ON SOLUTIONS OF THE BEHRENS-FISHER PROBLEM, BASED ON THE t-DISTRIBUTION

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1. The Problem. The problem [1, 2] is the interval estimation[1] of the difference of the means of two normal populations when the ratio of the variances of the populations is unknown. The reader who wishes to see the present solution before considering theoretical details will find it recapitulated in the Summary at the end and will want to refer to the following notation: $(x_1, x_2, \cdots, x_m)$ and $(y_1, y_2, \cdots, y_n)$ are random samples from normal populations with means α and β, and variances μ and ν, respectively. Define $\delta = \alpha - \beta$. We assume $m \leqq n$, and that the variates in each sample are in the order of observation, or else have been randomized.

Recently Neyman [3] has called attention to a solution which we shall designate as (B), and which is a special case of an unpublished solution of Bartlett[2]. It will be simpler to describe (B) later, but we mention now that it has the following advantages: (*i*) its validity does not depend on the values of unknown parameters, (*ii*) the required computations are simple, and (*iii*) only existing tables are needed,——the widely available Fisher t-tables. An unsatisfactory aspect of (B) is that when the sample sizes are unequal, $n - m$ of the variates y_i are completely discarded. The solution below shares with (B) the advantages (*i*), (*ii*), (*iii*); indeed, it is identical with (B) when $n = m$, but when $n \neq m$ it is free from the above objection.

2. Simple Solution. We begin with a simple restricted approach; later we will review the result from a somewhat broader standpoint. If random variables $d_1, d_2, \cdots, d_m$ are independently normally distributed with mean δ and variance σ^2, and if L and Q are defined from

$$L = \sum_{i=1}^{m} d_i/m, \qquad Q = \sum_{i=1}^{m} (d_i - L)^2,$$

then $m^{\frac{1}{2}}(L - \delta)/\sigma$ and Q/σ^2 are independently distributed; the former is a normal variable with zero mean and unit variance; the latter, $Q/\sigma^2 = \chi^2_{m-1}$, where χ^2_k is a generic notation for a random variable distributed according to the χ^2-law with k degrees of freedom. The quotient

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[1] We treat the problem from the standpoint of confidence intervals, rather than significance tests, since when the former are available for δ so is a whole class of the latter, namely for any hypothesis $\delta = \delta_0$, for all δ_0. Furthermore, questions of the existence of "best" tests and "best" confidence intervals are closely related [5a].

[2] How far Bartlett followed the path of this paper is not clear from the brief mention of his results by Welch [4], except that he did establish the sufficiency of certain orthogonality conditions.

$$m^{\frac{1}{2}}(L - \delta)/[Q/(m - 1)]^{\frac{1}{2}} = t_{m-1},$$

where t_k denotes generically a variable having the t-distribution with k degrees of freedom. Define $t_{k,\epsilon}$ from

$$\Pr(-t_{k,\epsilon} \leqq t_k \leqq t_{k,\epsilon}) = \epsilon. \tag{1}$$

Then a set of confidence intervals for δ with confidence coefficient ϵ is

$$|\delta - L| \leqq t_{m-1,\epsilon}\{Q/[m(m - 1)]\}^{\frac{1}{2}}. \tag{2}$$

Denote by $E(l)$ the expected length of the confidence interval (2),

$$E(l) = 2t_{m-1,\epsilon}[m(m - 1)]^{-\frac{1}{2}}\sigma E[(Q/\sigma^2)^{\frac{1}{2}}],$$

$$E(l) = t_{m-1,\epsilon}c_{m-1}\sigma/m^{\frac{1}{2}}, \tag{3}$$

where

$$c_k = 2k^{-\frac{1}{2}}E(\chi_k) = (8/k)^{\frac{1}{2}}\Gamma(\tfrac{1}{2}k + \tfrac{1}{2})/\Gamma(\tfrac{1}{2}k).$$

The symmetrical choice (1) of the limits on the t-distribution minimizes (3).

