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On some classes of mixed-super quasi-Einstein manifolds

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Abstract. Quasi-Einstein manifold and generalized quasi-Einstein manifold are the generalizations of Einstein manifold. The purpose of this paper is to study the mixed super quasi-Einstein manifold which is also the generalizations of Einstein manifold satisfying some curvature conditions. We define both Riemannian and Lorentzian doubly warped product on this manifold. Finally, we study the completeness properties of doubly warped products on $MS(QE)_4$ for both the Riemannian and Lorentzian cases.

1 Introduction

The notion of quasi-Einstein manifold was introduced by M. C. Chaki and R. K. Maity [7]. A non-flat Riemannian manifold (M^n, g), $(n \geq 3)$ is a *quasi-Einstein manifold* if its Ricci tensor S satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) \tag{1}$$

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and is not identically zero, where a, b are scalars, $b \neq 0$ and A is a non-zero 1-form such that

$$g(X, U) = A(X), \forall \ X \in \chi(M), \tag{2}$$

where, U being a unit vector field and $\chi(M)$ is the set of all differentiable vector fields on M.

Here a and b are called the *associated scalars*, A is called the associated 1-form and U is called the *generator* of the manifold. Such an n-dimensional manifold will be denoted by $(QE)_n$.

As a generalization of quasi-Einstein manifold, in [8], U. C. De and G. C. Ghosh defined the *generalized quasi-Einstein manifold.* A non-flat Riemannian manifold is called *generalized quasi-Einstein manifold* if its Ricci-tensor is non-zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y), \tag{3}$$

where a, b and c are non-zero scalars and A, B are two 1-forms such that

$$g(X, U) = A(X) \quad \text{and} \quad g(X, V) = B(X), \tag{4}$$

U and V being unit vectors which are orthogonal, i.e.,

$$g(U, V) = 0. \tag{5}$$

The vector fields U and V are called the generators of the manifold. This type of manifold will be denoted by $G(QE)_n$.

In [6], M. C. Chaki introduced the super quasi-Einstein manifold, denoted by $S(QE)_n$, where the Ricci tensor is not identically zero and satisfies the condition

$$\begin{aligned} S(X, Y) = ag(X, Y) &+ bA(X)A(Y) + c[A(X)B(Y) \\ &+ A(Y)B(X)] + dD(X, Y), \end{aligned} \tag{6}$$

where a, b, c and d are scalars such that b, c, d are nonzero, A, B are two nonzero 1-forms defined as (4) and U, V are mutually orthogonal unit vector fields, D is a symmetric $(0, 2)$ tensor with zero trace which satisfies the condition

$$D(X, U) = 0, \quad \forall X \in \chi(M). \tag{7}$$

Here a, b, c and d are called the *associated scalars*, A, B are called the *associated main and auxiliary 1-forms* respectively, U, V are called the main and the *auxiliary generators* and D is called the *associated tensor* of the manifold.

In [3], A. Bhattacharyya and T. De introduced the notion of *mixed generalized quasi-Einstein manifold*. A non-flat Riemannian manifold is called *mixed generalized quasi-Einstein manifold* if its Ricci tensor is non-zero and satisfies the condition

$$\begin{aligned} S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \\ + d[A(X)B(Y) + A(Y)B(X)], \end{aligned} \tag{8}$$

where a, b, c, d are non-zero scalars,

$$g(X,U) = A(X) \quad \text{and} \quad g(X,V) = B(X), \qquad \forall \;\; X \in \chi(M), \tag{9}$$

and also

$$g(U,V) = 0. \tag{10}$$

A, B are two non-zero 1-forms, U and V are unit vector fields corresponding to the 1-forms A and B respectively. If $d = 0$, then the manifold becomes to a $G(QE)_n$. This type of manifold is denoted by $MG(QE)_n$.

In [4], A. Bhattacharyya, M. Tarafdar and D. Debnath introduced the notion of $MS(QE)_n$.
A non-flat Riemannian manifold (M^n, g), $(n \geq 3)$ is called *mixed super quasi-Einstein manifold* if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$\begin{aligned} S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) + d[A(X)B(Y) \\ + A(Y)B(X)] + eD(X,Y), \end{aligned} \tag{11}$$

where a, b, c, d, e are scalars of which $b \neq 0$, $c \neq 0$, $d \neq 0$, $e \neq 0$ and A, B are two non zero 1-forms such that

$$g(X,U) = A(X) \text{ and } g(X,V) = B(X), \qquad \forall \;\; X \in \chi(M), \tag{12}$$

U, V being mutually orthogonal unit vector fields, D is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition

$$D(X,U) = 0, \qquad \forall \;\; X \in \chi(M). \tag{13}$$

Here a, b, c, d, e are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, U, V are called the main and the auxiliary generators and D is called the associated tensor of the manifold. We denote this type of manifold $MS(QE)_n$.

The notation of warped product generalizes that of a surface of revolution. Warped products were first defined by O'Neill and Bishop in [5]. They used this concept to construct Riemannian manifolds with negative sectional curvature. