

ON SOLVABLE GROUPS OF FINITE MORLEY RANK

ALI NESIN

ABSTRACT. We investigate solvable groups of finite Morley rank. We find conditions on G for G' to split in G . In particular, if G' is abelian and $Z(G) = 1$ we prove that $G = G' \rtimes T$ for some T and the ring $\mathbb{Z}[T]/\text{ann } G'$ is interpretable in G . We exploit the methods used in proving these results to find more information about solvable groups.

1. INTRODUCTION

The concept of Morley rank was introduced by Morley in 1965 [Mo]. Roughly speaking, in a structure of finite Morley rank every definable subset is endowed with two integers: the Morley rank and the Morley degree. The first number behaves like the dimension and the second one like the number of irreducible components of constructible sets of algebraic geometry (see, e.g., [Sp] for the definition of a constructible set). [La, Pi, Po1, Po2] give a detailed treatment of structures of finite Morley rank. In [Th1] and [Ne1], the reader may find a faster introduction to the subject as far as groups are concerned.

In the 1970s, Cherlin and Zil'ber independently discovered that groups of finite Morley rank behave like geometric groups and in view of Macintyre's result (infinite fields of finite Morley rank are algebraically closed) they made the following conjecture:

Conjecture. Infinite simple groups of finite Morley rank are algebraic groups over algebraically closed fields.

We are still far away from proving this conjecture. But considerable progress has been made in the last few years. First of all, Thomas proved the conjecture for locally finite groups using the classification of finite simple groups [Th2]. Second, Zil'ber showed that these groups look very much like algebraic groups [Zi2]. In §3 we will give a partial list of his results.

In an algebraic group a Borel subgroup is a maximal (algebraically) connected solvable subgroup. Borel subgroups play a very important role in the study of algebraic groups. Similarly, one may define a Borel subgroup of a group of finite Morley rank as a maximal (model theoretically) connected solvable subgroup and

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expect that one day they will be important in solving the above conjecture. We can justify our faith by saying that we know more about solvable groups of finite Morley rank than about simple groups (unless they are nilpotent, but Borels are supposed to be centerless). In this article we investigate these solvable groups and in the particular case where the group is solvable of class 2 we show strong resemblances with algebraic groups.

This paper may be seen as a sequel to [Ne2]. But it is self-contained.

In §2 we fix our group theoretical notation. We also state some basic theorems to refresh the model theorist's memory.

In §3 we do the same for model theory of groups. In that section we state some known facts that will be used in later sections.

In §4 we define the socle of a solvable group and investigate its basic properties; in particular, we look closely at the action of the group on its socle. This action was completely described in [Ne2]. We survey the results here.

In §5 we prepare the inductive step to prove two theorems and we prove them:

Theorem 1. *If G is a connected, solvable group of class 2 and of finite Morley rank and if $G' \cap Z(G) = 1$, then there is a definable subgroup T of G for which*

$$G = G' \rtimes T.$$

Theorem 2. *If G is a connected, solvable group of finite Morley rank and if for some $g \in G$, $C_{G'}(g) = 1$, then there is a definable subgroup T of G for which*

$$G = G' \rtimes T.$$

We give several examples to illustrate the theorems and Example 7 shows that the supplementary hypotheses of these two theorems ($G' \cap Z(G) = 1$ and $C_{G'}(g) = 1$) are crucial.

In the next section we investigate in detail solvable groups of class 2. We imbed each connected, centerless, solvable group of class 2 of finite Morley rank in another group with the same properties (Theorem 3). But this bigger group looks more like an algebraic group than the one we started with. We call it the full group of G and denote it by \tilde{G} . Then we look at \tilde{G} more attentively. We prove the following.

Theorem 4. *$\tilde{G} = \bigoplus_{i=1}^l G'_i \rtimes R_i^*$ where $G' = \bigoplus_{i=1}^l G'_i$ and each G'_i is a definable normal subgroup of G , R_i is a connected, local, Artinian and Noetherian ring with identity and of finite Morley rank.*

There is a complete classification of these rings (see Cohen [Co], also [Sa]).

The next theorem gives a smaller imbedding of G .

In §7, we define the Fitting subgroup of a group of finite Morley rank as being the maximal connected nilpotent normal subgroup. The main result of this section is the following theorem.

Theorem 6. *Let G be a solvable connected group of finite Morley rank. If F is the Fitting subgroup of G , then G/F is a divisible group.*

In §8 we look at the Fitting subgroup F of a centerless, connected solvable algebraic group. We show that the unipotent subgroup U is definable (Theorem 7) and if the language is the pure language of groups then $U = F$. This result is proved with Anand Pillay.

In the final section we show that if G is a connected, centerless, solvable group of finite Morley rank whose definable connected proper subgroups are nilpotent, then $G = k \rtimes T$ where $T < k^*$ and k is an algebraically closed field (Theorem 7). This generalizes Cherlin’s classification of centerless groups of Morley rank 2 (see [Ch]). We hope this result will be helpful in proofs by induction.

2. GROUP THEORETICAL AND ALGEBRAIC BACKGROUND

G will always stand for a group. We will denote its identity element by 1. $Z(G)$ stands for its center. If H is a subgroup of G we write $H < G$. If H is normal in G then we denote this fact by $H \triangleleft G$. If $x, h \in G$ then $x^h = h^{-1}xh$. Clearly $(x^h)^g = x^{hg}$. If $n \in \mathbf{N}$, x^{nh} means $(x^n)^h$ or $(x^h)^n$. If $X \subseteq G$, $H < G$ and if for all $x \in X$, $h \in H$, $x^h \in X$, then we will say that X is H -normal. $[x, h]$ stands for $x^{-1}h^{-1}xh$. If X is an abelian subgroup which is normalized by H then $[xy, h] = [x, h][y, h]$ for all $x, y \in X$, all $h \in H$. $[X, H]$ denotes the subgroup of G generated by the set $\{[x, h] : x \in X, h \in H\}$. The subgroups $G^i, G^{(i)}$ are defined inductively.

$$\begin{aligned} G^0 &= G^{(0)} = G, \\ G^{i+1} &= [G, G^i], \\ G^{(i+1)} &= [G^{(i)}, G^{(i)}]. \end{aligned}$$

These subgroups are normal. We have $G^1 = G^{(1)} = [G, G]$. We denote it by G' . G' is called the derived subgroup of G . It is the smallest subgroup H of G for which G/H is abelian. Any subgroup of G that contains G' is normal. We have $G^{i+1} \subseteq G^i$ and $G^{(i+1)} \subseteq G^{(i)}$. A group G is said to be solvable if $G^{(i)} = 1$ for some i . If i happens to be the least integer for which $G^{(i)} = 1$, then G is called solvable of class i . If G is solvable then $G \neq G'$. A group G is said to be nilpotent if $G^i = 1$ for some i . Define the i th center of G , $Z_i(G)$, by induction on i .

$$\begin{aligned} Z_0 &= Z_0(G) = \{1\}, \\ Z_{i+1} &= Z_{i+1}(G) = \{g \in G : [G, g] \subseteq Z_i(G)\}. \end{aligned}$$

Then $Z_i(G) \triangleleft G, Z_i \subseteq Z_{i+1}$, and

$$Z_{i+1}/Z_i = Z(G/Z_i).$$

G is nilpotent iff $Z_i(G) = G$ for some i . Thus a nilpotent group has nontrivial center. Every nontrivial normal subgroup of a nilpotent group intersects the center nontrivially. Subgroups and quotients of nilpotent (resp. solvable) groups

are nilpotent (resp. solvable). If H and G/H are solvable then so is G (here $H \triangleleft G$). Upper triangular matrices with nonzero determinant form a solvable group which is not nilpotent. Strictly upper-triangular matrices (1's on the diagonal) form a nilpotent group.

If $A \triangleleft H$ are subgroups of a group G , then $(H/A)' = H'A/A$.

If $X, Y \subseteq G$ then we denote by XY the subset $XY = \{xy : x \in X, y \in Y\}$.

If $(X_i)_{i \in I}$ are subsets of G , $\langle X_i \mid i \in I \rangle$ denotes the group generated by all the X_i 's. It is the smallest subgroup of G containing $\bigcup_i X_i$. If I is a finite set $\{1, \dots, n\}$, we write $\langle X_1, \dots, X_n \rangle$. If $H \triangleleft G$, $K < G$, then $\langle H, K \rangle = \{hk \mid h \in H, k \in K\}$. We denote it by HK . If $K \triangleleft G$ also (and H and K are nilpotent) then $HK \triangleleft G$ (and HK is nilpotent).

We say that a group G is the semidirect product of two subgroups H and K if

- (i) $H \triangleleft G$,
- (ii) $G = HK$,
- (iii) $H \cap K = \{1\}$.

In this case every element of g can be written uniquely as hk , for $h \in H$, $k \in K$. We write $G = H \rtimes K$ if G is the semidirect product of H and K (in that order). If $G = H \rtimes K$ then we have a homomorphism

$$\varphi : K \rightarrow \text{Aut } H$$

given by

$$\varphi(k)(h) = h^{k^{-1}}.$$

We say that a group K acts on a group H if a homomorphism

$$\varphi : K \rightarrow \text{Aut } H$$

is given. In that case we can construct a group G which is isomorphic to $H \rtimes K$ as follows. We define

$$G = H \times K$$

as a set. The product is defined as follows:

$$(h, k)(h', k') = (h \cdot \varphi(k)h', kk').$$

It is easy to check that G is a group and it is the semidirect product of $H \times \{1\}$ and $\{1\} \times K$. The best way to visualize G is as follows: think of G as the set

$$\left\{ \begin{pmatrix} k & h \\ 0 & 1 \end{pmatrix} : k \in K, h \in H \right\}$$

and the multiplication as the matrix multiplication (kh' meaning $\varphi(k)h'$). If $H = k^+$, $K = k \setminus \{0\}$ where k is a field, we obtain the affine group in this way.

In this case the action is multiplication:

$$\varphi(\lambda)a = \lambda a$$

for $\lambda \in k^*$, $a \in k$.

If $G = H \rtimes K$ for some subgroup K , we say that H splits in G . If H and K are abelian then G is solvable of class at most 2.

If G is a solvable, (algebraically) connected linear algebraic group then G is conjugate to upper-triangular matrices (Lie-Kolchin theorem; see, e.g., [Sp], Chapter 6). Suppose G is in upper-triangular form. Then the set of strictly upper-triangular matrices of G is denoted by U . U is called the unipotent subgroup (or radical) of G . It is clearly a normal subgroup of G . U is nilpotent and it contains G' . U is a connected subgroup of G . The set of diagonalizable matrices of G contains a closed connected subgroup T which is maximal for these properties. It is called a maximal torus. One has

$$G = U \rtimes T.$$

If $H \subseteq K$ are subgroups of G , $[K : H]$ will denote the index of H in K , i.e., $[K : H] = \text{Card}(K/H)$.

