

ON SOME 3-DIMENSIONAL COMPLETE RIEMANNIAN
MANIFOLDS SATISFYING $R(X, Y) \cdot R = 0$

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1. **Introduction.** Let (M, g) be a Riemannian manifold. By R we denote the Riemannian curvature tensor. By $T_x(M)$ and Exp_x we denote the tangent space to M at x and the exponential mapping of (M, g) at x . For $X, Y \in T_x(M)$, $R(X, Y)$ operates on the tensor algebra as a derivation at each point $x \in M$. In a locally symmetric space ($\nabla R = 0$), we have

$$(*) \quad R(X, Y) \cdot R = 0 \text{ for any point } x \in M \text{ and } X, Y \in T_x(M).$$

We consider the converse under some additional conditions.

THEOREM A (S. Tanno [8]). *Let (M, g) be a complete and irreducible 3-dimensional Riemannian manifold. If (M, g) satisfies $(*)$ and the scalar curvature S is positive and bounded away from 0 on M , then (M, g) is of constant curvature.*

Other results concerning this problem may be found in references. In this paper, we shall prove

THEOREM B. *Let (M, g) be a complete and irreducible 3-dimensional Riemannian manifold satisfying $(*)$. If the volume of (M, g) is finite, then (M, g) is of constant curvature, and hence, $\nabla R = 0$.*

COROLLARY B. *Let (M, g) be a compact and irreducible 3-dimensional Riemannian manifold satisfying $(*)$. Then (M, g) is of constant curvature.*

It may be noticed that $(*)$ implies in particular

$$(**) \quad R(X, Y) \cdot R_1 = 0,$$

where R_1 denotes the Ricci tensor of (M, g) .

In this paper, (M, g) is assumed to be connected, complete and of class C^∞ unless otherwise specified.

2. **Preliminaries.** Let (M, g) be a 3-dimensional Riemannian manifold. Assume $(*)$. $\dim M = 3$ implies

$$(2.1) \quad R(X, Y) = R^i X \wedge Y + X \wedge R^i Y - (S/2)X \wedge Y,$$

where

$$g(R^1X, Y) = R_1(X, Y) \quad \text{and} \quad (X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y.$$

Let (K_1, K_2, K_3) be eigenvalues of the Ricci transformation R^1 at a point x . Then (*) is equivalent to

$$(2.2) \quad (K_i - K_j)(2(K_i + K_j) - S) = 0.$$

Therefore we may have only three cases:

$$(K, K, K), \quad (K, K, 0), \quad (0, 0, 0) \quad \text{at each point.}$$

First, if (K, K, K) , $K \neq 0$, holds at some point x , then it holds on some open neighborhood U of x . Hence U is an Einstein space, and K is constant on U and on M . Therefore (M, g) is of constant curvature (cf. H. Takagi and K. Sekigawa [6]). From now, we assume that $\text{rank } R^1 \leq 2$ on M . Let $W = \{x \in M; \text{rank } R^1 = 2 \text{ at } x\}$. By W_0 we denote one component of W . On W_0 , we have two C^∞ -distributions T_1 and T_0 such that

$$T_1 = \{X; R^1X = KX\}, \\ T_0 = \{Z; R^1Z = 0\}.$$

For $X, Y \in T_1$ and $Z \in T_0$, by (2.1), we have

$$(2.3) \quad R(X, Y) = KX \wedge Y, \\ R(X, Z) = 0.$$

This shows that T_0 is the nullity distribution. Since the index of nullity at each point of M is 1 or 3, the nullity index of (M, g) is 1. Thus integral curves of T_0 are geodesics (and complete if (M, g) is complete) (cf. Y. H. Clifton and R. Maltz [2], etc.). Let $(E_1, E_2, E_3) = (E)$ be a local field of orthonormal frame such that $E_3 \in T_0$ (consequently, $E_1, E_2 \in T_1$) and

$$\nabla_{E_3} E_i = 0, \quad i = 1, 2, 3.$$

We call this (E) an adapted frame field. If we put $\nabla_{E_i} E_j = \sum_{k=1}^3 B_{ijk} E_k$, then we get $B_{ijk} = -B_{ikj}$ and

$$(2.4) \quad B_{3ij} = 0, \quad i, j = 1, 2, 3.$$

The second Bianchi identity and (2.3) give

$$(2.5) \quad E_3 K + K(B_{131} + B_{232}) = 0, \quad \text{or} \\ \text{div } E_3 = -E_3 K / K.$$

