

On Some Applications of a New Integral Transform

Hassan Eltayeb and Adem Kılıçman

Department of Mathematics and Institute for Mathematical Research
University Putra Malaysia, 43400 UPM
Serdang, Selangor, Malaysia
eltayeb@putra.upm.edu.my
akilicman@putra.upm.edu.my

Abstract. In this work a new integral transform, namely Sumudu transform was applied to solve linear ordinary differential equation with and without constant coefficients having convolution terms. In particular we apply Sumudu transform technique to solve Spring-Mass systems, Population Growth and financial problem.

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1. INTRODUCTION

The differential equations have played a central role in every aspects of applied mathematics for very long time and with the advent of the computer their importance has increased further. It is also well known that throughout science, engineering and far beyond, scientific computation is taking place in efforts to understand and control our natural environment in order to develop new technological processes. Thus investigation and analysis of differential equations arising in applications led to many deep mathematical problems therefore there are so many different techniques in order to solve differential equations.

Recently Watugala in [3],[4], [5], introduced a new integral transform and named it as Sumudu transform that is defined by the formula

$$F(u) = S[f(t); u] = \frac{1}{u} \int_0^{\infty} e^{-(\frac{t}{u})} f(t) dt, \quad u \in (-\tau_1, \tau_2).$$

and applied it to the solution of ordinary differential equations in control engineering problems. It appeared like the modification of the well known Laplace transform. However in [2], some fundamental properties of the Sumudu transform were established. By looking at the properties of this transform one can notice that the Sumudu transform has very special and useful properties and it can help with intricate applications in sciences and engineering. Once more, Watugala's work was followed by Weerakoon [1], [6] by introducing a complex inversion formula for the Sumudu transform.

Recently, A. Kılıçman and H. Eltayeb in [8], introduced a new method to produce a partial differential equation having polynomial coefficients by using the PDEs with constant coefficients and also studied the classification of the new partial differential equations. Later the same authors extended this setting in [13] to the finite product of convolution of hyperbolic and elliptic PDEs where the authors considered the positive coefficients of polynomials. In [11], the authors also applied double integral transforms to solve the partial differential equations.

In this study our purpose is to show the applicability of this interesting new transform as well as its efficiency in solving the linear ordinary differential equation with constant and non constant coefficients having convolutions terms, for more details see ([8]– [13]). We also provide some different examples by focusing our attention on solving Spring-mass and population growth problem.

Now let A be the set of single Sumudu transformable functions that is

$$A = \left\{ f(t) | \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{t}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}$$

Recall the following theorem that was given by Belgacem in [7], where they discussed the Sumudu transform of the derivatives:

Theorem 1. *Let $f(t)$ be in A , and let $G^{(n)}(u)$ denote the Sumudu transform of the n th derivative, $f^{(n)}(t)$ of $f(t)$, then for $n \geq 1$*

$$G^{(n)}(u) = \frac{G(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}}$$

more details see [7].

2. MAIN RESULTS

Before we prove the main results we provide the following lemma in order to show that $f(t)$ can be uniquely recovered from $F(u)$.

Lemma 1. *Let $f(t)$ and $g(t)$ be continuous functions defined for $t \geq 0$ have Sumudu transforms, $F(u)$ and $G(u)$, respectively. If $F(u) = G(u)$ almost everywhere then $f(t) = g(t)$ where u is a complex number.*

Proof. If α are sufficiently large, then the integral representation of $f(t)$ by

$$f(t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{\frac{t}{u}} F(u) du$$

and since $F(u) = G(u)$ all most every where then we have

$$f(t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{\frac{t}{u}} G(u) du$$

for the inverse Sumudu transform we replace u by $\frac{1}{s}$ to obtain

$$f(t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{st} G\left(\frac{1}{s}\right) \frac{ds}{s} = g(t)$$

and the theorem is proven. ■

In the next theorem we study the existence of Sumudu transform as follows.