If H, K are subgroups of G , $C_H(K)$ will mean the centralizer of K in H , i.e., $C_H(K) = \{h \in H : hk = kh \text{ for all } k \in K\}$. $C_H(K)$ is a subgroup of G . If K and H are normal in G then so is $C_H(K)$.

If R is a ring, I an ideal of R , we write $I \triangleleft R$. R^* stands for the set of invertible elements of R . It is a group under multiplication.

$\mathbf{Z}[G]$ denotes the group ring of G over \mathbf{Z} . It is defined as the formal set

$$\left\{ \sum_{i=1}^k n_i g_i : k \in \mathbf{N}, n_i \in \mathbf{Z}, g_i \in G \right\}$$

with the obvious addition and multiplication. If G is an Abelian group acting on another abelian group A , then we can extend this action to $\mathbf{Z}[G]$ by (denoting A additively)

$$\left(\sum n_i g_i \right) (a) = \sum n_i g_i a$$

where $g_i a$ is the element of A which is the result of the g_i action on a . If $\lambda = \sum n_i g_i$, the length of λ is $l(\lambda) = \sum |n_i|$.

Let M be an R -module. Let $X \subseteq M$. Then the annihilator of X in R is

$$\text{ann } X = \text{ann}_R X = \{r \in R \mid rX = 0\}.$$

This is a left-ideal of R . If X is a submodule then $\text{ann } X \triangleleft R$. Then X becomes an $R/\text{ann } X$ -module in the obvious way.

An ideal I of a ring R is said to be nilpotent if for some integer n , $I^n = 0$, i.e., if for any $x_1, \dots, x_n \in I$ we have $x_1 \cdots x_n = 0$.

A commutative ring R with identity is local if it has a unique maximal ideal M . Then $R \setminus M = R^*$ and R/M is a field.

If $I \triangleleft R$, I^n denotes the ideal generated by $x_1 \cdots x_n$ for all $x_1, \dots, x_n \in I$. Clearly

$$I^n = \left\{ \sum_{k=1}^m x_{k_1} \cdots x_{k_{n_k}} : m \in \mathbf{N}, x_{ij} \in I \right\}.$$

\mathbb{Z}_p^∞ denotes the Prüfer group, i.e.,

$$\mathbb{Z}_p^\infty = \{z \in \mathbb{C} : z^{p^n} = 1 \text{ for some } n\}.$$

3. MODEL THEORY OF GROUPS. BACKGROUND

If M is a structure for a language \mathcal{L} , by a definable subset of M we will mean a subset which is first-order definable using parameters from M .

Let G be a group of finite Morley rank. So to each definable subset X of G we attached two integers: r and d . r will be called the (Morley) rank, d the (Morley) degree of X . We will write $\text{rk } X = r$, $\text{deg } X = d$.

If $H \subseteq K$ are definable subsets of G then $\text{rk } H \leq \text{rk } K$. If they have the same rank then $\text{deg } H \leq \text{deg } K$. If H and K are also subgroups and if $\text{rk } H = \text{rk } K$, then $\text{deg } K = [K : H] \text{deg } H$. In particular, $K = H$ iff $\text{rk } K = \text{rk } H$, $\text{deg } K = \text{deg } H$. This shows that a group of finite Morley rank satisfies descending chain condition on definable subgroups. This has an important consequence: if H is a definable subgroup then

$$H^\circ = \bigcap \{K : K < G, K \text{ definable}, K \subseteq H, [H : K] < \infty\}$$

is the intersection of finitely many such subgroups, thus has finite index in H . H° is called the connected component of H . It is the smallest subgroup of H which is definable and has finite index in H . In other words, H° has no definable proper subgroups of finite index, and it is the biggest such definable subgroup of G . If $H < G$, then $H^\circ < G$ also. Cherlin [Ch] proved that $\text{deg } H$ is exactly $[H : H^\circ]$. Thus $\text{deg } H^\circ = 1$. H is connected iff $\text{deg } H = 1$, equivalently if $H = H^\circ$. The sentence “ H is connected” will always mean “ H is definable and connected.”

If $K < H$ are definable subgroups and if K and H/K are connected then H is connected. If $H < G$ is definable and infinite then $\text{rk } G > \text{rk } G/H$. Definable subgroups of G/H (in the G -language) are of the form K/H where K is a definable subgroup of G containing H .

We will now list some of the known facts that we will repeatedly use in the sequel. The proof of most of them can be found in [Po2]. Several of them are proved in more generality.

Fact 1 (Macintyre [B.Ch.Mac]). Let G be a group of finite Morley rank. Let H be a connected subgroup of G . Then any finite, H -normal subset of G is in the centralizer of H .

Fact 2 (Zil'ber [Z2]). Let $A_i < G$ be a family of connected subgroups of a group G of finite Morley rank. Then the subgroup

$$H = \langle A_i : i \in I \rangle$$

is connected. In fact, for some $i_1, \dots, i_n \in I$

$$H = A_{i_1} \cdots A_{i_n}.$$

If each A_i is Abelian and they commute with each other (i.e., if H is Abelian), we may denote the multiplication in H additively and write

$$H = A_{i_1} + \cdots + A_{i_n}.$$

Fact 3 (Zil'ber [Zi2]). Let G be a group of finite Morley rank. Let $H < G$ be a connected subgroup of G . If $X \subseteq G$ is any subset of G then the subgroup $[H, X]$ is connected.

Fact 4 (Zil'ber [Zi2]). If G is a connected group of finite Morley rank then G^i , $G^{(i)}$ are connected.

Fact 5 (Zil'ber [Zi1]/Nesin [Ne2]). Let G be a connected solvable group of finite Morley rank. Then G' is nilpotent (and connected by Fact 4).

Fact 6 (Macintyre [Mac2]). An infinite field of finite Morley rank is an algebraically closed field. It is also connected.

Fact 7 (Macintyre [Mac1]). An Abelian group G of finite Morley rank is of the form

$$G = D \oplus B$$

where D is a divisible subgroup and B is a subgroup of bounded exponent. Furthermore, for a given integer n , D has only finitely many elements of order n .

Fact 8 (Zil'ber [Zi3]; see also [Po2 or Ta]). Let G be a group of finite Morley rank. Let $H < G$ be an arbitrary subgroup of G . Denote by \overline{H} the smallest definable subgroup of G containing H . If H is nilpotent (resp. solvable) then so is \overline{H} .

Recall that an element $e \in R$ of a ring R is idempotent if $e \neq 0$ and $e^2 = e$. An idempotent is atomic if for any other idempotent f , either $ef = 0$ or $ef = e$.

Fact 9 (Cherlin-Reineke [Ch-R]). Let R be a commutative ring of finite Morley rank with an identity. Then R has finitely many atomic idempotents, say e_1, \dots, e_n . These idempotents give rise to a canonical decomposition of R (as a ring):

$$R = Re_1 \oplus \cdots \oplus Re_n.$$

Each $R_i = Re_i$ is a local noetherian ring with definable nilpotent maximal ideal M_i . M_i^n/M_i^{n+1} is a finite-dimensional vector space over R_i/M_i . If R_i is infinite then R_i/M_i is an algebraically closed field (by Fact 6).

4. THE SOCLE

For the rest of the article, unless otherwise stated, we assume that G is a connected, solvable group of finite Morley rank in which $Z(G) \cap G' = 1$. In particular, G is not nilpotent. Normal and definable subgroups of G which are

minimal for these properties will be called G -minimal subgroups. An infinite G -minimal subgroup is necessarily connected. If it is finite it must be central by Fact 1. Thus finite ones intersect G' trivially. For infinite ones: either they are in G' or they intersect G' trivially. In the latter case they are central.

Let $S = S_1(G)$ be the subgroup generated by all the G -minimal subgroups of G' . S will be called the socle of G . It is clearly a G -normal subgroup of G' .

Proposition 1. $S = A_1 \oplus \cdots \oplus A_n$ for some G -minimal subgroups A_i of G' . In particular, S is definable and connected. Furthermore $S \subseteq Z(G')$.

Proof. By definition of the socle we have $S = \langle A_i : A_i \text{ a } G\text{-minimal subgroup of } G' \rangle$. Since each A_i is infinite (if not $A_i \subseteq Z(G) \cap G' = 1$) they are connected (if not $A_i^\circ \triangleleft G$ and $A_i^\circ \not\subseteq A_i$). Since G' is nilpotent (Fact 5) $Z(G') \cap A_i \neq 1$. A_i being G -minimal, this forces A_i to be in $Z(G')$. Thus $S \subseteq Z(G')$. Now we apply Fact 2, to get

$$(1) \quad S = A_1 + \cdots + A_n$$

for some G -minimal A_i 's. For $i = 2, \dots, n$, consider the subgroup

$$A_i \cap (A_1 + \cdots + A_{i-1}).$$

It is definable and normal. Since A_i is G -minimal either $A_i \subseteq A_1 + \cdots + A_{i-1}$, in which case we may delete A_i from (1), or $A_i \cap (A_1 + \cdots + A_{i-1}) = 1$. Starting from $i = 2$ and deleting unnecessary A_i 's we get a direct sum. \square

We now fix our notation. For the rest of the article we will write

$$(2) \quad S = A_1 \oplus \cdots \oplus A_n$$

where each A_i is a G -minimal subgroup in $Z(G')$.

Lemma 2. Any nontrivial G -normal subgroup of G' intersects S nontrivially.

Proof. Let X be such a subgroup. Then the subgroup $[G, X]$ is connected by Fact 3. Since $X \triangleleft G$, $[G, X] \subseteq X$. Also $[G, X] \neq 1$ because if not $X \subseteq G' \cap Z(G) = 1$. Thus $[G, X]$ contains a G -minimal subgroup A which must be in S . Thus $A \subseteq S \cap X$. \square

Let us show by an example that even if G' is abelian S may be different from G' .

Example 1. Let K be an algebraically closed field. Let

$$G = (K \times K) \rtimes (K[x]/x^2)^*$$

where the action of $K[x]/x^2$ on $K \times K$ is given by

$$(\alpha + \beta x)(a, b) = (\alpha a, \beta a + \alpha b),$$

for $\alpha + \beta x \in K[x]$, $(a, b) \in K \times K$. G is an algebraic group. A matrix representation of G is

$$G = \left\{ \begin{pmatrix} \alpha & \beta & 0 & b \\ 0 & \alpha & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \alpha, \beta, a, b \in K, \alpha \neq 0 \right\}.$$

$Z(G) = 1$, $G' = K \times K$, the socle of G is $0 \times K$.

Zil'ber proved in [Zi2] that each A_i is isomorphic to the additive group $(F_i, +)$ of an algebraically closed field F_i . He also obtained the multiplication of F_i in G . We will now summarize the construction of the field $(F_i, +, \cdot)$.