By (2.4) and $R(E_i, E_3)E_3 = \nabla_{E_i} \nabla_{E_3} E_3 - \nabla_{E_3} \nabla_{E_i} E_3 - \nabla_{[E_i, E_3]} E_3 = 0$, we get

$$\begin{aligned}
 (2.6) \quad & E_3 B_{131} + (B_{131})^2 + B_{132} B_{231} = 0, \\
 & E_3 B_{132} + B_{131} B_{132} + B_{132} B_{232} = 0, \\
 & E_3 B_{231} + B_{231} B_{131} + B_{232} B_{231} = 0, \\
 & E_3 B_{232} + (B_{232})^2 + B_{231} B_{132} = 0.
 \end{aligned}$$

(2.5) and (2.6)₂, (2.5) and (2.6)₃, (2.5) and (2.6)_{1,4} imply

$$(2.7) \quad B_{132} = C_1(E)K, \quad B_{231} = C_2(E)K,$$

$$(2.8) \quad B_{131} - B_{232} = D(E)K,$$

where $C_1(E)$, $C_2(E)$ and $D(E)$ are functions defined on the same domain as (E) such that $E_3 C_1(E) = E_3 C_2(E) = E_3 D(E) = 0$.

By (2.5) and (2.8), we get

$$(2.9) \quad 2B_{131} = D(E)K - E_3 K/K.$$

Now, let $\gamma_x(s)$ be an integral curve of T_0 through $x = \gamma_x(0) \in W_0$ with arc-length parameter s , i.e., $\gamma_x(s) = \text{Exp}_x s(E_3)_x$. Then (2.6)₁, (2.7) and (2.9) give

$$(2.10) \quad \frac{1}{2} \frac{d}{ds} \left(\frac{1}{K} \frac{dK}{ds} \right) = HK^2 + \frac{1}{4} \left(\frac{1}{K} \frac{dK}{ds} \right)^2, \quad \text{along } \gamma_x(s),$$

where

$$H = H(E) = D(E)^2/4 + C_1(E)C_2(E).$$

(2.10) implies that H is independent of the choice of the adapted frame fields (E) . Solving (2.10), we get

$$(2.11) \quad K = \gamma, \quad (\text{for } H = 0), \quad \text{or}$$

$$(2.12) \quad K = \pm 1/((\alpha s - \beta)^2 - H/\alpha^2), \quad (\text{for } H \neq 0),$$

where α , β and γ are constant along $\gamma_x(s)$, $\alpha \neq 0$.

With respect to our arguments, without loss of essentiality, we may assume that M is orientable. Let (E) be any adapted frame field which is compatible with the orientation. We call it an oriented adapted frame field. Then we see that $f = (C_1(E) - C_2(E))K$ is independent of the choice of oriented adapted frame fields, and hence f is a function of class C^∞ on W_0 . $f = 0$ holds on an open set $U \subset W_0$, if and only if T_1 is integrable on U . This is a geometric meaning of f . In the sequel, we assume that the volume of (M, g) is finite. We can see that $H = H(E) = D(E)^2/4 + C_1(E)C_2(E)$ is a function of class C^∞ on W_0 . Let $W(H) = \{x \in W_0; H \neq 0 \text{ at } x\}$. We assume that $W(H) \neq \emptyset$. Let $W(H)_0$ be one component of $W(H)$. By (2.12) and completeness of (M, g) , H must be negative on $W(H)_0$. For each point $x \in W(H)_0$, consider $\gamma_x(s)$. Then $\gamma_x(s) \in W(H)_0$,

for all s . Let $x_0 = \gamma_x(\beta/\alpha)$. For $(E_1)_{x_0}, (E_2)_{x_0} \in T_1(x_0)$, there exists a 2-dimensional submanifold, $\{\varphi(u_1, u_2) \in W(H)_0; (u_1, u_2) \in (-\varepsilon, \varepsilon)^2, \varepsilon > 0\}$, such that $\varphi(0, 0) = x_0$ and $(\partial\varphi/\partial u_1)(0, 0) = (E_1)_{x_0}, (\partial\varphi/\partial u_2)(0, 0) = (E_2)_{x_0}$. Now, we define a mapping

$$\begin{aligned} \Phi: (-\varepsilon, \varepsilon)^2 \times (-\delta, \delta) &\rightarrow W(H)_0 \quad \text{by} \\ (2.13) \quad \Phi(u_1, u_2, w_3) &= \text{Exp}_{\varphi(u_1, u_2)} w_3 E_3, \quad \text{for some } \delta > 0. \end{aligned}$$

Then Φ is of class C^∞ and furthermore, for small ε, δ , $V(\varepsilon, \delta) = \{\Phi(u_1, u_2, w_3) \in W(H)_0; (u_1, u_2, w_3) \in (-\varepsilon, \varepsilon)^2 \times (-\delta, \delta)\}$ is a local coordinate neighborhood with origin at x_0 . In $V(\varepsilon, \delta)$, by (2.12), we get

$$(2.14) \quad K = \pm 1 / ((Aw_3 - B)^2 - H/A^2),$$

where A and B are functions of class C^∞ on $V(\varepsilon, \delta)$ such that $\partial A / \partial w_3 = \partial B / \partial w_3 = 0$ on $V(\varepsilon, \delta)$ and $A = \alpha, B = 0$ at x_0 .

By continuity of A and B in (2.14), there is $\varepsilon_0, 0 < \varepsilon_0 < \varepsilon$ such that $-\delta/4 < B/A < \delta/4$, for $(u_1, u_2) \in (-\varepsilon_0, \varepsilon_0)^2$.

Now, we define a mapping $\psi: (-\varepsilon_0, \varepsilon_0)^2 \rightarrow V(\varepsilon, \delta)$ by

$$(2.15) \quad \psi(u_1, u_2) = \text{Exp}_{\varphi(u_1, u_2)} (B(u_1, u_2)/A(u_1, u_2)) E_3.$$

And furthermore, we define a mapping $\Psi: (-\varepsilon_0, \varepsilon_0)^2 \times (-\delta_0, \delta_0) \rightarrow V(\varepsilon, \delta)$ by

$$(2.16) \quad \Psi(u_1, u_2, u_3) = \text{Exp}_{\psi(u_1, u_2)} u_3 E_3, \quad \delta_0 = \delta/4.$$

Then Ψ is of class C^∞ and

$$U(\varepsilon_0, \delta_0) = \{\Psi(u_1, u_2, u_3) \in V(\varepsilon, \delta); (u_1, u_2, u_3) \in (-\varepsilon_0, \varepsilon_0)^2 \times (-\delta_0, \delta_0)\}$$

is a local coordinate neighborhood with origin at x_0 .

Between w_3 in $V(\varepsilon, \delta)$ and u_3 in $U(\varepsilon_0, \delta_0)$, the following relation holds:

$$(2.17) \quad w_3 = u_3 + B/A, \quad \text{in } U(\varepsilon_0, \delta_0).$$

Thus (2.14) and (2.17) imply

$$(2.18) \quad K = \pm 1 / ((Au_3)^2 - H/A^2), \quad \text{on } U(\varepsilon_0, \delta_0).$$