Theorem 2. (Existence of the Sumudu transform) *If f is of exponential order, then its Sumudu transform $S[f(t)] = F(u)$ exists and is given by*

$$F(u) = \int_0^\infty e^{-\frac{t}{u}} f(t) dt,$$

where $\frac{1}{u} = \frac{1}{\eta} + \frac{i}{\tau}$. The defining integral for F exists at points $\frac{1}{u} = \frac{1}{\eta} + \frac{i}{\tau}$ in the right half plane $\eta > K$ and $\zeta > L$.

Proof. By using $\frac{1}{u} = \frac{1}{\eta} + \frac{i}{\tau}$ and, we can express $F(u)$ as

$$\begin{aligned} F(u) &= \int_0^\infty f(t) \cos\left(\frac{t}{\tau}\right) e^{-\frac{t}{\eta}} dt \\ &\quad - i \int_0^\infty f(t) \sin\left(\frac{t}{\tau}\right) e^{-\frac{t}{\eta}} dt. \end{aligned}$$

Then for values of $\frac{1}{\eta} > \frac{1}{K}$, we have

$$\begin{aligned} \int_0^{\infty} |f(t)| \left| \cos\left(\frac{t}{\tau}\right) \right| e^{-\frac{t}{\eta}} dt &\leq M \int_0^{\infty} e^{(\frac{1}{K} - \frac{1}{\eta})t} dt \\ &\leq \left(\frac{M\eta K}{\eta - K} \right) \end{aligned}$$

and

$$\begin{aligned} \int_0^{\infty} |f(t)| \left| \sin\left(\frac{t}{\tau}\right) \right| e^{-\frac{t}{\eta}} dt &\leq M \int_0^{\infty} e^{(\frac{1}{K} - \frac{1}{\eta})t} dt \\ &\leq \left(\frac{M\eta K}{\eta - K} \right) \end{aligned}$$

which imply that the integrals defining the real and imaginary parts of F exist for value of $\text{Re}(\frac{1}{u}) > \frac{1}{K}$, completing the proof. ■

Before we study the Sumudu transform of differential equations, let us give some concept of Sumudu transform of higher derivatives as follow.

Proposition 1. (Sumudu transform of higher derivatives) *Let f be n times differentiable on $(0, \infty)$ and $f(t) = 0$ for $t < 0$. Further suppose that $f^{(n)} \in L_{loc}$. Then $f^{(k)} \in L_{loc}$ for $0 \leq k \leq n-1$, $\text{dom}(Sf) \subset \text{dom}(Sf^{(n)})$ and, for any polynomial P of degree n ,*

$$(2.1) \quad P(u)S(y)(u) = S(f)(u) + M_P(u)\varphi(y, n)$$

for $u \in \text{dom}(Sf)$. In particular

$$(2.2) \quad (Sf^{(n)})(u) = \frac{1}{u^n}(Sf)(u) - \left(\frac{1}{u^n}, \frac{1}{u^{n-1}}, \dots, \frac{1}{u} \right) \varphi(f; n)$$

(where with $\varphi(f; n)$ we mean a column vector). For particular case, for $n = 2$ we have

$$(2.3) \quad (Sf'')(u) = \frac{1}{u^2}(Sf)(u) - \frac{1}{u^2}f(0+) - \frac{1}{u}f'(0+)$$

The proof of this proposition is given in [12].

In general, if f is a differentiable on (a, b) with $a < b$, and $f(t) = 0$ for $t < a$ or $t > b$ and $f^{(n)} \in L_{loc}$ then, for all u

$$(2.4) \quad S \left[P(\dot{D})f \right] (u) = P(u)(Sf)(u) - M_P(u) \left[e^{-\frac{a}{u}} \varphi(f; a; n) - e^{-\frac{b}{u}} \phi(f; b; n) \right].$$