A_i is a normal abelian subgroup of G . Thus G acts on A_i by conjugation. We can extend this action to the group ring $\mathbf{Z}[G]$ by

$$(3) \quad \left(\sum_{i=1}^k n_i g_i \right) (a) = \prod_{i=1}^k a^{n_i g_i}$$

for $\sum n_i g_i \in \mathbf{Z}[G]$, $a \in A_i$. Since A_i is abelian, the order of multiplication in formula (3) is not important. Now A_i becomes a $\mathbf{Z}[G]$ -module. Since A_i is G -minimal, A_i is an irreducible $\mathbf{Z}[G]$ -module. Let us divide $\mathbf{Z}[G]$ by the annihilator of A_i . Thus A_i is an irreducible, faithful $\mathbf{Z}[G]/\text{Ann } A_i$ -module. Set

$$(4) \quad F_i = \mathbf{Z}[G]/\text{Ann } A_i.$$

F_i is a commutative ring with 1. By Schur's lemma F_i has no 0-divisors. In fact, if $\lambda \in F_i$, $\text{Ker } \lambda = \{a \in A_i : \lambda a = 0\}$ is a G -normal subgroup of A_i . Thus if $\lambda \neq 0$, $\text{Ker } \lambda$ is finite thus 0. Therefore each $\lambda \in F_i \setminus \{0\}$ is an automorphism of A_i and is determined uniquely by λa for a fixed $a \in A_i \setminus \{0\}$. Zil'ber proved that F_i is an interpretable field in G . More precisely, he proved that for some fixed integer k_i , any $\lambda \in F_i$ has a representative μ in $\mathbf{Z}[G]$ of length $\leq k_i$ (we remind the reader that the length of $\mu = \sum n_i g_i \in \mathbf{Z}[G]$, $l(\mu)$, is $\sum |n_i|$). Thus by Fact 6, F_i is an algebraically closed field. In the sequel, the phrase "by Zil'ber" will refer to the above result.

Since for $a \in A_i \setminus \{0\}$, $F_i a$ is a definable G -normal subgroup of A_i , we have $F_i a = A_i$. Since $\text{Ann}_{F_i} a = 0$ we have

$$(F_i, +) \simeq A_i.$$

Clearly $G/C_G(A_i)$ imbeds in F_i^* and generates F_i additively. The conjecture is that $G/C_G(A_i) = F_i^*$. This is saying that the fixed integer k_i is 1.

Now we give three examples to illustrate Zil'ber's method of finding the field. The examples will show that the construction may give rise to a field even if the group is unstable.

Example 2. This is the easiest example to check Zil'ber's construction; let K be any field and define

$$G = K \rtimes K^* = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} : a \in K, b \in K^* \right\}.$$

Then

$$A_i = S = G' = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in K \right\}.$$

Thus $n = 1$. If T is the set of diagonal matrices we have $T \simeq K^*$. In this example the integer k is 1. G is ω -stable iff K is algebraically closed, in which case the Morley rank of G is 2 (assuming the language is $\{\cdot, {}^{-1}, \text{id}\}$).

Example 3. Let $H = \{\rho e^{2\pi i\sigma} : \rho \in \mathbf{R}^*, \sigma \in \mathbf{Q}\}$. H is a subgroup of (\mathbf{C}^*, \cdot) . Notice that $\mathbf{R}^*, i\mathbf{R}^* \subseteq H$. Thus any element of \mathbf{C} can be written (not uniquely) as a sum of two elements of H . How consider the group

$$G = \mathbf{C} \rtimes H = \left\{ \begin{pmatrix} 1 & a \\ 0 & h \end{pmatrix} : a \in \mathbf{C}, h \in H \right\}.$$

Again $A_i = S = G' \simeq \mathbf{C}$, $n = 1$. But now the integer k is 2. However, G can be shown not to have finite Morley rank. It is even unstable (one can define the field \mathbf{R}), as James Loveys and David Marker have recently shown.

Example 4. Let S^1 be the unit circle in \mathbf{C} , i.e.,

$$S^1 = \{\beta \in \mathbf{C} : |\beta| = 1\}.$$

Then $S^1 < (\mathbf{C}, \cdot)$ and every element of \mathbf{C} can be written as a sum of finitely many elements of S^1 . But there is no bound in the number of elements of S^1 that we need to sum in order to get \mathbf{C} . Define

$$G = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} : \alpha \in \mathbf{C}, \beta \in S^1 \right\} \simeq \mathbf{C} \rtimes S^1.$$

If we do Zil'ber's construction we find $A_i = S = G'$, thus $n = 1$, but now $k = \infty$. Thus G does not have finite Morley rank. In this case one can prove that G is unstable by interpreting the field \mathbf{R} in G . Recently Marker generalized this result [Ma].

Above we explained G -actions on G -minimal subgroups. We may also look at the G -action on $S = A_1 \oplus \dots \oplus A_n$. We know that if we restrict the action to any of the G -minimal subgroups A_i we find a field. But it may be that these restricted actions on A_i are not independent. In [Ne2] we found all possible actions of G on S . We will now explain this result.

We extend the action of G on S to $\mathbf{Z}[G]$ as in (3). Now S becomes a $\mathbf{Z}[G]/\text{Ann } S$ -module. Let

$$(5) \quad F = \mathbf{Z}[G]/\text{ann } S.$$

F also acts on each A_i . We clearly have natural isomorphisms

$$(6) \quad F/\text{ann}_F A_i \simeq F_i.$$

But this is not enough to conclude that F is interpretable in G . We proved in [Ne2] that F is interpretable anyway. We need some notation before going further. Let

$$(7) \quad M_i = \text{ann}_F A_i.$$

Since $F/M_i \simeq F_i$ and F_i is a field, M_i is a maximal ideal of F . Define an equivalence relation on the index set $\{1, \dots, n\}$ by

$$(8) \quad i \sim j \Leftrightarrow M_i = M_j \Leftrightarrow M_i \subseteq M_j.$$

thus if $i \sim j$ then $F_i = F_j$. (The reader should be aware that this is an equality, more than an isomorphism.) Furthermore, in this case F_i acts on $A_i \oplus A_j$ by scalar multiplication; i.e., for $(a_i, a_j), (a'_i, a'_j)$ in $A_i \oplus A_j$, for $\lambda \in F$, $\lambda(a_i, a_j) = (a'_i, a'_j)$ iff for some $\mu \in F_i = F_j$ we have $a'_i = \mu a_i, a'_j = \mu a_j$.

On the other hand, if $i \not\sim j$ then F acts on $A_i \oplus A_j$ "as freely as possible"; i.e., for $(a_i, a_j), (a'_i, a'_j) \in A_i \oplus A_j$, there is a $\lambda \in F$ for which $\lambda(a_i, a_j) = (a'_i, a'_j)$ iff for some $\mu_i \in F_i, \mu_j \in F_j$

$$a'_i = \mu_i a_i, \quad a'_j = \mu_j a_j.$$

In other words,

$$\begin{aligned} i \sim j &\Leftrightarrow F/\text{ann}(A_i \oplus A_j) \simeq F_i = F_j, \\ i \not\sim j &\Leftrightarrow F/\text{ann}(A_i \oplus A_j) \simeq F_i \times F_j. \end{aligned}$$

Let us give examples of both cases.

Example 5. Let $G = (K \times K) \rtimes K^*$ where $\alpha \in K^*$ acts on $K \times K$ by

$$\alpha(a, b) = (\alpha a, \alpha^2 b).$$

Then G is an algebraic group. It is connected, solvable, and centerless. $G' = K \times K, S = G', n = 2, A_1 = K \times \{0\}, A_2 = \{0\} \times K, F_1 \simeq K \simeq F_2$. Let us find F . Let us first suppose that $\text{char} K \neq 2$. Let $i \in K$ be an element for which $i^2 = -1$. Imbed K^* in G and $\mathbf{Z}[K^*]$ in $\mathbf{Z}[G]$. Consider the element $(i) + (1)$ of $\mathbf{Z}[G]$ (which is different from the element $(1 + i) \in K$ of $\mathbf{Z}[G]$). If $(a, b) \in K \times K$ then

$$((i) + (1))(a, b) = i(a, b) + 1(a, b) = (ia, -b) + (a, b) = ((1 + i)a, 0).$$

Thus $(i) + (1)$ acts as the 0 endomorphism on $A_2 = \{0\} \times K$ but not on A_1 .

Next consider the element $(1) + (-1)$ of $\mathbf{Z}[G]$. If $(a, b) \in K \times K$ then

$$((1) + (-1))(a, b) = 1(a, b) + (-1)(a, b) = (a, b) + (-a, b) = (0, 2b).$$

Thus $(1) + (-1)$ acts as the 0 endomorphism on A_1 but not on A_2 . Therefore in this case we have two equivalence classes. By using these two endomorphisms it is easy to show that $F \simeq K \times K$ and it acts on $S = K \times K$ componentwise.

If $\text{Char } K = 2$, then G is isomorphic to the group described in the next example.

Example 6. Let $G = (K \times K) \rtimes K^*$ with the following action:

$$\alpha(a, b) = (\alpha a, \alpha b)$$

for $\alpha \in K^*$, $(a, b) \in K \times K$. We have $S = G' = K \times K$, $n = 2$, $A_1 = \{0\} \times K$, $A_2 = K \times \{0\}$. $F_1 \simeq K \simeq F_2$. But in this case $F \simeq K$ also (because we have only one equivalence class).

In [Ne2] we showed that these two examples are essentially the only possible actions of $\mathbf{Z}[G]$ on S . Let us make this more precise.

We first reindex

$$(9) \quad S = (A_{11} \oplus \cdots \oplus A_{1m_1}) \oplus \cdots \oplus (A_{l1} \oplus \cdots \oplus A_{lm_l})$$

in such a way that if

$$(10) \quad B_i = A_{i1} \oplus \cdots \oplus A_{im_i}$$

then

$$(11) \quad A_i \subseteq B_i$$

and

$$(12) \quad A_j, A_k \subseteq B_i \Leftrightarrow j \sim k.$$

In other words, we assemble in a block all the A_i 's which have the same annihilators in $\mathbf{Z}[G]$. Thus (with the conventions (11) and (12)), $\text{Ann}(A_i) = \text{Ann}(B_i) = M_i$. Now F_i acts on B_i and B_i becomes a vector space over F_i .

Fact (see [Ne2], claims 8, 9, and 10).

(A) Let $a = (a_{i1}, \dots, a_{im_i})$, $a' = (a'_{i1}, \dots, a'_{im_i})$ be two elements of B_i . Then there is a $\beta \in F$ for which $\beta a = a'$ iff there is a $\gamma \in F_i$ for which $\gamma a_{ij} = a'_{ij}$ for all $j = 1, \dots, m_i$.