Let $\gamma(u_1, u_2)$ be the integral curve of T_0 starting from $\psi(u_1, u_2)$, $(u_1, u_2) \in (-\varepsilon_0, \varepsilon_0)^2$, i.e., $\gamma(u_1, u_2)(s) = \text{Exp}_{\psi(u_1, u_2)} s E_3$. Then, in $U(\varepsilon_0, \delta_0)$, u_3 can be considered as the arc-length parameter of $\gamma(u_1, u_2)$. We put $L(u_1, u_2) = \{\gamma(u_1, u_2)(s) \in M; -\infty < s < \infty\}$. Since $\dim T_0 = 1$, taking account of (2.12) and (2.18), we can see that $\gamma(u_1, u_2)(s_1) \neq \gamma(u_1, u_2)(s_2)$ for $s_1 \neq s_2$. From (2.12) and (2.18), $dK/ds = 0$ for $s = 0$ and otherwise $dK/ds \neq 0$ along $L(u_1, u_2)$, for any $(u_1, u_2) \in (-\varepsilon_0, \varepsilon_0)^2$. Thus, we can see that if $(u_1, u_2) \neq (v_1, v_2)$, $(u_1, u_2), (v_1, v_2) \in (-\varepsilon_0, \varepsilon_0)^2$, then $L(u_1, u_2) \cap L(v_1, v_2) = \emptyset$.

Now, we put

$$U(\varepsilon_0) = \{\hat{\Psi}(u_1, u_2, u_3) \in M; (u_1, u_2) \in (-\varepsilon_0, \varepsilon_0)^2, -\infty < u_3 < \infty\},$$

where $\hat{\Psi}$ denotes an extension of Ψ defined by

$$\hat{\Psi}(u_1, u_2, u_3) = \text{Exp}_{\psi(u_1, u_2)} u_3 E_3, \text{ on } (-\varepsilon_0, \varepsilon_0)^2 \times (-\infty, \infty).$$

Then, from the above arguments, we have the following

LEMMA 2.1. $U(\varepsilon_0)$ is a local coordinate neighborhood with origin at x_0 .

For any $G > 0$, we put

$$V_G = \{\hat{\Psi}(u_1, u_2, u_3) \in U(\varepsilon_0); (u_1, u_2) \in (-\varepsilon_0/2, \varepsilon_0/2)^2, 0 < u_3 < G\}.$$

Then $\bar{V}_G \subset U(\varepsilon_0)$. Let $\text{vol}(M, g)$ and $\text{vol}(V_G)$ denote the volumes of (M, g) and the open subspace V_G of (M, g) , respectively. Then, by the assumption, we have

$$(2.19) \quad \text{vol}(V_G) < \text{vol}(M, g) < \infty, \text{ for any } G > 0.$$

On the other hand, since $E_3 = \partial/\partial u_3$ on $U(\varepsilon_0)$, we have

$$\text{div } E_3 = (1/\sqrt{g_0})(\partial\sqrt{g_0}/\partial u_3) \text{ on } U(\varepsilon_0),$$

where

$$g_0 = \det(g_{ij}), \quad g_{ij} = g(\partial/\partial u_i, \partial/\partial u_j), \quad i, j = 1, 2, 3.$$

Thus, by (2.5), we get

$$(2.20) \quad (1/\sqrt{g_0})(\partial\sqrt{g_0}/\partial u_3) + (1/K)(\partial K/\partial u_3) = 0 \text{ on } U(\varepsilon_0).$$

Solving (2.20), we get

$$(2.21) \quad \sqrt{g_0} = C/K,$$

where $C = C(u_1, u_2)$ is a function of class C^∞ on $U(\varepsilon_0)$.

Thus, from (2.18) and (2.21), we get

$$\begin{aligned} \text{vol}(V_G) &= \int_{V_G} dM = \int_{-\varepsilon_0/2}^{\varepsilon_0/2} \int_{-\varepsilon_0/2}^{\varepsilon_0/2} \int_0^G (C/K) du_1 du_2 du_3 \\ &\geq a(\varepsilon_0)^2 G, \text{ for any } G > 0, \end{aligned}$$

where

$$a = \underset{\substack{-\varepsilon_0/2 \leq u_1, u_2 \leq \varepsilon_0/2 \\ u_3 = 0}}{\text{Min}} C/K > 0.$$

But, this contradicts (2.19). Thus we have the following

LEMMA 2.2. If $\text{vol}(M, g)$ is finite, then, for each point $x \in W$, $S = 2K$ is constant along $\gamma_x(s)$, $-\infty < s < \infty$.