We consider using in connection with the confidence intervals (2) linear functions

$$d_i = x_i - \sum_{j=1}^{n} c_{ij}y_j, \qquad i = 1, 2, \cdots, m. \tag{4}$$

The variables d_i have a multivariate normal distribution. Necessary and sufficient conditions that the d_i all have the same mean δ, equal variances σ^2, and zero covariances, are easily found to be

$$\sum_{j=1}^{n} c_{ij} = 1, \qquad \sum_{k=1}^{n} c_{ik}c_{jk} = c^2\delta_{ij}, \tag{5}$$

where $\delta_{ii} = 1$, $\delta_{ij} = 0$ if $i \neq j$. If (4) are used in (2), $E(l)$ is given by (3) with $\sigma^2 = \mu + c^2\nu$. Hence to minimize $E(l)$ we must find an $m \times n$ matrix $C = (c_{ij})$ satisfying (5), and for which c^2 is minimum. The minimum value of c^2 is m/n: this is easily proved by the use of vector algebra.

Let γ_i be the i-th row of C, and let ψ be the $1 \times n$ matrix $(1, 1, \cdots, 1)$. Denote the transpose of a matrix by a prime. Then the conditions (5) read

$$\gamma_i\psi' = 1, \qquad \gamma_i\gamma_j' = c^2\delta_{ij}. \tag{6}$$

First suppose vectors $\gamma_1, \gamma_2, \cdots, \gamma_m$ satisfy (6). Then it is possible to adjoin $n - m$ orthogonal vectors $\gamma_{m+1}, \cdots, \gamma_n$, so that the complete set satisfies the second group of conditions (6). Since this set is a basis in n-space,

$$\psi = \sum_{k=1}^{n} g_k\gamma_k,$$

where the g_k are scalars. Now

$$1 = \gamma_i \psi' = \sum_{k=1}^{n} g_k \gamma_i \gamma_k' = g_i c^2, \qquad i = 1, 2, \cdots, m,$$

and thus $g_1 = g_2 = \cdots = g_m = c^{-2}$. But

$$n = \psi\psi' = \sum_{k=1}^{n} g_k^2 \gamma_k \gamma_k' = mc^{-2} + \sum_{k=m+1}^{n} g_k^2 c^2 \geqq mc^{-2},$$

and hence $c^2 \geqq m/n$. On the other hand this lower bound for c^2 may be attained by taking any set $\beta_1, \beta_2, \cdots, \beta_m$ of orthogonal vectors with norms m/n, that is, $\beta_i\beta_j' = \delta_{ij}m/n$, and rotating them so that their equal angles vector $\lambda = (n/m)(\beta_1 + \beta_2 + \cdots + \beta_m)$ coincides with ψ. Then $\lambda S = \psi$, where $SS' = I$. For $\gamma_i = \beta_i S$,

$$\gamma_i\psi' = \beta_i SS'\lambda' = \beta_i\lambda' = 1,$$
$$\gamma_i\gamma_j' = \beta_i SS'\beta_j' = \beta_i\beta_j' = \delta_{ij}m/n,$$

so that equations (6) are satisfied with $c^2 = m/n$.

An especially neat solution of this minimum problem was obtained by the above method; its validity may easily be verified directly. It is

$$c_{ij} = \begin{cases} \delta_{ij}(m/n)^{\frac{1}{2}} - (mn)^{-\frac{1}{2}} + 1/n, & j \leqq m, \\ 1/n, & j > m. \end{cases}$$

Then

$$d_i = x_i - (m/n)^{\frac{1}{2}} y_i + (mn)^{-\frac{1}{2}} \sum_{j=1}^{m} y_j - \sum_{j=1}^{n} y_j/n,$$

and L and Q become simply $L = \bar{x} - \bar{y}$ and

$$Q = \sum_{i=1}^{m} (u_i - \bar{u})^2, \tag{7}$$

where

$$\bar{x} = \sum_{i=1}^{m} x_i/m, \qquad \bar{y} = \sum_{i=1}^{n} y_i/n, \qquad u_i = x_i - (m/n)^{\frac{1}{2}} y_i, \qquad \bar{u} = \sum_{i=1}^{m} u_i/m. \tag{8}$$

We may now write (2) as[3]

$$\bar{x} - \bar{y} - t_{m-1,\epsilon}\{Q/[m(m-1)]\}^{\frac{1}{2}} \leqq \delta \leqq \bar{x} - \bar{y} + t_{m-1,\epsilon}\{Q/[m(m-1)]\}^{\frac{1}{2}}. \tag{9}$$

The solution (B) mentioned at the beginning, consists of taking $c_{ij} = \delta_{ij}$ in (4), so that the conditions (5) are satisfied with $c^2 = 1$. Hence for both (B) and (9) the expected length of the confidence interval is given by (3), but with $\sigma^2 = \mu + \nu$ for (B), while $\sigma^2 = \mu + (m/n)\nu$ for (9).