(B) Let $b = (b_1, \dots, b_l)$, $b' = (b'_1, \dots, b'_l) \in S$ with $b_i, b'_i \in B_i$. Then there is a $\beta \in F$ for which $\beta b = b'$ iff there are $\gamma_i \in F_i$ ($i = 1, \dots, l$) for which $\gamma_i b_i = b'_i$ for all $i = 1, \dots, l$.

(C) There is an integer k such that any $\beta \in F$ has a representative $\gamma \in \mathbf{Z}[G]$ of length $\leq k$. In other words, the ring F is interpretable in G .

Facts A and B say that $F \simeq F_1 \times \cdots \times F_l$. We can state these two facts in one compact form:

(D) $S \rtimes F^* = \bigoplus_{i=1}^l (B_i \rtimes F_i^*)$ and

$$B_i \rtimes F_i^* \simeq \left\{ \left(\begin{array}{c} a_{i1} \\ \text{Id} \quad \vdots \\ a_{im_i} \\ \hline 0 \cdots 0 \quad \beta \end{array} \right) a_{ij} \in A_{ij}, \beta \in F_i^* \right\}.$$

In particular, each group $B_i \rtimes F_i^*$ is an algebraic group over an algebraically closed field.

Next, we state some immediate consequences of Facts A, B, C, and D.

Lemma 3. *Every G -normal subgroup of S is definable and connected.*

Proof. Since $F = F_1 \times \cdots \times F_l$ and each F_i is connected (Fact 6), F is a connected ring.

Now let U be a G -normal subgroup of S . Let $x \in U$. Clearly $Fx \simeq F/\text{ann}_F(x)$. Thus Fx is a definable connected subgroup. Since F has an identity element, $x \in Fx$. Thus

$$U = \langle Fx : x \in U \rangle.$$

By Fact 2, U is definable and connected. \square

Lemma 4. *S is a completely reducible F -module.*

Proof. Let U be an F -submodule of S . In particular, U is G -normal. Thus by Lemma 3, it is definable. We will find a complement V to U in S . If for all G -minimal A , $A \subseteq U$, then there is nothing to do, because then $U = S$, $V = \{0\}$. If not, for some G -minimal A , $A \cap U = \{0\}$. Thus $\langle U, A \rangle = U \oplus A$. Since $\text{rk}(S/U \oplus A)$ is strictly less than $\text{rk}(S/U)$, by induction, $U \oplus A$ has a complement in S . \square

In §6 we will generalize Lemma 3 to G' in case G' is abelian (see Corollary 19).

5. SPLITTING G'

The purpose of this section is to prove Theorems 1 and 2 stated in the Introduction. We keep the same hypothesis about G : it is connected, solvable, has finite Morley rank, and $G' \cap Z(G) = 1$. We will use the notation of §4.

Lemma 5. *Suppose G is ω_1 -saturated. Then there is an element $g \in G$ whose order modulo each $C_G(A_i)$ is infinite.*

Proof. If the lemma does not hold then for all $g \in G$ there is an integer $p = p(g)$ such that $g^p \in \bigcup_{i=1}^n C_G(A_i)$. Since G is saturated the same integer p will work for all $g \in G$. Thus $g^p \in \bigcup_{i=1}^n C_G(A_i)$ for all $g \in G$. Let

$$G_i = \{g : g^p \in C_G(A_i)\}.$$

Thus

$$G = G_1 \cup \cdots \cup G_n.$$

Therefore $\text{rk } G = \text{rk } G_i$ for some i . But G_i is a subgroup (because $G' \subseteq C_G(A_i) \subseteq G_i$ and $G/C_G(A_i)$ is abelian). Since G is connected $G = G_i$. But then $G/C_G(A_i)$ is a group of exponent p . Also $G/C_G(A_i) < F_i^*$. Since F_i is a field it has finitely many elements of order p . Hence $G/C_G(A_i)$ is finite. G being connected, this implies $G = C_G(A_i)$, i.e., $A_i \subseteq G' \cap Z(G) = 1$, a contradiction. This proves the lemma. \square

Remark. Since $C_G(A_i) \neq G$ and G is connected, $G \neq \bigcup_{i=1}^n C_G(A_i)$. So it is easy to find a g not in any of the centralizers. Lemma 6 says that, if G is ω_1 -saturated, we can find a $g \in G \setminus \bigcup_{i=1}^n C_G(A_i)$ with the property that $g^p \notin C_G(A_i)$ for any $i = 1, \dots, n$, any $p \geq 1$.

Lemma 6. *Let $g \in G \setminus \bigcup_{i=1}^n C_G(A_i)$. Then $C_S(g) = 1$.*

Proof. Since $S = \bigoplus_{i=1}^n A_i$ and $A_i \triangleleft G$, we have $C_S(g) = \bigoplus_{i=1}^n C_{A_i}(g)$. Thus we just need to show that $C_{A_i}(g) = 1$. We will show that $C_{A_i}(g) \triangleleft G$. Since A_i is G -minimal this will imply that either $C_{A_i}(g) = A_i$ or $C_{A_i}(g) = 1$. But $C_{A_i}(g) = A_i$ means $g \in C_G(A_i)$, which contradicts the choice of g . So let us show $C_{A_i}(g) \triangleleft G$. Let $a \in C_{A_i}(g)$, $h \in G$. Since by Proposition 1 $G' \subseteq C_G(A_i)$ we have

$$a^h = a^{gh} = a^{hg}.$$

Thus $a^h \in C_{A_i}(g)$ and we are done. \square

From now on we fix such a g .

We would like to be able to do induction on the rank of G . But if $\bar{G} = G/S$ we may have $Z(\bar{G}) \cap \bar{G}' \neq 1$. This will keep us away from the hypothesis. Let us consider $Z(\bar{G})$ and see what conditions we need on G for $Z(\bar{G}) \cap \bar{G}'$ to be the trivial subgroup of \bar{G} . Let

$$X = \{x \in G' \mid [G, x] \subseteq S\}.$$

Then X is a G -normal definable subgroup of G' containing S . In fact,

$$X/S = (G/S)' \cap Z(G/S).$$

(Recall that $(G/S)' = G'S/S = G'/S$.)

Lemma 7. $X = S \oplus C_X(g)$.

Proof. Since $S \subseteq Z(G')$ and $C_X(g) \subseteq G'$ the additive notation \oplus makes sense. We will switch from multiplicative to additive notation without any comment.

Clearly, $S + C_X(g) \subseteq X$. By Lemma 6, $S + C_X(g) = S \oplus C_X(g)$.

Conversely, let $x \in X$. Then by definition of X , $[g, x] \in S$. On the other hand, by the previous lemma $C_S(g) = 1$; thus the group homomorphism

$$\tilde{g} : S \rightarrow S$$

defined by

$$\tilde{g}(s) = [g, s]$$

is an automorphism. Thus

$$[g, x] = [g, s]$$

for some $s \in S$, i.e.,

$$s^{-1}x \in C_X(g)$$

for some $s \in S$. Hence $x = ss^{-1}x \in S \oplus C_X(g)$. This proves the lemma. \square

Lemma 8. *If $C_X(g) \triangleleft G'$ then $C_X(g) = 1$.*

Proof. We will prove that $C_X(g) \triangleleft G'$ implies $C_X(g) \triangleleft G$. This will prove the lemma in view of Lemmas 2 and 6. Let $c \in C_X(g)$, $h \in G$. Since $[h, g] \in G'$, by the definition of X ,

$$[c, [h, g]] \in C_X(g) \cap S = 1.$$

Thus

$$c^{hg} = c^{gh} = c^h.$$

This shows that $c^h \in C_X(g)$. Since this is true for all $c \in C_X(g)$, $h \in G$, we have $C_X(g) \triangleleft G$. \square

Corollary 9. *If G is solvable of class 2 then $Z(G/S) \cap (G/S)' = 1$.*

Proof. By hypothesis G' is abelian. So $C_X(g) \triangleleft G'$. By Lemma 8, $C_X(g) = 1$. By Lemma 7, $X = S$. Since $1 = X/S = Z(G/S) \cap G'/S$, we have the lemma. \square

Lemma 10. *If for some $t \in G$, $C_{G'}(t) = 1$, then $C_{\overline{G}}(t) = 1$ where $\overline{G} = G/S$.*

Proof. Let $Y = \{y \in G' : [y, t] \in S\}$. Y is a definable subgroup of G . In fact

$$Y/S = C_{\overline{G}}(t).$$

Let $y \in Y$. So $[y, t] \in S$. On the other hand,

$$C_S(t) \subseteq C_{G'}(t) = 1;$$

thus the homomorphism \tilde{i} defined by

$$\tilde{i} : S \rightarrow S, \quad \tilde{i}(s) = [s, t]$$

is in fact an automorphism. Therefore for some $s \in T$

$$[y, t] = [s, t],$$

i.e., $s^{-1}y \in C_{G'}(t) = 1$, so $s = y$. This shows that $Y = S$, so $C_{\overline{G}}(t) = 1$. \square

We next show that in order to split G' in G , it is enough to split $(G/S)'$ in G/S . We say that G' splits definably in G if there is a definable subgroup T for which $G = G' \rtimes T$.

Proposition 11. *If $(G/S)'$ splits definably in G/S then G' splits definably in G .*

Proof. Suppose $G/S = (G/S)' \rtimes (T/S)$ for some definable subgroup T containing S . Since $(G/S)' = G'/S$ we have

- (i) $G = G'T$,
- (ii) $G' \cap T = S$.

We will split S definably in T . More specifically, we will show that

$$T = S \rtimes C_T(t)$$

for some t in T . Let t be an element of T for which $C_S(t) = 1$ (we know there is such an element in G ; by (i) we may take it in T). As in the proof of Lemma 10 we have

$$S = \{[s, t] : s \in S\}.$$

Let $t_1 \in T$ be any element. Then

$$[t_1, t] \in T' \subseteq G' \cap T = S.$$

Thus $[t_1, t] \in S$. So for some $s \in S$

$$[t_1, t] = [s, t],$$

i.e., for some $s \in S$, $s^{-1}t_1 \in C_T(t)$. Now we can write

$$t_1 = ss^{-1}t_1$$

with $s \in S$, $s^{-1}t_1 \in C_T(t)$. So

$$T = SC_T(t).$$

Clearly

$$S \cap C_T(t) = C_S(t) = 1.$$

This proves the proposition, because now $G = G' \rtimes C_T(t)$. \square

We can now prove the first two theorems stated in the introduction.

Theorem 1. *Let G be a connected, solvable group of class ≤ 2 and of finite Morley rank in which $Z(G) \cap G' = 1$. Then for some definable subgroup T , $G = G' \rtimes T$.*

Proof. The theorem easily follows from Corollary 9 and Proposition 11 by induction on the rank of G . We first split G'/S in G/S (using Corollary 9) and apply Proposition 11. \square

Theorem 2. *Let G be a connected, solvable group of finite Morley rank. Assume for some $t \in G$, $C_{G'}(t) = 1$. Then there is a definable subgroup T of G such that*

$$G = G' \rtimes T.$$

Proof. Using the inductive method of proof on the rank of G , the theorem follows from Lemma 10 and Proposition 11. We first split G'/S in G/S (using Lemma 10) and apply Proposition 11. \square

Remark 1. T of the above theorems is necessarily connected.

Remark 2. Since $T' \subseteq T \cap G' = 1$, T is Abelian.

Remark 3. If G is an algebraic group, the subgroup T of the theorems may be bigger than a maximal torus; i.e., it may contain unipotent elements. In Example 1, e.g.,

$$T = (K[x]/x^2)^*$$

and the elements of the form

$$1 + \alpha x$$

of T are unipotent elements. The algebraic torus in this case is the set

$$\{\beta + 0 \cdot x : \beta \in K^*\} \simeq K^* .$$

We suspect that if $\text{Char } K = 0$ the torus is not definable. However, it is interpretable as a quotient of G by the maximal nilpotent subgroup (see §8 for the definability of this maximal nilpotent subgroup).

Corollary 12. *Suppose G is as in Theorem 1. Assume furthermore that $Z(G) = 1$ and $G' = S$. Then G can be interpretably imbedded in a finite product \tilde{G} of interpretable algebraic groups. \tilde{G} has the form of the group $S \rtimes F^*$ described in Fact D.*

Proof. Let $G = G' \rtimes T$. Since T is an abelian subgroup, $C_T(G')$ commutes with T . It obviously commutes with G' . Thus

$$C_T(G') \subseteq Z(G) = 1 .$$

Thus $T < F^*$ (notation as in §4). In fact, $Z[T]/\text{ann } S$ is naturally isomorphic to F . Since $G = G' \rtimes T \subseteq G' \rtimes F^*$, we obtain the corollary from Facts C and D. \square

Remark. Even if G is an algebraic group satisfying the hypothesis of Corollary 1, \tilde{G} may be strictly bigger than G . In Example 5, e.g., $\tilde{G} = (K \times K) \rtimes (K^* \times K^*)$ with the componentwise action.

Conjecture. Any centerless, connected, solvable group of finite Morley rank can be interpretablely imbedded in a finite product of interpretable, (centerless, connected, solvable), algebraic groups over algebraically closed fields.

See [E-N] for a partial solution of the Conjecture.

Corollary 13. *Suppose G is as in Theorem 1. Then $C_G(G') = G'$.*

Proof. We have $G = G' \rtimes T$. Since G' is abelian, $G' \subseteq C_G(G')$. Thus $C_G(G') = G' \rtimes (T \cap C_G(G'))$. But $T \cap C_G(G')$ commutes with T (because T is abelian). It also commutes with G' ; thus $C_G(G') \cap T \subseteq Z(G) = 1$. Hence $C_G(G') = G'$. \square

Let us show by an example that the hypotheses of Theorems 1 and 2 are crucial. We would like to thank Xialong Wu for providing the example.

Example 7. Let G be the subgroup of invertible upper-triangular 6×6 matrices over a field K with $a_{11} = a_{22} = a_{33}$ and $a_{44} = a_{55} = a_{66}$, i.e.,

$$G = \left\{ \left(\begin{pmatrix} a & & & & & \\ & a & & * & & \\ & & a & & & \\ & & & b & & \\ & & & & b & \\ 0 & & & & & b \end{pmatrix} \mid a, b \in K^* \right\} .$$

Then G' is the set of strictly-triangular matrices with $a_{12} = a_{23} = a_{45} = a_{56} = 0$, i.e.,

$$G' = \left\{ \left(\begin{array}{cccc} 1 & 0 & * & \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & \\ \hline & & 1 & 0 & * \\ 0 & 0 & 1 & 0 \\ & & 0 & 0 & 1 \end{array} \right) \right\}.$$

G' has no complement in G (definable or not). This is an example where G' is not abelian, $C_{G'}(g) \neq 1$ for all g , and $Z(G) \cap G' = 1$. To find a centerless counterexample divide G by its center.

Definition. Let G be a connected solvable group of finite Morley rank in which $Z(G) \cap G' = 1$. Assume furthermore that either

- (1) G' is abelian or
- (2) for some $g \in G$, $C_{G'}(g) = 1$.

In Lemmas 9 and 10, we showed that G/S satisfies the same properties; in particular $Z(G) \cap G' = 1$ where $G = G/S$. Thus we can define $S_2 = S_2(G)$ by

$$S_2/S_1 = S(G/S_1)$$

where $S_1 = S$. In general we set

$$S_{i+1}/S_i = S(G/S_i)$$

and we call S_i the i th socle of G . Since each S_i is connected and we have $S_i \subseteq S_{i+1}$, for some integer j , $S_j = G'$. By Corollary 12, if T is a complement of G' in G , the group $(S_{i+1}/S_i) \rtimes T$ modulo a subgroup of T imbeds interpretably in a product of interpretable algebraic groups (described in Fact D).

6. SOLVABLE GROUPS OF CLASS 2

We assume in this section that G is a connected, centerless 2-step solvable group of finite Morley rank. Notation is as in the previous sections. S denotes the socle of G . We have $S = A_1 \oplus \dots \oplus A_n$ for G -minimal subgroups A_i . T is a complement of G' in $G : G = G' \rtimes T$. $\mathbf{Z}[T]/\text{ann } A_i = F_i$ is an algebraically closed field. $\mathbf{Z}[T]/\text{ann } S = F$ is a ring which is isomorphic to a direct sum of some of the F_i 's. The group $S \rtimes F^*$ was described in Facts A, B, C, and D.

We now consider the action of T on G' . Since G' is an Abelian group we may make $\mathbf{Z}[T]$ act on G' . Let $R = \mathbf{Z}[T]/\text{ann } G'$. Clearly R also acts on S and A_i . It should be clear that we have the following canonical isomorphisms:

$$R/\text{ann}_R S \simeq F, \quad R/\text{ann}_R A_i \simeq F_i.$$

By Corollary 13, $T < R^*$. Therefore our group $G = G' \rtimes T$ imbeds naturally in $\tilde{G} = G' \rtimes R^*$. We call \tilde{G} the full group of G . We will show that the group \tilde{G} is interpretable in G . Then \tilde{G} will also have finite Morley rank.

Let us find the full group of some of the examples we gave.

In Examples 1, 2, and 6, $\tilde{G} = G$.

In Example 3, $\tilde{G} = \mathbb{C} \rtimes \mathbb{C}^* = \{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} : a, b \in \mathbb{C}, b \neq 0 \}$.

In Example 5, $\tilde{G} = (\mathbb{C} \times \mathbb{C}) \rtimes (\mathbb{C}^* \times \mathbb{C}^*)$.

Already Example 3 shows that \tilde{G} may look more like an algebraic group than G . Clearly $\tilde{G} = \tilde{G}$ and $\tilde{G}' = G'$.

The section is devoted to the proof of the fact that R is interpretable in G . By trying to prove this we obtain a lot of information about the action of R on G' .

Lemma 14. *Let $x \in S$. Then*

- (i) $Rx = Fx$,
- (ii) Rx is a connected normal subgroup of G ,
- (iii) Rx is a direct sum of G -minimal subgroups.

Proof. Since $x \in S$ and $R/\text{ann}_R S \simeq F$ (naturally), (i) follows. (ii) and (iii) are immediate from Lemmas 3 and 4. \square

Since S_i is G -normal, S_i is also an R -module.

Lemma 15. *Let $x \in S_{i+1}$. Then*

- (i) $(S_i + Rx)/S_i$ is a connected, G/S_i -normal subgroup of G'/S_i ,
- (ii) $(S_i + Rx)/S_i$ is a direct sum of G/S_i -minimal subgroup of G'/S_i .

Proof. We may proceed as in Lemma 14 (replacing G by G/S_i), if we can show that G/S_i satisfies the hypothesis of Lemmas 3 and 4. This is done in Corollary 9. \square

Lemma 16. *Let $x \in G'$. Then Rx is a connected, G -normal subgroup of G' .*

Proof. We will prove by induction on i that $Rx \cap S_i$ is connected. Since each $x \in G'$ is in some S_i this will prove the first claim of the lemma. The second claim is obvious.

Since $1 \in R$ we have

$$(S_{i-1} + (Rx \cap S_i))/S_{i-1} = \langle R\bar{y} \mid y \in Rx \cap S_i \rangle$$

where \bar{y} is the image of y in G/S_{i-1} . By Lemma 15, each $R\bar{y}$ is connected. By Fact 2, the group generated by these $R\bar{y}$'s is connected. Thus

$$(S_{i-1} + (Rx \cap S_i))/S_{i-1} \simeq (Rx \cap S_i)/(Rx \cap S_{i-1})$$

is connected. But by induction $Rx \cap S_{i-1}$ is connected. These easily imply that $Rx \cap S_i$ is connected. \square

The next lemma gives explicitly the definition of Rx (or equivalently of $R/\text{ann}_R x$).

Lemma 17. *Let $x \in G'$. There is an integer k for which*

$$Rx = \{r(x) : r \in \mathbb{Z}[T] \text{ and } l(r) \leq k\}.$$

Hence $R/\text{ann } x$ is an interpretable ring.

Proof. Suppose $x \in S_i \setminus S_{i-1}$. Let $\bar{x} \in G/S_{i-1}$ be the image of x . By Fact C, for some integer k_0 ,

$$R\bar{x} = \{r(\bar{x}) : r \in \mathbf{Z}[T] \text{ and } l(r) \leq k_0\}.$$

Thus for all $r \in \mathbf{Z}[T]$, there is an $r' \in \mathbf{Z}[T]$ of length $\leq k_0$ such that

$$r(x) - r'(x) \in Rx \cap S_{i-1}.$$

On the other hand, by Fact 2 and Lemma 16,

$$Rx \cap S_{i-1} = \langle Ry : y \in Rx \cap S_{i-1} \rangle = Ry_1 + \cdots + Ry_u$$

for some $y_1, \dots, y_u \in Rx \cap S_{i-1}$. Write $y_j = r_j(x)$, so

$$Rx \cap S_{i-1} = Rr_1(x) + \cdots + Rr_u(x).$$

Thus $r(x) - r'(x) \in Rr_1(x) + \cdots + Rr_u(x)$. But $r_j(x) \in S_{i-1}$, so by induction (on i) for some integer k_j ,

$$\begin{aligned} Rr_j(x) &= \{rr_j(x) : l(r) \leq k_j, r \in \mathbf{Z}[T]\} \\ &= \{s(x) : l(s) \leq k_j + l(r_j), s \in \mathbf{Z}[T]\}. \end{aligned}$$

Hence $r(x) = r'(x) + s_1(x) + \cdots + s_n(x)$ where $r', s_j \in \mathbf{Z}[T]$, $l(r') \leq k_0$, $l(s_j) \leq k_j + l(r_j)$. If $r'' = r' + s_1 + \cdots + s_n$ we have $r(x) = r''(x)$ and $l(r'') \leq k_0 + \cdots + k_n + l(r_1) + \cdots + l(r_n)$. Set k to be the integer $k_0 + \cdots + k_n + l(r_1) + \cdots + l(r_n)$.

