_p N_p = \langle 1 \rangle$. Since $\pi(M)$ is finite, the claims of Proposition 1 follow. Clearly the rank r of G can be bounded in terms of n, $\pi(M)$ and for each $p \in \pi(M)$ by the upper bound r_p of the ranks of the elementary abelian p-sections of G. □

Consider a module X over the commutative ring R. If X is Noetherian, then the $\mathbb{Z}$-torsion submodule T of X has finite exponent e say. If Y is an R-submodule of X with X/Y irreducible of characteristic $p > 0$ and Y irreducible of characteristic 0, then $pX = Y$. Thus, if $P = \{x \in X\colon px = 0\}$, then $X/P \cong Y$, $P \cap Y = \{0\}$ and $P \cong X/Y$. Suppose X has a composition series (as R-module) of finite length. The above implies that the composition factors of X/T all have characteristic 0. Necessarily those of T have characteristics dividing e. It follows that $\pi(X)$ is equal to the set of characteristics of the composition factors of X. Also if φ is any R-homomorphism of X, then $T\varphi$ is the $\mathbb{Z}$-torsion submodule of $X\varphi$, $\pi(X\varphi) \subseteq \pi(X)$ and $e(X\varphi)$ divides $e(X)$.

Lemma 3. *Let* $\mathbf{X}$ *be a class of* R*-modules that is closed under taking homomorphic images and direct sums of finitely many modules. Let* M *be an* R*-module and* G *a finitely generated subgroup of* $\mathrm{Aut}_R M$ *such that* $M/C_M(G) \in \mathbf{X}$. *Then* $[M, G] \in \mathbf{X}$.

Proof. Let $G = \langle x_1, x_2, \ldots, x_s\rangle$ and $N = [M, G]$. Now each $M(x_i - 1) \cong M/C_M(x_i)$ is an image of $M/C_M(G)$ and hence each $M(x_i - 1) \in \mathbf{X}$. Now $N = \sum\limits_i M(x_i - 1)$ since

$$M(xy - 1) \leqslant M(x - 1) + M(y - 1), \quad x, y \in G$$

and hence N is an image of $\bigoplus\limits_i M(x_i - 1)$. Therefore $N \in \mathbf{X}$. □

Proof of Proposition 2 (a) and 2 (b). Set $N = C_M(G)$ and $C = C_G(M/N)$. By Proposition 1 the group G/C is soluble-by-finite and of finite rank. If $g \in C$, then $g\alpha g - 1$ determines an embedding of C into the additive group of $H = \mathrm{Hom}_R(M/N, [M, C])$. In particular, G is soluble-by-finite.

If $g \in G$, then $M(g - 1)$ is an image of M/N. Suppose M/N has finite R-composition length. If $e = e(M/N)$ and if T is the $\mathbb{Z}$-torsion submodule of $[M, G]$, then $e(T \cap [M, H]) = \{0\}$ for every finitely generated subgroup H of G by Lemma 3 and hence $eT = \{0\}$. If C contains an element of a prime order p, then so does H and hence so does $[M, C]$. Therefore p divides e and the p-component of C has, by hypothesis, finite rank. If C contains an element of infinite order, then so does H. But T has finite exponent and hence M/N contains an element of infinite order. Therefore $0 \in \pi(M/N)$ and so the $\mathbb{Z}$-torsion-free quotient of C has finite rank. Hence C has finite rank and consequently so does G. This settles Part (a).

For Part (b) if C contains an element of prime order p, then so does H and hence so does $[M, C]$. Consequently $p \in \pi([M, G])$ and hence the p-component of C has finite rank. If C contains an element of infinite order, then so does H. But M/N is finitely R-generated. Thus, $[M, C]$ contains an element of infinite order as $\mathbb{Z}$-module and consequently the full torsion-free quotient of C, C and G itself have finite rank. $\square$

Lemma 4. *Let e be a positive integer and π a set of primes and possibly zero. Let $\mathbf{X}$ denote the class of all R-modules with finite composition length such that $e(M)$ divides e and $\pi(M) \subseteq \pi$. If M is an R-module and G a subgroup of $\operatorname{Aut}_R(M)$ of finite rank such that $M/C_M(G) \in \mathbf{X}$, then $[M, G] \in \mathbf{X}$.*

Lemma 4 completes our proof of the theorem from [3]. Unlike in Propositions 1, 2 (b) and 2 (c), in Lemma 4 we cannot weaken having finite composition length to being Noetherian (example later).

P r o o f. Let r denote the rank of G and l the composition length $M/C_M(G)$. We prove that $[M, G]$ has composition length at most lr.

Consider a subgroup $H = \langle x_1, x_2, \dots, x_r \rangle$ of G. Then $[M, H] = \sum\limits_i M(x_i - 1)$ has composition length $l_H \leqslant lr$. Choose such $H \leqslant G$ so that l_H is maximal. Let $x \in G$ and set $K = \langle H, x \rangle$. Since $\operatorname{rank} G = r$, K too can be generated by r elements and trivially $[M, K] \geqslant [M, H]$ and $l_K \geqslant l_H$. Therefore $l_K = l_H$ and $[M, K] = [M, H]$ for every x in G. Consequently $[M, G] = [M, H]$ and $l_G = l_H \leqslant lr$. Finally $[M, H]$ and hence $[M, G]$ lie in $\mathbf{X}$ by Lemma 3. $\square$

Lemma 5. *Let G be a subgroup of $\operatorname{Aut}_R M$ of finite rank r and s a positive integer. If $M/\operatorname{Ann}_M(\mathbf{g}^s)$ has finite R-composition length l, then $M\mathbf{g}^s$ has finite R-composition length at most lr^s. Also $e(M\mathbf{g}^s)$ divides $e(M/\operatorname{Ann}_M(\mathbf{g}^s))$ and $\pi(M\mathbf{g}^s)$ is contained in $\pi(M/\operatorname{Ann}_M(\mathbf{g}^s))$.*

Proposition 3 (a) follows at once from Lemma 5.

P r o o f. We induct on s. The case where $s = 1$ is covered by Lemma 4 (and the bound obtained in its proof). Suppose $s > 1$ and set $N = \operatorname{Ann}_M(\mathbf{g}^s)$. Apply the case $s = 1$ to $M/N\mathbf{g}$. This yields that $M\mathbf{g}/N\mathbf{g}$ has composition length at most lr, has $e(M\mathbf{g}/N\mathbf{g})$ dividing $e(M/N)$ and has $\pi(M\mathbf{g}/N\mathbf{g})$ contained $\pi(M/N)$. Now apply induction to $M\mathbf{g}$, $N\mathbf{g}$, $(M\mathbf{g})\mathbf{g}^{s-1}$ and $(N\mathbf{g})\mathbf{g}^{s-1} = \{0\}$. $\square$

P r o o f of Proposition 2 (c). Set $C = C_G(M\mathbf{g})$. By Proposition 1 the group G/C is soluble-by-finite and of finite rank. There is a standard embedding of C into $H = \operatorname{Hom}_R(M/M\mathbf{g}, M\mathbf{g})$. In particular, C is abelian and G is soluble-by-finite.

If $e = e(M\mathbf{g})$, then the $\mathbb{Z}$-torsion submodule of H has an exponent dividing e. Hence, if C contains an element of prime order p, then p divides e, $p \in \pi(M\mathbf{g})$ and the maximal p-subgroup of C has finite rank. Therefore the torsion subgroup of C has finite rank. If C is not a torsion, then neither is H or $M\mathbf{g}$. Then $0 \in \pi(M\mathbf{g})$, so by hypothesis the $\mathbb{Z}$-torsion-free quotient of C has finite rank. Therefore C and G have finite rank. □

Lemma 6. *Let* $\mathbf{Y}$ *be a class of* R*-modules that is closed under taking submodules and direct sums of finitely many modules. Let* M *be an* R*-module and* G *a finitely generated subgroup of* $\mathrm{Aut}_R M$ *such that* $[M,G] \in \mathbf{Y}$*. Then* $M/C_M(G) \in Y$*.*

Proof. Let $G = \langle x_1, x_2, \ldots, x_s\rangle$. Then for each i we have $M/C_M(x_i) \cong M(x_i - 1) \leqslant M\mathbf{g}$. Thus, we have embeddings $M/C_M(G) \to \bigoplus_i M/C_M(x_i) \to (M\mathbf{g})^{(s)}$, which lie in $\mathbf{Y}$. Therefore $M/C_M(G) \in \mathbf{Y}$. □

Lemma 7. *Let* e *and* s *be positive integers and* π *a set of primes and possibly zero. Let* $\mathbf{Y}$ *denote the class of all* R*-modules with finite composition length such that* $e(M)$ *divides* e *and* $\pi(M) \subseteq \pi$*. If* M *is an* R*-module and* G *a subgroup of* $\mathrm{Aut}_R(M)$ *of finite rank such that* $M\mathbf{g}^s \in \mathbf{Y}$*, then* $M/\mathrm{Ann}_M(\mathbf{g}^s) \in \mathbf{Y}$*.*

Proposition 3 (b) follows at once from Lemma 7 and the bound below.

Proof. Let r denote the rank of G and l the composition length of $M\mathbf{g}^s$ (as R-module of course). We prove that $M/\mathrm{Ann}_M(\mathbf{g}^s)$ has composition length at most lr^s.

Consider first the case where $s = 1$. Choose $H = \langle x_1, x_2, \ldots, x_r\rangle \leqslant G$ such that the composition length l^H of $M/C_M(H)$ is maximal. By Lemma 6 we have $M/C_M(H) \in \mathbf{Y}$ and, cf. the proof of Lemma 6, clearly $l^H \leqslant lr$. If $x \in G$ and $K = \langle H, x\rangle$, then $C_M(K) \leqslant C_M(H)$, so $l^H \leqslant l^K$. By the choice of H we have $l^H = l^K$, $C_M(K) = C_M(H)$ for all x in G. Hence, $C_M(G) = C_M(H)$ and the case where $s = 1$ follows.

Now assume that $s > 1$. Apply the case where $s = 1$ to $M\mathbf{g}^{s-1} \geqslant M\mathbf{g}^{s-1}\mathbf{g} \in \mathbf{Y}$. Hence, $M\mathbf{g}^{s-1}/A \in \mathbf{Y}$ and has composition length at most lr, where $A = \mathrm{Ann}_M(\mathbf{g}) \cap M\mathbf{g}^{s-1}$. Now apply induction on s to $M/A \geqslant (M/A)\mathbf{g}^{s-1} = M\mathbf{g}^{s-1}/A \in \mathbf{Y}$. Thus, $M/B \in \mathbf{Y}$ and has composition length at most lrr^{s-1}, where $B/A = \mathrm{Ann}_{M/A}(\mathbf{g}^{s-1})$. Then $B\mathbf{g}^{s-1} \leqslant A$, so $B\mathbf{g}^s = \{0\}$. But then $M/\mathrm{Ann}_M(\mathbf{g}^s)$ as an image of M/B lies in $\mathbf{Y}$. The proof is complete. □

To prove Proposition 3 (c) we need to recall the part of the theory of Krull dimension. All we use can be found, for example, in Sections 6.1 and 6.2 of [4].

Suppose M is a nonzero Noetherian R-module. Then M has Krull dimension, an ordinal $\kappa(M)$, and a critical composition series $M = M_0 > M_1 > \ldots > M_n = \{0\}$ of finite length, where if $\alpha_i = \kappa(M_{i-1}/M_i)$ for each i, then $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_n$. We denote this sequence of ordinals by $\mathrm{sp}(M)$. It does depend only on M, see [4], 6.2.21. Now any nonzero submodule of an α-critical is α-critical ([4], 6.2.11). Thus, if N is a nonzero submodule of M, then $\{M_i \cap N\colon\ 0 \leqslant i \leqslant n$ with the repetitions removed$\}$ is a critical composition series of N and $\mathrm{sp}(N)$ is a subsequence of $\mathrm{sp}(M)$.

Now suppose that $M = M_0 > M_1 > \ldots > M_r = N = N_0 > N_1 > \ldots > N_s = \{0\}$, where the M_{i-1}/M_i form a critical composition series of M/N and the N_j form a critical composition series of N with $\mathrm{sp}(M/N) = \{\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_r\}$ and $\mathrm{sp}(N) = \{\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_s\}$. If $\alpha_r \geqslant \beta_1$, then the above series of M is a critical composition series of M and

$$\mathrm{sp}(M) = \{\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_r \geqslant \beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_s\}.$$

Suppose $\alpha_{t-1} \geqslant \beta_1 > \alpha_t$. Let $K \geqslant N_1$ be a submodule of M_{t-1} maximal subject to $K \cap N = N_1$. Then M_{t-1}/K is β_1-critical and $\kappa(K) \leqslant \beta_1$. Hence, if $\mathrm{sp}(K) = \{\gamma_1 \geqslant \gamma_2 \geqslant \ldots \geqslant \gamma_u\}$, then

$$\mathrm{sp}(M) = \{\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_{t-1} \geqslant \beta_1 \geqslant \gamma_1 \geqslant \ldots \geqslant \gamma_u\}.$$

P r o o f of Proposition 3 (c). Consider first the case where $s = 1$. Let $H = \langle x_1, x_2, \ldots, x_r\rangle \neq \langle 1\rangle$ be an r-generated subgroup of G. Then $M/C_M(H)$ embeds into the Noetherian R-module $(M\mathbf{g})^{(r)}$ and in particular $\mathrm{sp}(M/C_M(H)) = \{\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_m\}$ is a subsequence of the finite sequence $\mathrm{sp}((M\mathbf{g})^{(r)}) = \{\delta_1 \geqslant \delta_2 \geqslant \ldots \geqslant \delta_n\}$. Consider those H with $\alpha_1 = \delta_j$ maximal. Of these H consider those with the number of α_i equal to δ_j maximal. Then of these H consider those with the number of α_i equal to δ_{j+1} maximal. Keep going like this right through to and including the final δ_n.

Let $x \in G$ and set $K = \langle H, x\rangle$. Then $C_M(K) \leqslant C_M(H)$. Also K is r-generator since G has finite rank r and therefore K is one of the subgroups of G considered during the choice of H. Suppose $C_M(K) < C_M(H)$ and set $\mathrm{sp}(C_M(H)/C_M(K) = \{\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_s\}$. If

$$\mathrm{sp}(M/C_M(K)) = \{\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_m \geqslant \beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_s\}$$

or if $\alpha_{t-1} \geqslant \beta_1 > \alpha_t$ with

$$\mathrm{sp}(M/C_M(K)) = \{\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_{t-1} \geqslant \beta_1 \geqslant \gamma_1 \geqslant \ldots \geqslant \gamma_u\}$$

for some γ_j, then we have a contradiction to our choice of H. Therefore $C_M(K) = C_M(H)$ and this is for all x in G. Consequently, $C_M(G) = C_M(H)$. But $M/C_M(H)$ is R-Noetherian (apply Lemma 6 with $\mathbf{Y}$ being the class of Noetherian R-modules). Hence, $M/C_M(G)$ is also R-Noetherian, which settles the '$s = 1$' case of Proposition 3 (c). The proof is now completed by an easy induction on s, cf. the final paragraph of the proof of Lemma 7, again with $\mathbf{Y}$ being the class of Noetherian R-modules. □

Remark. If we apply the above proof to the case where $R=\mathbb{Z}$, we obtain the following. If $M\mathbf{g}^s$ is additively finitely l-generated, then $M/\mathrm{Ann}_M(\mathbf{g}^s)$ is additively lr^s-generated. A similar remark applies if R is any principal ideal domain. The main step is that $M/C_M(G)$ embeds into the R-module $(M\mathbf{g})^{(r)}$.

P r o o f of Proposition 4. Set $N=\mathrm{Ann}_M(\mathbf{g}^s)$ and suppose $\dim_F(M/N)$ is finite. If $C=C_G(M/N)$, then G/C is soluble-by-finite and of finite rank by Proposition 1. Also C stabilizes the series

$$M\geqslant N\geqslant N\mathbf{g}\geqslant N\mathbf{g}^2\geqslant\ldots\geqslant N\mathbf{g}^s\geqslant\{0\},$$

each factor of which is additively an elementary abelian p-group (torsion-free abelian if $p=0$). By standard stability theory, e.g. see [9], 1.21, the group C is nilpotent of class at most s and has a series of lengths s whose factors are elementary abelian p-groups (torsion-free abelian if $p=0$). Therefore C has finite rank (at most r_ps in our earlier notation). Consequently, G is soluble-by-finite and of finite rank.

Now assume that $\dim_F(M\mathbf{g}^s)$ is finite and set $C=C_G(M\mathbf{g}^s)$. Our proof here is similar to that above. We deduce that C is nilpotent of finite rank (at most r_ps), that G/C is soluble-by-finite and of finite rank and that G is soluble-by-finite and of finite rank.

The index in G of its maximal soluble normal subgroup is bounded in terms of $\dim_F(M/N)$ or $\dim_F(M\mathbf{g}^s)$ only and $\operatorname{rank} G$ is bounded in terms of r_p, s and $\dim_F(M/N)$ or $\dim_F(M\mathbf{g}^s)$, respectively, only. □

Examples. (1) Although in Propositions 1, 2 (b) and 2 (c) and in Lemma 3 we can work with Noetherian modules rather than modules with finite composition length, this is not the case with Proposition 3 (a) or for that matter Lemmas 4 and 5, even if R is the integers.

Let $M=\mathbb{Z}\oplus C$, where C is an additive Prüfer p-group for some prime p (and $\mathbb{Z}$ denotes the integers). If $a\in C$, let (a) denote the automorphism of M given by $(n,c)(a)=(n,na+c)$. Then $G=\{(a)\colon a\in C\}$ is a subgroup of $\mathrm{Aut}_{\mathbb{Z}}(M)$ isomorphic to C. Clearly G centralizes C, M/C is $\mathbb{Z}$-Noetherian and $[M,G]=C$, which is $\mathbb{Z}$-Artinian, but not $\mathbb{Z}$-Noetherian.

(2) In Proposition 2 (a) we cannot weaken the hypothesis on $M/C_M(G)$ to just being R-Noetherian. For, repeat the construction of Example (1), but now with C being the direct sum of infinitely many Prüfer p-groups. Defining G in the same way, again G centralizes C and $M/C\cong\mathbb{Z}$ is $\mathbb{Z}$-Noetherian with $\pi(M/C)=\{0\}$. However now G is abelian of infinite rank, being isomorphic to C. Also G is periodic, so every elementary p-section of G is trivial for every p in $\pi(M/C)$.

(3) Although Proposition 2 is the basis of the proof of Proposition 3, Proposition 2 is not the '$s = 1$' case of a more general result along the lines of Proposition 3. As a trivial example let $R = \mathbb{Z}$ and $M = F^{(2)}$, where F is an infinite field of characteristic $p > 0$. Let $G = \mathrm{Tr}_1(2, F)$, the full (lower) unitriangular group of degree 2 over F. If we set $N = M$, then $N\mathbf{g}^2 = \{0\}$ and $\pi(M/N)$ is empty and yet G is an elementary abelian p-group of infinite rank. Trivially, M/N has finite composition length. Further, $M\mathbf{g}^2$ is R-Noetherian being $\{0\}$. Thus, there is no '$s = 2$' version for any of the three cases of Proposition 2. If you feel that having these modules $\{0\}$ is a bit of a cheat, set $M_1 = \mathbf{F}_q^{(2)} \oplus M$, where $\mathbf{F}_q$ denotes the field of q-elements, q a prime other than p, and $G_1 = \mathrm{GL}(2, q) \times G$, acting on M_1 in the obvious way. If $\mathbf{g}_1$ is the augmentation ideal of G_1, then M_1 modulo the annihilator of $(\mathbf{g}_1)^2$ and $M_1(\mathbf{g}_1)^2$ are both isomorphic to $\mathbf{F}_q^{(2)}$, the set $\pi(\mathbf{F}_q^{(2)}) = \{q\}$, the group G_1 has finite q-rank and yet G_1 has infinite rank.

References

[1] *R. Brauer, W. Feit*: An analogue of Jordan's theorem in characteristic p. Ann. Math. (2) *84* (1966), 119–131. zbl MR doi

[2] *M. R. Dixon, L. A. Kurdachenko, J. Otal*: Linear analogues of theorems of Schur, Baer and Hall. Int. J. Group Theory *2* (2013), 79–89. zbl MR

[3] *L. A. Kurdachenko, I. Ya. Subbotin, V. A. Chupordia*: On the relations between the central factor-module and the derived submodule in modules over group rings. Commentat. Math. Univ. Carol. *56* (2015), 433–445. zbl MR doi

[4] *J. C. McConnell, J. C. Robson*: Noncommutative Noetherian Rings. With the Cooperation of L. W. Small. Pure and Applied Mathematics. A Wiley-Interscience Publication, John Wiley & Sons, Chichester, 1987. zbl MR

[5] *B. A. F. Wehrfritz*: Infinite Linear Groups. An Account of the Group-Theoretic Properties of Infinite Groups of Matrices. Ergebnisse der Mathematik und ihrer Grenzgebiete 76, Springer, Berlin, 1973. zbl MR doi

[6] *B. A. F. Wehrfritz*: Automorphism groups of Noetherian modules over commutative rings. Arch. Math. *27* (1976), 276–281. zbl MR doi

[7] *B. A. F. Wehrfritz*: On the Lie-Kolchin-Mal'cev theorem. J. Aust. Math. Soc., Ser. A *26* (1978), 270–276. zbl MR doi

[8] *B. A. F. Wehrfritz*: Lectures around Complete Local Rings. Queen Mary College Mathematics Notes, London, 1979. MR

[9] *B. A. F. Wehrfritz*: Group and Ring Theoretic Properties of Polycyclic Groups. Algebra and Applications 10, Springer, Dordrecht, 2009. zbl MR doi

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