[3] Obvious modifications of (9) will make it suitable for "one-sided" estimation.

3. More General Solutions. We now generalize our approach to the following extent: Let L be a linear form and Q a quadratic form in the variates $x_1, \cdots, x_m, y_1, \cdots, y_n$, with coefficients independent of the parameters (i. of p.). If for some constant h i. of p., and some function f of the parameters, $h(L - \delta)/f$ and Q/f^2 are independently distributed, the former according to the normal law with zero mean and unit variance, the latter according to the χ^2-law with $k - 1$ degrees of freedom, then the quotient

$$h(L - \delta)/[Q/(k - 1)]^{\frac{1}{2}} \tag{10}$$

will have the t-distribution with $k - 1$ degrees of freedom, no matter what the values of the parameters.

We note that necessarily then

$$E(L) = \delta, \tag{11}$$

$$f^2 = h^2E[(L - \delta)^2]. \tag{12}$$

The t-distribution of (10) leads to the confidence intervals

$$| \delta - L | \leqq t_{k-1,\epsilon}[Q/(k - 1)]^{\frac{1}{2}}/h, \tag{13}$$

where $t_{k-1,\epsilon}$ is defined by (1), and the confidence coefficient is ϵ. Proceeding as toward (3), we find that the expected length of (13) is

$$E(l) = t_{k-1,\epsilon}c_{k-1}f/h. \tag{14}$$

If
$$L = \sum_{i=1}^{m} a_i x_i - \sum_{i=1}^{n} b_i y_i,$$

$$E(L) = \alpha \sum_{i=1}^{m} a_i - \beta \sum_{i=1}^{n} b_i. \tag{15}$$

Since a_i, b_i are i. of p., it follows from (11) and (15) that

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_i = 1. \tag{16}$$

Writing
$$\xi_i = x_i - \alpha, \qquad \eta_i = y_i - \beta,$$

$$L - \delta = \sum_{i=1}^{m} a_i \xi_i - \sum_{i=1}^{n} b_i \eta_i, \tag{17}$$

$$E[(L - \delta)^2] = \mu \sum_{i=1}^{m} a_i^2 + \nu \sum_{i=1}^{n} b_i^2 = f^2/h^2 \tag{18}$$

from (12); thus (14) may be written

$$E(l) = t_{k-1,\epsilon}c_{k-1}\left[\mu \sum_{i=1}^{m} a_i^2 + \nu \sum_{i=1}^{n} b_i^2\right]^{\frac{1}{2}}. \tag{19}$$

From (18) we also have

$$f^2 = a^2\mu + b^2\nu, \tag{20}$$

where

$$a^2 = h^2 \sum_{i=1}^{m} a_i^2, \qquad b^2 = h^2 \sum_{i=1}^{n} b_i^2$$

are i. of p.

4. Lemma. We propose to prove next that the maximum value of k is m, that is to say, it is impossible to obtain a t-distribution for a quotient (10) with more than $m - 1$ degrees of freedom. For this we need a lemma to the effect that certain well known sufficient conditions for a quadratic form to have a χ^2-distribution are also necessary.

Since under our assumptions $h^2(L - \delta)^2/f^2 = \chi_1^2$ and $Q/f^2 = \chi_{k-1}^2$ are independent, therefore $Q^*/f^2 = \chi_k^2$, where

$$Q^* = h^2(L - \delta)^2 + Q.$$

To shorten the notation, write

$$z_i = \begin{cases} x_i, & i = 1, 2, \cdots, m, \\ y_{i-m}, & i = m + 1, \cdots, m + n, \end{cases}$$

$$\alpha_i = E(z_i), \qquad \zeta_i = z_i - \alpha_i, \qquad \sigma_i^2 = E(\zeta_i^2).$$