Since an element $r \in R/\text{ann } x$ is uniquely defined by $r(x)$, the last claim is now obvious. \square

The above lemma does not say that R is interpretable in G . It just states that for any $x \in G'$, $R/\text{ann}_R x$ is interpretable in G . We will prove that R is indeed interpretable in G . Let us first give an example where $G' \neq Rx_1 \oplus \cdots \oplus Rx_u$ for any $x_1, \dots, x_u \in G'$.

Example 8. Let $G = (K \times K \times K) \rtimes (K[x]/x^2)^*$ with the following action:

$$(\alpha + \beta x)(a, b, c) = (\alpha a, \alpha b, \alpha c + \beta a + \beta b).$$

Then

$$\begin{aligned} S &= 0 \times 0 \times K, \\ G' &= K \times K \times K, \\ R &= K[x]/x^2, \\ R(1, 0, 0) &= K \times 0 \times K, \\ R(0, 1, 0) &= 0 \times K \times K, \\ G' &= R(1, 0, 0) + R(0, 1, 0). \end{aligned}$$

Since $\dim G' = 3$, $\dim R = 2$, there is no $x \in G'$ for which $Rx = G'$. For the same reason there are no $x_1, x_2 \in G'$ for which $G' = Rx_1 \oplus Rx_2$.

Lemma 18. $G' = Rx_1 + \dots + Rx_u$ for some $x_1, \dots, x_u \in G'$.

Proof. Since $1 \in R$, $G' = \langle Rx \mid x \in G' \rangle$. By Fact 2 (using Lemma 16),

$$G' = Rx_1 + \dots + Rx_u$$

for some $x_1, \dots, x_u \in G'$. \square

The next result is an unexpected property of G :

Corollary 19. Any G -normal subgroup of G' is definable and connected.

Proof. Let $X \subseteq G'$ be a G -normal subgroup. Since $1 \in R$, we have

$$X = \langle Rx \mid x \in X \rangle.$$

By Lemma 16, each Rx is connected. So by Fact 2, X is definable and connected. \square

We say that $x \in G'$ is a *minimal* element (of S_i) if $x \in S_i \setminus S_{i-1}$ and if $(Rx + S_{i-1})/S_{i-1}$ is a G/S_{i-1} -minimal subgroup. Notice that if x is a minimal element of S_i then $(Rx + S_{i-1})/S_{i-1}$ is isomorphic to the additive structure of the algebraically closed field R/M_x where

$$M_x = \{r \in R : r(x) \in S_{i-1}\}.$$

This is the content of Zil'ber's construction.

We say that $x \in G'$ is a *local* element if x is minimal and if $R/\text{ann } x$ is a local ring.

By Fact 9 and Lemma 17 if x is a local element then the maximal ideal of $R/\text{ann } x$ is definable and nilpotent. Thus $R/\text{ann } x$ cannot have a nontrivial idempotent.

Notice that the local elements of S_1 are exactly the minimal elements of S_1 .

Lemma 20. If $x \in S_i \setminus S_{i-1}$ is minimal then there is a local element $y \in Rx$ such that

$$(Rx + S_{i-1})/S_{i-1} = (Ry + S_{i-1})/S_{i-1}.$$

Proof. Let $\bar{R} = R/\text{ann } x$. We know by Lemma 17 that \bar{R} is interpretable in G . Thus it has finite Morley rank. Hence by Fact 9, $\bar{R} = \bar{R}\bar{e}_1 \oplus \dots \oplus \bar{R}\bar{e}_t$ for orthogonal atomic idempotents $\bar{e}_i \in \bar{R}$, each $\bar{R}\bar{e}_j$ being a local ring. Now we can decompose Rx accordingly:

$$Rx = Re_1x \oplus \dots \oplus Re_t x.$$

Since $(Rx + S_{i-1})/S_{i-1}$ is G/S_{i-1} -minimal, all but one of the elements e_jx are in S_{i-1} , say $e_1x \notin S_{i-1}$. Then

$$(Rx + S_{i-1})/S_{i-1} = (Re_1x + S_{i-1})/S_{i-1}.$$

It remains to show that $R/\text{ann}(e_1x)$ is a local ring. This is the same as showing $\bar{R}/\text{ann}(\bar{e}_1\bar{x})$ is a local ring. But clearly $\bar{R}/\text{ann}(\bar{e}_1\bar{x}) \simeq \bar{R}\bar{e}_1$. \square

Lemma 21. *Let $x \in G'$ be a local element. Then the maximal ideal of $R/\text{ann } x$ is $M_x/\text{ann } x$. Also $M_x/\text{ann } x$ is an interpretable nilpotent ideal.*

Proof. $M_x/\text{ann } x$ is certainly a proper ideal of $R/\text{ann } x$. Since

$$(R/\text{ann } x)/(M_x/\text{ann } x) \simeq R/M_x$$

and the latter is a field, $M_x/\text{ann } x$ is the maximal ideal of $R/\text{ann } x$. The last statement follows from Lemma 17 and Fact 9. \square

Lemma 22. *Let $X \subseteq G'$ be a G -normal subgroup of G' . Then there are finitely many local elements x_1, \dots, x_u for which $X = Rx_1 + \dots + Rx_u$.*

Proof. We prove by induction that $X \cap S_i$ satisfies this property. For $i = 1$ the result follows from Lemma 3 (minimal elements of S_1 are local elements of S_1 also). Similarly, $X \cap S_i/X \cap S_{i-1} = (Rx_1 + \dots + Rx_u)/X \cap S_{i-1}$ for some minimal elements of S_i . By Lemma 20, we may choose the elements x_i to be local elements. Now the lemma follows by induction. \square

The next lemma is a variant of Nakayama's lemma (see, e.g., [McD]).

Lemma 23. *Let $x \in G'$ be a local element. Then*

$$M_x(S_i \cap Rx) \subseteq S_{i-1}$$

for all i . Also, for any minimal element $y \in Rx$, $M_x = M_y$. Thus any minimal element of Rx is also a local element.

Proof. Let $y \in Rx$ be a minimal element. By Fact 9 and Lemma 17 for some integer n , $M_x^n \subseteq \text{ann } x \subseteq \text{ann } y \subseteq M_y$. But M_y is maximal, thus prime, so $M_x = M_y$. This proves the second claim. The third claim follows.

Let $y \in S_i \cap Rx$. Say $y = r_0x$. If $y \in S_{i-1}$ then $M_x y \subseteq S_{i-1}$. Assume therefore that $y \in S_i \setminus S_{i-1}$. Consider the ideal

$$M_y = \{r \in R : r(y) \in S_{i-1}\}.$$

Since $y \notin S_{i-1}$, $M_y \neq R$. Since $\text{ann } x \subseteq M_y$ and $R/\text{ann } x$ is a local ring, by Lemma 21 we have $M_y \subseteq M_x$. Conversely, let $r \in M_x$. Assume $r(y) \in S_i$. Since $0 \neq (Ry + S_{i-1})/S_{i-1}$ is G/S_{i-1} -minimal (because $Ry \subseteq Rx$ and x is minimal), for some $s \in R$

$$sr(y) \equiv y(S_{i-1}).$$

But $r \in M_x$, so $r^n \in \text{ann}(x)$ for some natural number n . Therefore

$$r^n(y) = r^n r_0(x) = r_0 r^n(x) = 0.$$

But also

$$s^n r^n(y) \equiv y(S_{i-1}).$$

Thus $y \in S_{i-1}$, a contradiction. This lemma is proved. \square

Corollary 24. *Let $x, y \in G'$ be two local elements. If $Rx \cap Ry \neq 0$ then $M_x = M_y$.*

Proof. Let $z \in Rx \cap Ry \setminus \{0\}$ be a minimal element. Assume $z \in S_i \setminus S_{i-1}$. If $M_x \neq M_y$, then $R = M_x + M_y$. Thus

$$1 = \alpha + \beta$$

for some $\alpha \in M_x, \beta \in M_y$. So

$$z = \alpha(z) + \beta(z).$$

But by Lemma 23 $\alpha(z), \beta(z) \in S_{i-1}$. Hence $z \in S_{i-1}$, a contradiction. \square

Lemma 25. *Let $x, y \in G'$ be two local elements. Then the following are equivalent:*

- (i) $\text{ann } x \subseteq M_y$;
- (ii) $M_x = M_y$;
- (iii) $\text{ann } y \subseteq M_x$.

Proof. It is enough to show that (i) \Leftrightarrow (ii). (i) \Rightarrow (ii). Since $M_x/\text{ann } x$ is a nilpotent ideal $M_x^n \subseteq \text{ann } x \subseteq M_y$ for some n . But M_y is a prime ideal. Thus $M_x^n \subseteq M_y$ implies $M_x \subseteq M_y$. Since they are maximal $M_x = M_y$.

(ii) \Rightarrow (i). $\text{ann}(x) \subseteq M_x = M_y$. \square

Now we define an equivalence relation on the set of local elements of G' (which is a definable subset, but we do not need this): If $x, y \in G'$ are local, we say that x is equivalent to y , and we write $x \sim y$, if $\text{ann } x \subseteq M_y$. By Lemma 25 this is an equivalence relation. By Lemma 22, $G' = Rx_1 + \dots + Rx_u$ for some local elements x_1, \dots, x_u . Let us fix these elements once for all and restrict our equivalence relation to the set $\{x_1, \dots, x_u\}$.

Let

$$R_j = \bigcap_{x_i \not\sim x_j} \text{ann } x_i.$$

Let us assume, for the sake of the simplicity of notation and without loss of generality, that x_1, \dots, x_l are unique representatives of each equivalence class. Each R_j is an ideal of R . Since we work modulo $\text{ann } G'$, $R_j \cap R_k = 0$ if $j \neq k$. We will denote M_{x_j} by M_j .

Proposition 26. $R = \bigoplus_{j=1}^l R_j$.

Proof. Clearly $R_j R_k = 0$ if $j \neq k$. Also, if $x_j \not\sim x_k$, $R = M_j + \text{ann } x_k$. Thus

$$1 = \alpha_{jk} + \beta_{jk} \quad (j \neq k)$$

for $\alpha_{jk} \in M_j, \beta_{jk} \in \text{ann}(x_k)$. Let us first fix j and vary x_k over the complement of the equivalence class of x_j . Multiply them:

$$1 = \prod_{x_k \not\sim x_j} (\alpha_{jk} + \beta_{jk}) = \gamma_j + \delta_j$$

where $\gamma_j \in M_j$, $\delta_j = \prod_{x_k \not\sim x_j} \beta_{jk} \in R_j$. Now vary j over the set of equivalence classes and multiply then:

$$1 = \prod_{j=1}^l (\gamma_j + \delta_j) = \eta + \eta_1 + \dots + \eta_l$$

where $\eta = \gamma_1 \cdots \gamma_l \in M_1 \cap \dots \cap M_l$ and $\eta_j \in R_j$. Since each $M_i/\text{ann } x_j$ is nilpotent if $x_i \sim x_j$, some power of $M_1 \cap \dots \cap M_l$ is the zero ideal. Let n be this power. Then

$$1 = (\eta + \eta_1 + \dots + \eta_l)^n = \eta^n + \varphi_1 + \dots + \varphi_n = \varphi_1 + \dots + \varphi_n$$

where $\varphi_j \in R_j$. Thus $R = \sum R_j$. Since $R_j \cap \sum_{j \neq k} R_k = 0$ we have

$$R = \bigoplus_{j=1}^l R_j. \quad \square$$

Lemma 27. $G' = \bigoplus_{j=1}^l R_j G'$.

Proof. Since $R = \bigoplus_{j=1}^l R_j$, we have $G' = \sum_{j=1}^l R_j G'$. Write $1 = e_1 + \dots + e_l$ with $e_j \in R_j$. Since $R_j R_k = 0$ if $j \neq k$ we have $e_i e_j = 0$ if $i \neq j$ and $e_i^2 = 1$. Let us show that they direct-sum: if

$$\beta_k x_k \in R_k G' \cap \sum_{j \neq k} R_j G' \quad (\beta_k \in R_k),$$

multiplying both sides by $1 = e_1 + \dots + e_l$, we get

$$\beta_k x_k = 0.$$

This proves the lemma. \square

Notice that by Corollary 19, $R_j G'$ is definable. But we do not yet know if R_j is interpretable in G .

Lemma 28. *If $x, y \in R_j G'$ are minimal elements then $M_x = M_y$ (i.e., $x \sim y$).*

Proof. Since for the (fixed) local elements x_1, \dots, x_u we have

$$G' = R x_1 + \dots + R x_u$$

and since $R_j x_i = 0$ if $x_i \not\sim x_j$ we have

$$R_j G' = R_j x_{j_1} + \dots + R_j x_{j_t}$$

for some $x_{j_1}, \dots, x_{j_t} \in \{x_1, \dots, x_u\}$ for which $M_{x_{j_1}} = \dots = M_{x_{j_t}} = M_j$. For each i some power of M_j is in $\text{ann } x_{j_i}$. Thus for some integer n

$$M_j^n \subseteq \bigcap_{i=1}^t \text{ann } x_{j_i} = \text{ann } R_j G'.$$

Now if $z \in R_j G'$ then $\text{ann } R_j G' \subseteq M_z$. Thus

$$M_j^n \subseteq M_z.$$

If z is minimal then M_z is a prime ideal; thus the above inclusion gives

$$M_j \subseteq M_z.$$

But M_j is also a maximal ideal, thus $M_j = M_z$. \square

To prove that R is interpretable, in view of Proposition 26 and Lemma 27 we need only show that the action of R_j on $R_j G'$ is interpretable (this is saying that R_j is interpretable). Thus we may assume for a while that $R = R_j$, $G' = R_j G'$. With these hypotheses we have the following property: if $x, y \in G'$ are minimal then $M_x = M_y$. We may therefore write M for M_x . Clearly if $x \in G'$, then $\text{ann } x \subseteq M$.

Lemma 29 (with the above additional hypothesis). *R is a local ring with M as the maximal ideal. Furthermore, M is nilpotent.*

Proof. Let $G' = Rx_1 + \dots + Rx_u$, $x_i \in G'$ local, $r \in R \setminus M$. Then for each $i = 1, \dots, u$ there is an $s_i \in R$ for which

$$rs_i \equiv 1 \pmod{\text{ann } x_i}$$

(because $R/\text{ann } x_i$ is a local ring with $M/\text{ann } x_i$ as its maximal ideal). Thus

$$\prod_{i=1}^u (1 - rs_i) \in \bigcap_{i=1}^u \text{ann } x_i = \text{ann } G' = 0.$$

The left-hand side is of the form $1 - rt$. Thus $rt = 1$ and r is invertible.

Let us now show that M is nilpotent. Since $M/\text{ann } x_i$ is nilpotent, for some integer n_i , $M^{n_i} \subseteq \text{ann } x_i$. Thus

$$M^{n_1 \dots n_u} \subseteq \bigcap_{i=1}^u \text{ann } x_i = 0. \quad \square$$

To prove that R is interpretable we need some more notation. Assume that $G' = S_l$ ($l \in \mathbf{N}$). Then we know by Lemma 23 that $MG' \subseteq S_{l-1}$. Define

$$M(i) = \{r \in R : rG' \subseteq S_{l-i}\}.$$

$M(i)$ is an ideal of R . $M(i) \supseteq M(i+1)$, $M = M(1)$, $M^i \subseteq M(i)$ (by Lemma 23 again). Since $MS_{l-i} \subseteq S_{l-i-1}$ (Lemmas 22 and 23) we have

$$M.M(i) \subseteq M(i+1).$$

This shows that $M(i)/M(i+1)$ is a vector space over R/M . We would like to show that it is finite dimensional. Since $R/\text{ann } x_j$ is an interpretable ring (Lemma 17), it is ω -stable; thus

$$(M(i) + \text{ann } x_j / \text{ann } x_j) / (M(i+1) + \text{ann } x_j / \text{ann } x_j)$$

is a finite-dimensional vector space over $(R/\text{ann } x_j) / (M \text{ann } x_j)$. Let us record this in a nicer form.

Lemma 30 (hypothesis as in Lemma 29). $M(i)/(M(i + 1) + (M(i) \cap \text{ann } x))$ is a finite-dimensional vector space over R/M for any $x \in G'$.

Proof.

$$\begin{aligned} & (M(i) + \text{ann } x / \text{ann } x) / (M(i + 1) + \text{ann } x / \text{ann } x) \\ & \simeq M(i) + \text{ann } x / M(i + 1) + \text{ann } x \\ & = M(i) + M(i + 1) + \text{ann } x / M(i + 1) + \text{ann } x \\ & \simeq M(i) / M(i) \cap (M(i + 1) + \text{ann } x) \\ & = M(i) / M(i + 1) + (M(i) \cap \text{ann } x) \end{aligned}$$

with natural isomorphisms. Also

$$(R / \text{ann } x) / (M / \text{ann } x) \simeq R / M,$$

again with natural isomorphism. The lemma follows from the previous remarks. \square

Lemma 31 (hypothesis as in Lemma 29). $M(i) / M(i + 1)$ is a finite-dimensional vector space over R/M .

Proof. Let $G' = Rx_1 + \dots + Rx_u$ be as usual. We first show that

$$\bigcap_{j=1}^u (M(i + 1) + (M(i) \cap \text{ann } x_j)) = M(i + 1).$$

Let r be in the left-hand side. Thus $r = \alpha_j + \beta_j$ where $\alpha_j \in M(i + 1)$, $\beta_j \in \text{ann } x_j$. Thus

$$r(x_j) = \alpha_j(x_j).$$

But $\alpha_j \in M(i + 1)$, so $r(x_j) \in S_{l-(i+1)}$. This is true for all j . Hence $r(G') \subset S_{l-(i+1)}$, so $r \in M(i + 1)$. This proves the claim.

Next we consider the product of natural maps:

$$M(i) / M(i + 1) \rightarrow \prod_{j=1}^u M(i) / (M(i + 1) + (M(i) \cap \text{ann } x_j)).$$

By the above claim, this R/M -vector space homomorphism is 1-1. Since by the previous lemma each component of the right-hand side is a finite-dimensional vector space over R/M , the lemma is proved. \square

Lemma 32 (hypothesis as in Lemma 29). R is an interpretable ring.

Proof. Choose a base $\bar{x}_{i_1}, \dots, \bar{x}_{i_{i_1}}$ of $M(i) / M(i + 1)$ over R/M . Let $x_{i_1}, \dots, x_{i_{i_1}}$ be a set of representatives in $M(i)$. Let $r \in R$. By Zil'ber R/M is interpretable: there is an $s \in \mathbf{Z}[T]$ of bounded length such that $r - s \in M$. Now each element of M can be written as linear combinations of x_{i_j} 's with coefficients $r_{ij} \in R$ of bounded length. The lemma follows. \square

In Lemma 32, we proved more:

Lemma 33 (hypothesis as in Lemma 29). *There is a fixed integer k such that if $r \in R$ then there is an element $\gamma \in \mathbf{Z}[T]$ of length $\leq k$ for which*

$$r = \gamma \in \mathbf{Z}[T]/\text{ann } G'.$$

We do not need the hypothesis of Lemma 29 anymore:

Theorem 3. *R is interpretable in G (its interpretation is given in Lemma 33). The full group $\tilde{G} = G' \rtimes R^*$ is a connected solvable group of class 2 and of finite Morley rank.*

Proof. With the notation of Proposition 26, Lemma 33 says that R_j is interpretable. Hence R is interpretable. Thus \tilde{G} is a solvable group of class 2 which has finite Morley rank. To show that it is connected we need to show that R is. We leave this to the next theorem. \square

Theorem 4. *Let G be a connected centerless solvable group of class 2 and of finite Morley rank. Let \tilde{G} be the full group of G . Then $\tilde{G} \simeq \bigoplus_{i=1}^l (G'_i \rtimes R_i^*)$ where*

- (i) R_i is an interpretable connected local ring, with nilpotent maximal ideals. $G'_i < G'$ and $G'_i \triangleleft G$. Furthermore, if $i \neq j$ then R_i^* acts as identity on G'_j .
- (ii) R_i is commutative, has an identity element, and every ideal is definable and connected.
- (iii) R_i is Artinian and Noetherian.
- (iv) $\tilde{G}' = \bigoplus G'_i = G'$.

Proof. At this point we may use Cherlin-Reineke (Fact 9) but we have all the information in our possession already. Let G'_i be $R_i G'$; then $G' = \bigoplus G'_i$. We know that $R = \bigoplus R_i$. Thus $\tilde{G} = G' \rtimes R^* = (\bigoplus G'_i) \rtimes (\bigoplus R_i^*) \simeq \bigoplus (G'_i \rtimes R_i^*)$.

By Lemma 32, R_i is interpretable. R_i is connected because R_i/M_i is an algebraically closed field (so is connected) and M_i^j/M_i^{j+1} are vector spaces over R_i/M_i (so are connected). This proves (i).