Let y be n times differentiable on $(0, \infty)$, zero on $(-\infty, 0)$ and satisfy the following equation

$$(2.5) \quad P(\dot{D})y = f * g$$

under the initial condition

$$y(0) = y_0, y'(0) = y_1, \dots, y^{(n-1)}(0) = y_{n-1}$$

Then $y^{(k)}$ is locally integrable and Sumudu transformable for $0 \leq k \leq n$ and, for every such k , then Sumudu transform of Eq(2.5) given by Eq(2.1) where

$$P(u) = \frac{a_n}{u^n} + \frac{a_{n-1}}{u^{n-1}} + \dots + a_0,$$

$$(2.6) \quad M_P(u)\varphi(y, n) = \begin{pmatrix} \frac{1}{u} & \frac{1}{u^2} & \dots & \frac{1}{u^n} \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \cdot & \cdot & \cdot & a_n \\ a_2 & a_3 & \cdot & \cdot & a_n & 0 \\ a_3 & \cdot & \cdot & a_n & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_{n-1} \end{pmatrix}$$

and the non homogeneous term is single convolution defined by

$$f * g = \int_0^x f(t-x)g(x)dx$$

In particular for, $n = 2$ we have

$$\left(\frac{a_2}{u^2} + \frac{a_1}{u} + a_0\right)S(y)(u) = S(f * g)(u) + \begin{pmatrix} \frac{1}{u} & \frac{1}{u^2} \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_2 & 0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$$

In order to get the solution of Eq(2.5), we taking inverse Sumudu transform for Eq(2.1) as follows

$$(2.7) \quad y(t) = S^{-1} \left[\frac{(f * g)(u)}{P(u)} \right] + S^{-1} \left[\frac{M_P(u)}{P(u)} \phi(y, n) \right]$$

provided that the inverse exist for each terms in the right hand side of Eq (2.7).

Now, let us multiply the right hand side of Eq(2.5) by polynomial $\Psi(t) = \sum_{k=0}^n t^k$, we obtain the non constant coefficients in the form of

$$(2.8) \quad \Psi(t) * [P(D)y] = f * g$$

under the same initial conditions used above. By taking Sumudu transform for Eq(2.8) and using the initial condition, after arrangement we have

$$S[y](u) = \frac{F(u)G(u)}{k!u^k P(u)} + \frac{1}{P(u)} \begin{pmatrix} \frac{1}{u} & \frac{1}{u^2} & \dots & \frac{1}{u^{n+1}} \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \cdot & \cdot & \cdot & a_n \\ a_2 & a_3 & \cdot & \cdot & a_n & 0 \\ a_3 & \cdot & \cdot & a_n & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_{n-1} \end{pmatrix}$$

By taking inverse Sumudu transform we have

$$(2.9) \quad y = S^{-1} \left[\frac{F(u)G(u)}{k!u^k P(u)} + \frac{1}{P(u)} \left(\frac{1}{u} \frac{1}{u^2} \dots \frac{1}{u^{n+1}} \right) \begin{pmatrix} a_1 & a_2 & \cdot & \cdot & \cdot & a_n \\ a_2 & a_3 & \cdot & \cdot & a_n & 0 \\ a_3 & \cdot & \cdot & a_n & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ y_{n-1} \end{pmatrix} \right]$$

Here we assume that the inverse exist. Now, if we substitute Eq(2.9) into Eq(2.8), we obtain the non homogeneous term of Eq(2.8) that is $f * g$ and polynomial in the form of $\Phi(t) = -\sum_{k=1}^n \frac{1}{k!} t^k$.

The solution of Eq(2.5) by using Laplace transform is similar to solution by using Sumudu transform just we replace $\frac{1}{u}$ by s , and Eq(2.6) written in the form of

$$(2.10) \quad M_P(s)\varphi(y, n) = (1 \ s \dots s^{n-1}) \begin{pmatrix} a_1 & a_2 & \cdot & \cdot & \cdot & a_n \\ a_2 & a_3 & \cdot & \cdot & a_n & 0 \\ a_3 & \cdot & \cdot & a_n & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ y_{n-1} \end{pmatrix}$$

by using Laplace transform and inverse laplace transform we have

$$(2.11) \quad y = L^{-1} \left[\frac{F(s)G(s)}{P(s)} + \frac{1}{P(s)} (1 \ s \dots s^{n-1}) \begin{pmatrix} a_1 & a_2 & \cdot & \cdot & \cdot & a_n \\ a_2 & a_3 & \cdot & \cdot & a_n & 0 \\ a_3 & \cdot & \cdot & a_n & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ y_{n-1} \end{pmatrix} \right]$$