Let

$$Q = \sum_{s,t} q_{st} z_s z_t,$$

where the indices s and t range from 1 to $m + n$ throughout. Then q_{st} is i. of p., and

$$Q = \sum_{s,t} q_{st} \zeta_s \zeta_t + 2 \sum_s q_s \zeta_s + q,$$

where

$$q_s = \sum_t q_{st} \alpha_t, \qquad q = \sum_s q_s \alpha_s.$$

From (17)

$$h^2(L - \delta)^2 = \sum_{s,t} p_{st} \zeta_s \zeta_t,$$

where p_{st} are i. of p. Putting $q_{st}^* = q_{st} + p_{st}$, q_{st}^* are i. of p., and

$$Q^* = \sum_{s,t} q_{st}^* \zeta_s \zeta_t + 2 \sum_s q_s \zeta_s + q. \tag{21}$$

The moment-generating function of Q^*/f^2 is

$$\phi(\theta) = E[\exp(\theta Q^*/f^2)] = C_1 \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left(\theta Q^*/f^2 - \tfrac{1}{2} \sum_s \zeta_s^2/\sigma_s^2\right) \prod_s d\zeta_s .$$

There exists a non-singular linear transformation from the ζ's to v's such that

$$\sum_s \zeta_s^2/\sigma_s^2 = \sum_s v_s^2,$$

(22)
$$\sum_{s,t} q_{st}^* \zeta_s \zeta_t = \sum_s \lambda_s v_s^2 .$$

Then

(23)
$$\sum_s q_s \zeta_s = \sum_s p_s v_s ,$$

$$\phi(\theta) = C_2 e^{\theta q/f^2} \prod_s \int_{-\infty}^{+\infty} \exp\{-\tfrac{1}{2}[v^2 - 2\theta(\lambda_s v^2 + 2p_s v)/f^2]\}\, dv$$
$$= e^{\theta q/f^2} \prod_s (1 - 2\theta\lambda_s/f^2)^{-\frac{1}{2}} \exp\{2\theta^2 p_s^2/(f^4 - 2\theta\lambda_s f^2)\}.$$

Now $Q^*/f^2 = \chi_k^2$ if and only if

$$\phi(\theta) = (1 - 2\theta)^{-\frac{1}{2}k}.$$

Hence

(24)
$$p_s = 0, \qquad q = 0,$$

and k of the λ_s must be equal to f^2 while the remaining λ_s vanish. No generality is lost in assuming

(25)
$$\lambda_1 = \lambda_2 = \cdots = \lambda_k = f^2, \qquad \lambda_{k+1} = \cdots = \lambda_{m+n} = 0.$$

Let $w_i = fv_i$, $i = 1, 2, \cdots, k$. From equations (21) to (25) we deduce that

(26)
$$Q^* = \sum_{s,t} q_{st}^* \zeta_s \zeta_t = \sum_{i=1}^{k} w_i^2 ,$$

where q_{st}^* is i. of p., and the w_i are linear combinations of the ζ_s such that

(27)
$$E(w_i w_j) = f^2 \delta_{ij} .$$

That the conditions (26) are necessary[4] for $Q^*/f^2 = \chi_k^2$ constitutes the desired lemma.

5. Maximum Number of Degrees of Freedom. We have seen that the w_i in (26) must be of the form

(28)
$$w_i = \sum_{j=1}^{m} a_{ij} \xi_j + \sum_{j=1}^{n} b_{ij} \eta_j .$$

We substitute (28) and (20) into (27) and write the result in matrix form,

(29)
$$\mu AA' + \nu BB' = (a^2\mu + b^2\nu) I_k ,$$

where I_j is the identity matrix of order j, $A^{k\times m} = (a_{ij})$, $B^{k\times n} = (b_{ij})$, and whenever a new matrix is introduced, a superscript $r \times c$ indicates that it has r rows

[4] We have incidentally proved sufficiency.

and c columns. Now if we knew that AA' and BB' were i. of p., then we could equate coefficients of μ and ν in (29) and immediately draw the desired conclusion $k \leqq m$. But that AA', BB' are i. of p. is not obvious, since this need not be true of A and B. However, we do know that the matrices