3. Proof of Theorem B. In the sequel, we shall assume that $\text{vol}(M, g)$ is finite and $\text{rank } R^1$ is at most 2 on M and $\text{rank } R^1 = 2$ at some point of M . From Lemma 2.2, $H = 0$ on W_0 . Let $V = \{x \in W_0; f(x) \neq 0\}$. Now, we assume that $V \neq \emptyset$. Let V_0 be one component of V . $H = H(E) = 0$ implies $D(E)^2 = -4C_1(E)C_2(E)$. Put $\cos 2\theta(E) = K(C_1(E) + C_2(E))/f$ and $\sin 2\theta(E) = KD(E)/f$. Define (E^*) by $E_3^* = E_3$ and

$$\begin{aligned} E_1^* &= \cos \theta(E)E_1 - \sin \theta(E)E_2, \\ E_2^* &= \sin \theta(E)E_1 + \cos \theta(E)E_2. \end{aligned}$$

Then we have $D(E^*) = 0$. Furthermore, for (E) and (E') , we have $E_1^*(E) = \pm E_1^*(E')$ and $E_2^*(E) = \pm E_2^*(E')$. $H = 0$ and $D(E^*) = 0$ imply $C_1(E^*)C_2(E^*) = 0$. So we can assume that $C_2(E^*) = 0$ (otherwise, change $(E_1^*, E_2^*, E_3^*) \rightarrow (E_2^*, -E_1^*, E_3^*)$). Then we get

$$(3.1) \quad B_{1\ 32}^* \neq 0, \quad B_{1\ 31}^* = B_{2\ 31}^* = B_{2\ 32}^* = 0,$$

where

$$\nabla_{E_1^*} E_j^* = \sum_{k=1}^3 B_{i\ jk}^* E_k^*.$$

$R(E_1^*, E_3^*)E_2^* = 0$ implies

$$(3.2) \quad E_3^* B_{1\ 21}^* = 0.$$

$R(E_1^*, E_2^*)E_3^* = 0$ implies $B_{2\ 21}^* = 0$ and

$$(3.3) \quad E_2^* B_{1\ 32}^* + B_{1\ 21}^* B_{1\ 32}^* = 0.$$

$R(E_1^*, E_2^*)E_1^* = -KE_2^*$ implies

$$(3.4) \quad E_2^* B_{1\ 21}^* + (B_{1\ 21}^*)^2 = -K.$$

By $B_{2\ ij}^* = 0$, each trajectory of E_2^* is a geodesic. Put $h = B_{1\ 21}^*$ and $F = (E_1^* f)^2$. Then F is a function of class C^∞ on V_0 . From Lemma 2.2, and (3.3), we get

$$\begin{aligned} E_3^*(E_1^* f) &= E_1^*(E_3^* f) + [E_3^*, E_1^*]f \\ &= -B_{1\ 32}^*(E_2^* f) = f^2 h, \quad \text{i.e.,} \\ (3.5) \quad d(E_1^* f)/ds &= f^2 h, \quad \text{along } \gamma_x(s), \quad x \in V_0. \end{aligned}$$

From Lemma 2.2, for each point $x \in V_0$, $\gamma_x(s) \in V_0$, $-\infty < s < \infty$. Taking account of (3.2) and solving (3.5), we get

$$(3.6) \quad F = (f(x)^2 h(x)s + c)^2, \quad \text{along } \gamma_x(s), \quad -\infty < s < \infty,$$

where c is constant along $\gamma_x(s)$.