Let $I \triangleleft R_i$. Since R_i has an identity element

$$I = R_i I = \langle R_i a : a \in I \rangle.$$

Since $R_i a \simeq R_i/\text{ann } a$, $R_i a$ is connected. by Zil'ber, I is definable and connected. This proves (ii). (iii) follows from (ii). \square

Notice that all fields found by Zil'ber's method in $G'_i \rtimes R_i^*$ is isomorphic to R_i/M_i . Notice also that R_i acts on S_j/S_{j-1} by scalar multiplication, where S_j is the j th socle of G'_i .

Kathryn Enochs and the author have recently shown [E-N] that $G'_i \rtimes R_i^*$ is an algebraic group over the field R_i/M_i .

We can find a smaller imbedding of G .

Theorem 5. *Let G be a connected centerless solvable group of class 2 and finite Morley rank. Then there are interpretable abelian subgroups T_i acting on G' ; there are finitely many, definable G -normal subgroups $G'_i < G'$ such that*

- (i) $G \subseteq \bigoplus_{i=1}^m (G'_i \rtimes T_i)$.
- (ii) $\mathbf{Z}[T_i]/\text{ann } G'_i$ is an interpretable, local, Noetherian ring with nilpotent maximal ideals and algebraically closed residue fields.
- (iii) T_i is a homomorphic image of T where T is a definable complement of G' in G .

Proof. Let R be as before. Let e_1, \dots, e_m be atomic idempotents of R . (Then $R_i = Re_i$ with the above notation.) Set $T_i = Te_i, G'_i = R_iG'$. \square

Remark. Most of the results of this section also hold for modules (with two sorted languages R and M) over commutative rings.

7. FITTING SUBGROUP

Let G be a connected solvable group of finite Morley rank. By Facts 4 and 5, G' is connected and nilpotent. Let F be a nilpotent, normal, connected subgroup of G which is maximal for these properties. If F_1 and F_2 are two such groups, then $\langle F_1, F_2 \rangle = F_1F_2$ is nilpotent and connected. Thus $F_1 = F_2$. Therefore such a subgroup is unique. We call it the Fitting subgroup of G . It contains G' .

If H is a nilpotent, definable, normal subgroup of G then $H^0 \subseteq F$. Thus FH/F is finite.

If G is algebraic (say over K), G/F is a quotient of a torus ($\simeq (K^*)^n$, some n). Thus in this case G/F is divisible. We generalize this.

Theorem 6. *Let G be a solvable connected group of finite Morley rank, F its Fitting subgroup. Then G/F is divisible.*

Proof. G/F is abelian (because $G' \subseteq F$). Thus by Fact 7,

$$G/F = B/F \oplus D/F$$

where D/F is divisible and B/F has bounded exponent. If n is this exponent, then

$$D = \{g^n : g \in G\}F.$$

Thus D is definable. Since D/F has finite Morley rank, for any fixed integer m , D/F has only finitely many elements of order m (if not we can find infinitely many copies of \mathbf{Z}_{p^∞} for some prime number p).

B may not be definable, but the set

$$C = \{g \in G : g^n \in F\}$$

is definable and contains B . Since G/F is abelian, C is a subgroup of G . It is clearly normal. By the above discussion, $C \cap D$ is finite, thus $[C : B] < \infty$. Notice also that $F \subseteq C^0$.

We will show that C^0 is nilpotent. Since $C' \subseteq F$, we need only show that

$$\underbrace{[C^0, [C^0, \dots, [C^0, F] \dots]]}_{m \text{ times}} = 1$$

for some m .

By considering socles and centers we can find a sequence of definable normal subgroups $(F_i)_i$:

$$0 = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_n = F$$

in such a way that either F_i/F_{i-1} is G/F_{i-1} -minimal and $Z(G/F_{i-1}) \cap F_i/F_{i-1} = 1$ or

$$F_i/F_{i-1} = Z(G/F_{i-1}).$$

We want to show that $[C^0, F_i] \subseteq F_{i-1}$ for all i . This will clearly imply that C^0 is nilpotent.

If $F_i/F_{i-1} = Z(G/F_{i-1})$ then $[G, F_i] \subseteq F_{i-1}$ by definition of $Z(G/F_{i-1})$.

If not, the action of C on F_i/F_{i-1} is a vector-space action; i.e., F_i/F_{i-1} is a 1-dimensional vector space over an algebraically closed field and C acts on it as it should. Since elements of C have the same finite-order modulo F , these actions are automorphisms of order n . Thus we have finitely many possible actions. Therefore C^0 acts trivially (i.e. as identity) on F_i/F_{i-1} ; in other words, $[F_i, C^0] \subseteq F_{i-1}$.

Thus C^0 is nilpotent. So $C^0 \subseteq F \subseteq C^0$. Thus $C^0 = F$.

Since $F \subseteq B$ and $|C/B|$ is finite, this shows that B/F is finite. But G/F is connected. So $B/F = 1$ and $G/F = D/F$ is divisible. \square

Corollary. *Given any integer n , G/F has only finitely many elements of order n .*

Recently, the author and Kathryn Enochs proved that in case G' is abelian and G is centerless, then $F = C_G(S)^0$. The proof relies heavily on the methods, results, and classification of local rings of the previous section. It would be interesting to find a simpler proof of this fact. We do not know if the statement $F = C_G(S_i)^0$ (for some i) is true without any solvability class assumption.

Conjecture. Let G be a connected solvable group of finite Morley rank. Let F be the Fitting subgroup of G . Then F splits (not necessarily definably) in G .

The reason to believe in this conjecture will be more apparent in the next section.

The conjecture has recently been proved by the author and Kathryn Enochs (see [E-N]) in case G is centerless and solvable of class 2.

8. LINEAR ALGEBRAIC GROUPS

Let G be a linear algebraic group over an algebraically closed field K . If G is algebraically connected, we say that G is a -connected. If G is model theoretically connected, we say that G is m -connected.

The following theorem was proved with Anand Pillay.

Theorem 7. *Let G be a centerless a -connected solvable linear algebraic group. Then the unipotent subgroup U is definable and if the language is the pure groups language U is also m -connected.*

Proof. We define the following subgroups:

- $T =$ a maximal torus of G .
- $\bar{U} =$ definable closure of U ,
- $\overline{\bar{U}} =$ Zariski closure of \bar{U} ,
- $\bar{U}^0 = a$ -connected component of \bar{U} .

By Fact 8, \bar{U} is nilpotent. It is known that $\overline{\bar{U}}$ is nilpotent (see Borel [Bo, §2.4, Corollary]). Then $\bar{U}^0 \cap T$ is central in \bar{U}^0 (see [Sp, Corollary 6.8(i), p. 145]). Thus $\bar{U}^0 \cap T$ commutes with U (because $U \subseteq \bar{U}^0$). Since T is abelian, $\bar{U}^0 \cap T$ also commutes with T . Thus $\bar{U}^0 \cap T$ commutes with $G (= U \rtimes T)$. Thus $\bar{U}^0 \cap T = 1$. Since $U \subseteq \bar{U}^0$, this implies that $U = \bar{U}^0$. Thus $[\bar{U} : U] < \infty$. Since $U \subseteq \bar{U} \subseteq \overline{\bar{U}}$, we also have $[\overline{\bar{U}} : U] < \infty$. Thus $\bar{U} = U \rtimes (\bar{U} \cap T)$ with $|\bar{U} \cap T| < \infty$.

Let q be the order of $\bar{U} \cap T$. Let p be the characteristic of the field. Since T is isomorphic to $(K^*)^n$ we have $(p, q) = 1$. Let $ut \in \bar{U} = U \rtimes (\bar{U} \cap T)$. Then

$$(ut)^q = vt^q = v$$

for some $v \in U$. Thus

$$\bar{U}^q \subseteq U.$$

On the other hand, since $(p, q) = 1$ we have

$$U = U^q \subseteq \bar{U}^q,$$

thus

$$U = \bar{U}^q$$

and U is definable.

We now assume that the language is pure group language. Let U^0 be the m -connected component of U . U^0 is a constructible set in the Zariski topology of G (i.e., finite union of intersections of an open and closed set). It contains an open in its closure [Po2]. Thus U^0 is an open subgroup of U . Thus it is also closed, hence $U = U^0$. \square

Fitting subgroup has the meaning we gave to it in §7.

Corollary (G and U as above and in the pure language of groups). *U is the Fitting subgroup of G .*

Proof. Let F be the Fitting subgroup of G . By Theorem 7, $U \subseteq F$. Replacing U by F in the first part of the proof of Theorem 7, we see that $F = U$. \square

Question. Does the last statement of Theorem 7 hold if the language is richer (but G is still ω -stable)?

9. MINIMAL NONNILPOTENT GROUPS

Cherlin proved in [Ch] that a centerless connected group G of Morley rank 2 is an affine group over an algebraically closed field, i.e.,

$$G \simeq \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} : a \in K, b \in K^* \right\} = K \rtimes K^*$$

for some algebraically closed field K . In this section we generalize this result.

Theorem 8. *Let G be a connected centerless solvable group of finite Morley rank. Assume all definable proper connected subgroups of G are nilpotent. Then for some algebraically closed field K and some definable $T < K^*$ for which $T + \dots + T = K$ (finite number) we have $G = K \rtimes T$ with T acting on K by field multiplications.*

Proof. Let S be the socle of G , $g \in G$ be as in Lemma 5. Let X be as in Lemma 7:

$$X = \{x \in G' : [G, x] \subseteq S\}.$$

By Lemma 7,

$$X = S \oplus C_x(g).$$

Assume $C_x(g) \neq 1$. Let $x \in C_x(g) - \{1\}$. Then $g^m \in C_G(x)^0$ for some integer m . Replacing g by g^m , we may suppose that $g \in C_G(x)^0$. By hypothesis $C_G(x)^0 \neq G$, so $C_G(x)^0$ is nilpotent. But $S \subseteq C_G(x)^0$, $g \in C_G(x)^0$, thus $[g[g, \dots, [g, S] \dots]] = 1$; this is a contradiction because $C_x(g) = 1$. Thus $X = S$. This means that

$$G'/S \cap Z(G/S) = 1.$$

In particular, G/S is not nilpotent. Thus G/S satisfies the hypothesis of our statement. Thus by induction $G/S = G'/S \rtimes T/S$. By Proposition 11, $G = G' \rtimes T_1$ for some definable T_1 (T is definable by induction hypothesis). $S \rtimes T_1 \subseteq G$ and $S \rtimes T_1$ is not nilpotent. Thus $G = S \rtimes T_1$. If $A_1 \subseteq S$ is a minimal normal then $G = A_1 \rtimes T_1$ for the same reason. Since G is centerless we get the result by Zil'ber. \square

The natural question is now:

Question. Suppose G is a centerless connected group of finite Morley rank. Assume every proper definable connected subgroup of G has a nontrivial center. What can we say about G ?

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556

Current address: Department of Mathematics, University of California, Irvine, California 92717