$$F = A'A, \qquad G = A'B, \qquad H = B'B$$

are i. of p. because the matrix (q^*_{st}) of (26) is

$$\begin{pmatrix} F & G \\ G' & H \end{pmatrix}.$$

Multiplying (29) on the left by A' and on the right by A, we obtain

$$\mu F^2 + \nu GG' = (a^2\mu + b^2\nu)F. \tag{30}$$

(30) must hold identically in μ, ν. Since the coefficients of μ, ν are now i. of p., we may equate them, hence $GG' = b^2F$. Similarly multiplying (29) by B' and B, we get $G'G = a^2H$. Now[5] for any matrix M, rank M = rank $M'M$ = rank MM'. Thus rank F = rank $H = r$, say. Again, $F = A'A$, therefore r = rank $A \leqq m$. Since F is a positive[5] matrix, i. of p., there exists a non-singular $P^{m\times m}$, i. of p., such that

$$F = P'I_{m,r}P = A'A, \tag{31}$$

where $I_{j,r}$ is the $j \times j$ matrix the first r of whose diagonal elements are unity and all other elements zero. Let $T^{k\times m} = AP^{-1}$. Then

$$A = TP, \qquad T'T = I_{m,r} \tag{32}$$

from (31). Likewise we can write

$$B = U^{k\times n}R^{n\times n}, \qquad U'U = I_{n,r}, \tag{33}$$

where R is non-singular and i. of p. Then $G = A'B = P'T'UR$, hence $T'U = (P')^{-1}GR^{-1}$ is i. of p. We note

$$T = (T_1^{k\times r}, 0^{k\times(m-r)}), \qquad U = (U_1^{k\times r}, 0^{k\times(n-r)}),$$

where

$$T_1'T_1 = U_1'U_1 = I_r.$$

Since

$$T'U = \begin{pmatrix} T_1' \\ 0 \end{pmatrix}(U_1, 0) = \begin{pmatrix} T_1'U_1 & 0 \\ 0 & 0 \end{pmatrix}$$

[5] A simple proof [5b] of this useful theorem is the following: Let r = rank M, p = rank $M'M$. $p \leqq r$ since the rank of the product cannot exceed the rank of a factor. M contains r independent column vectors; the Grammian matrix of these vectors is non-singular and appears as an $r \times r$ minor in $M'M$. Hence $p \geqq r$. Furthermore, all principal minors of $M'M$ are Grammian matrices (which always have non-negative determinants), hence $M'M$ is always positive—we use this below.

is i. of p., so is its minor $V^{r\times r} = T_1'U_1$. Write

$$P = \begin{pmatrix} P_1^{r\times m} \\ P_2^{(m-r)\times m} \end{pmatrix}, \qquad R = \begin{pmatrix} R_1^{r\times n} \\ R_2^{(n-r)\times n} \end{pmatrix}.$$

Then from (32), (33),

$$A = T_1P_1, \qquad B = U_1R_1. \tag{34}$$

Substituting (34) in (29), we get

$$\mu T_1P_1P_1'T_1' + \nu U_1R_1R_1'U_1' = (a^2\mu + b^2\nu)I_k, \tag{35}$$

and multiplying by T_1' on the left, T_1 on the right,

$$\mu P_1P_1' + \nu VR_1R_1'V' = (a^2\mu + b^2\nu)I_r.$$

Again the coefficients of μ, ν are i. of p., so

$$P_1P_1' = a^2I_r,$$

$$VR_1R_1'V' = b^2I_r. \tag{36}$$

Similarly we find

$$R_1R_1' = b^2I_r. \tag{37}$$

From (36), (37), $VV' = I_r$. (35) now becomes

$$a^2\mu T_1T_1' + b^2\nu U_1U_1' = (a^2\mu + b^2\nu)I_k. \tag{38}$$

Multiplication of (38) on the right by U_1 gives

$$a^2\mu T_1V + b^2\nu U_1 = (a^2\mu + b^2\nu)U_1.$$

Hence $T_1V = U_1$, therefore $U_1U_1' = T_1T_1'$, and putting this back into (38) we have $I_k = T_1T_1'$, rank I_k = rank T_1T_1' = rank $T_1'T_1$ = rank I_r, $k = r \leqq m$.

6. Minimum Expected Length of Confidence Intervals. We now point out that of all confidence intervals (13) with $k = m$, the confidence intervals (9) have the minimum expected length. Recalling that the a_i, b_i in (19) are subject to the conditions (16), we easily find

$$\sum_{i=1}^{m} a_i^2 \geqq 1/m, \qquad \sum_{i=1}^{n} b_i^2 \geqq 1/n. \tag{39}$$

From (39) and (19) we have

$$E(l) \geqq t_{m-1,\epsilon}c_{m-1}[\mu + (m/n)\nu]^{\frac{1}{2}}/m^{\frac{1}{2}},$$

and referring to the statement at the end of section 2, the property of (9) asserted above is now obvious.