3. SOME APPLICATIONS OF SUMUDU TRANSFORM

The Sumudu transforms are very useful method in applications. In this section we focus on three examples in the following problems:

Example 1. Spring-Mass systems Consider the Spring-Mass systems given by the following second order initial value problem

$$my'' + cy' + ky = f(t), \quad (t > 0); \quad y(0) = \alpha, \quad y'(0) = \beta$$

where m represents the mass, c the damping coefficient, k the spring constant determined by Hooke's law and $f(t)$ the forcing function. In particular we consider the initial value problem that having a discontinuous forcing function and $m = 1$, $c = -2$, $k = -3$ given by

$$(3.1) \quad y'' - 2y' - 3y = f(t), \quad (t > 0); \quad y(0) = 1, \quad y'(0) = 0$$

where

$$f(t) = \begin{cases} 0, & \text{if } 0 < t \leq 1 \\ t-1 & \text{if } 1 < t \leq 4 \\ 3 & \text{if } t > 4 \end{cases},$$

First $f(t)$ can be write in the form of

$$\begin{aligned} f(t) &= (t-1)[H(t-1) - H(t-4)] + 3H(t-4) \\ &= (t-1)H(t-1) - (t-4)H(t-4). \end{aligned}$$

Then Eq(3.1) can be written in the form

$$(3.2) \quad y'' - 2y' - 3y = (t-1)H(t-1) - (t-4)H(t-4), \quad (t > 0); \quad y(0) = 1, \quad y'(0) = 0.$$

By using Sumudu transform for Eq(3.2) we have

$$\left(\frac{1}{u^2} - \frac{2}{u} - 3\right) Y(u) = ue^{-\frac{1}{u}} - ue^{-\frac{4}{u}} + \left(\frac{1}{u} \frac{1}{u^2}\right) \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

by simplifying above equation, we have

$$(3.3) \quad Y(u) = \frac{u^3 e^{-\frac{1}{u}}}{1-2u-3u^2} - \frac{u^3 e^{-\frac{4}{u}}}{1-2u-3u^2} + \frac{-2u+1}{1-2u-3u^2}$$

by replacing u by $\frac{1}{s}$ then we have

$$(3.4) \quad Y\left(\frac{1}{s}\right) = \frac{e^{-s}}{s^2(s^2-2s-3)} - \frac{e^{-4s}}{s^2(s^2-2s-3)} + \frac{s-2}{(s^2-2s-3)},$$

by using inverse Sumudu transform for Eq(3.4) we obtain the solution of Eq(3.1)

$$\begin{aligned} y(t) &= \left(\frac{2}{9} - \frac{1}{3}(t-1) - \frac{1}{4}e^{-(t-1)} + \frac{1}{36}e^{3(t-1)}\right) H(t-1) \\ &\quad - \left(\frac{2}{9} - \frac{1}{3}(t-4) - \frac{1}{4}e^{-(t-4)} + \frac{1}{36}e^{3(t-4)}\right) H(t-4) \\ &\quad + \frac{1}{4}(3e^{-t} + e^{3t}) H(t). \end{aligned}$$

In the region $t > 0$ we have $H(t) = 1$ so that after a little simplification, for $t > 0$,

$$\begin{aligned} y(t) &= \left(\frac{5}{9} - \frac{1}{3}t - \frac{1}{4}e^{-(t-1)} + \frac{1}{36}e^{3(t-1)} \right) H(t-1) \\ &\quad - \left(\frac{14}{9} - \frac{1}{3}t - \frac{1}{4}e^{-(t-4)} + \frac{1}{36}e^{3(t-4)} \right) H(t-4) \\ &\quad + \frac{1}{4} (3e^{-t} + e^{3t}). \end{aligned}$$

and $y(t)$ may also be written in the form of

$$y(t) = \begin{cases} \frac{1}{4}(3e^{-t} + e^{3t}) & \text{if } 0 < t \leq 1 \\ \frac{5}{9} - \frac{1}{3}t + \frac{3}{4}e^{-t} + \frac{1}{4}e^{3t} \\ \quad - \frac{1}{4}e^{1-t} + \frac{1}{36}e^{3t-3} & \text{if } 1 < t \leq 4 \\ -1 + \frac{3}{4}e^{-t} + \frac{1}{4}e^{3t} \\ \quad - \frac{1}{4}e^{1-t} + \frac{1}{36}e^{3t-3} & \text{if } t > 4 \\ \quad + \frac{1}{4}e^{4-t} - \frac{1}{36}e^{3t-12} \end{cases}.$$

Example 2. Population Growth

The growth of a population is usually modelled with an equation of the form

$$y'(t) = ky(t)$$

where y represents the number of individuals at a given time t and k is carry capacity of environment. This model assumes that the rate of growth of population is proportional to the population size. The Sumudu transforms can be used to solve the Population problems as follows. Let $y(t)$ represent the Population of certain country where the rate of Population increases depends on the growth rate of the country as well as the rate at which people are being added to or subtracted from the Population due to immigration. Now let us consider Population problem given by the first order differential equation having forcing function as

$$(3.5) \quad y'(t) + ky(t) = f(t), \quad y(0) = y_0$$

where the force function given by $f(t) = 1000(1 + \lambda \sin(t))$, k and λ are constants by using Sumudu transform for (3.5) we have

$$(3.6) \quad S(y)(u) = \frac{100u}{1+ku} \left(\frac{u^2 + \lambda u + 1}{u^2 + 1} \right) + \frac{y_0}{1+ku},$$

on using inverse Sumudu transform for Eq(3.6) we obtain the solution of Eq(3.5) in the form of

$$(3.7) \quad y(t) = \frac{1000}{k} (1 - e^{-kt}) + \frac{1000\lambda}{(k^2 + 1)} (e^{-kt} - \cos(t) + \sin(t)k) + y_0 e^{-kt}.$$

Example 3. Financial Application

Suppose that money deposited in a bank increases with continuous compounded interest. This mean the rate of growth is proportional to the amount present. Assume that the amount of money present given by $y(t)$ at time t and let μ be proportionality constant. Consider the $y(t)$ the first order differential equation

$$(3.8) \quad \frac{dy(t)}{dt} + \mu y(t) = f(t), \quad y(0) = \theta$$

where $f(t)$ represent the forcing function and initial amount given by $y(0) = \theta$. In particular we consider the problem with a discontinuous forcing function given by

$$(3.9) \quad \frac{dy(t)}{dt} + \mu y(t) = \begin{cases} \sin(t), & 0 \leq t < \pi \\ 0, & t \geq \pi \end{cases}, \quad y(0) = \theta$$

Eq(3.9) can be rewrite in the form of

$$(3.10) \quad \begin{aligned} \frac{dy(t)}{dt} + \mu y(t) &= \sin(t) [H(t - 0) - H(t - \pi)] \\ &= \sin(t) + \sin(t - \pi)H(t - \pi). \end{aligned}$$

On using Sumudu transform for Eq(3.10) we have

$$(3.11) \quad Y(u) = \frac{u^2 (1 + e^{-\frac{\pi}{u}})}{(1 + u^2)(1 + \mu u)} + \frac{\theta}{(1 + \mu u)},$$

in order to obtain the solution of Eq(3.10) we apply inverse Sumudu transform, thus

$$(3.12) \quad y(t) = \frac{(1 + e^{\pi\mu})e^{-\mu t}}{(\mu^2 + 1)} + \theta e^{-\mu t}$$

if we consider the proportionality constant $\mu = 1$ and initial amount $y(0) = 0$ then Eq(3.12) becomes

$$y(t) = \frac{1}{2} (1 + e^{\pi}) e^{-t}.$$

Conclusion 1. Application of the Sumudu transform to the solution of Spring-Mass systems, Population Growth and financial problem has been demonstrated.

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