Let $V^* = \{x \in V_0; h(x) \neq 0\}$. From (3.4), we see that $V^* \neq \emptyset$. Let V_0^* be one component of V^* . Then, by (3.2), we see that, for each point $x \in V_0^*$, $\gamma_x(s) \in V_0^*$, $-\infty < s < \infty$. For each point $x \in V_0^*$, consider $\gamma_x(s)$. Let $x_0 = \gamma_x(-c/f(x)^2 h(x))$ in (3.6).

Then we have

$$(3.7) \quad F = ((f^2 h)w_3 + k)^2, \text{ on } V(\epsilon, \delta) \cap V_0^*,$$

where $k = k(u_1, u_2)$ is a function of class C^∞ on $V(\epsilon, \delta) \cap V_0^*$ such that $k(0, 0) = 0$, and $V(\epsilon, \delta)$ is a local coordinate neighborhood with origin at x_0 constructed by the similar fashion as in § 2. From (3.7), by applying the similar arguments as in the proof of Lemma 2.1, to the function F instead of K , we can construct a local coordinate neighborhood

$$U(\epsilon^*) = \{\Psi^*(u_1, u_2, u_3) \in V_0^*; (u_1, u_2) \in (-\epsilon^*, \epsilon^*)^2, -\infty < u_3 < \infty\},$$

with origin at x_0 such that $F = ((f^2 h)u_3)^2$ on $U(\epsilon^*)$, where $\epsilon^* > 0$, and Ψ^* is a mapping of class C^∞ defined by the similar way as $\hat{\Psi}$ in § 2. For any $G > 0$, let $V_G^* = \{\Psi^*(u_1, u_2, u_3) \in U(\epsilon^*); (u_1, u_2) \in (-\epsilon^*/2, \epsilon^*/2)^2, 0 < u_3 < G\}$. From Lemma 2.2, and (2.5), we have $\text{div } E_3^* = 0$. Thus, we can see that if $G \rightarrow \infty$, then $\text{vol}(V_G^*) \rightarrow \infty$. But, this is a contradiction. Thus, we can conclude that $f = 0$ on W_0 and hence T_1 is integrable on W_0 . Thus, T_1 and T_0 are parallel on W_0 (cf. S. Tanno [7]). If W is dense in M , the restricted homogeneous holonomy group of (M, g) is reducible. If W is not dense in M , then the interior of the complement of W in M is flat. Hence, also in this case, the restricted homogeneous holonomy group of (M, g) is reducible. Lastly, if $\text{rank } R^1 = 0$ on M , then (M, g) is flat. Therefore, this completes a proof of Theorem B.

From our arguments in this paper, we can also show the following

THEOREM C. *Let (M, g) be a complete and simply connected 3-dimensional Riemannian manifold satisfying (*). If the volume of (M, g) is finite, then (M, g) is isometric to a 3-dimensional sphere.*

REFERENCES

- [1] K. ABE, A characterization of totally geodesic submanifolds in S^N and CP^N by an inequality, Tôhoku Math. J., 23 (1971), 219-244.
- [2] Y. H. CLIFTON AND R. MALTZ, The K -nullity spaces of the curvature operator, Michigan Math. J., 17 (1970), 85-89.
- [3] S. KOBAYASHI AND K. NOMIZU, Foundations of differential geometry, Vol. I, Wiley-Interscience, New York, 1963.
- [4] A. ROSENTHAL, Riemannian manifolds of constant nullity, Michigan Math. J., 14 (1967), 469-480.
- [5] K. SEKIGAWA, On some 3-dimensional Riemannian manifolds, Hokkaido Math. J., 2(1973), 259-270.

- [6] H. TAKAGI AND K. SEKIGAWA, On some 3-dimensional Riemannian manifolds satisfying a certain condition on the curvature tensor, *Sci. Rep. Niigata Univ.*, 7 (1969), 23-27.
- [7] S. TANNO, A theorem on totally geodesic foliations and its applications, *Tensor N. S.*, 24 (1972), 116-122.
- [8] S. TANNO, 3-dimensional Riemannian manifolds satisfying $R(X, Y) \cdot R = 0$, to appear.

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