7. Asymptotic Shortness of Confidence Intervals. In conclusion we wish to compare our results with the case where the ratio of the variances, $\theta = \nu/\mu$, is known. If

$$S_x = \sum_{i=1}^{m} (x_i - \bar{x})^2, \qquad S_y = \sum_{i=1}^{n} (y_i - \bar{y})^2,$$

$$L = \bar{x} - \bar{y}, \qquad \sigma_L^2 = (\mu/m) + (\nu/n),$$

then $(L - \delta)/\sigma_L$, S_x/μ, S_y/ν are mutually independently distributed, the first normally with zero mean and unit variance, and $S_x/\mu = \chi^2_{m-1}$, $S_y/\nu = \chi^2_{n-1}$. Hence

$$(L - \delta)[(\mu m^{-1} + \nu n^{-1})(S_x\mu^{-1} + S_y\nu^{-1})/(m + n - 2)]^{-\frac{1}{2}} = t_{m+n-2}.$$

TABLE I

Values of R for $\epsilon = .95$

$m-1$ \ $n-1$	5	10	20	40	∞
5	1.15	1.20	1.23	1.25	1.28
10		1.05	1.07	1.09	1.11
20			1.03	1.03	1.05
40				1.01	1.02

TABLE II

Values of R for $\epsilon = .99$

$m-1$ \ $n-1$	5	10	20	40	∞
5	1.27	1.36	1.42	1.47	1.52
10		1.10	1.13	1.16	1.20
20			1.05	1.06	1.09
40				1.02	1.04

This relation yields the confidence intervals

(40) $$|\delta - L| \leqq t_{m+n-2,\epsilon}(m + n - 2)^{-\frac{1}{2}}(m^{-1} + \theta n^{-1})^{\frac{1}{2}}(S_x + S_y/\theta)^{\frac{1}{2}},$$

where the confidence coefficient is again ϵ. The confidence intervals (40) are known to be highly efficient; for instance they are of the shortest unbiased type [5a]. We calculate their expected length to be

$$E(l) = t_{m+n-2,\epsilon}c_{m+n-2}[\mu + (m/n)\nu]^{\frac{1}{2}}/m^{\frac{1}{2}}.$$

The ratio R of $E(l)$ for (9) to $E(l)$ for (40) is thus

(41) $$R = (t_{m-1,\epsilon}c_{m-1})/(t_{m+n-2,\epsilon}c_{m+n-2}).$$

As $k \to \infty$, $c_k \to 2$, $t_{k,\epsilon} \to t_{\infty,\epsilon}$, hence as $m \to \infty$, $R \to 1$ no matter what the values of $n \geqq m$. For small values of m the ratio of the t values in (41) is considerably >1, but this is partly offset by c_k approaching its limiting value 2 from below so that the ratio of the c's is <1. The behaviour of R for finite m is indicated in Tables I and II. Table I (II) tells us for example that with $m > 10$, and $\epsilon = .95$ (.99), the expected length of the confidence intervals (9) is at most 11 per cent (20%) longer than that of the optimum confidence intervals (40) available when the ratio θ is known. While we may conclude from $R \to 1$ as $m \to \infty$, that our solution (9) is asymptotically extremely efficient, we cannot conclude from Tables I, II that for small m (9) is inefficient, since we do not know what the lengthening effect of the extra nuisance parameter in the Behrens-Fisher problem would be on "best" confidence intervals.

8. Summary. In the terminology of the first paragraphs of sections 1 and 3 we have proved that there do not exist a linear form L and a quadratic form Q in the observations such that the quotient (10) will have the t-distribution (for *all* values of the parameters) with more than $m - 1$ degrees of freedom. We have further shown that of all confidence intervals (13) based on the t-distribution with $m - 1$ degrees of freedom, and with confidence coefficient ϵ, (9) has the minimum expected length. The quantities needed to apply our solution (9) are given by (1), (7) and (8). Finally, by comparing this solution with a known highly efficient solution for the case when the ratio of the population variances is known, it has been possible to show that at least asymptotically our confidence intervals (9) are very short.

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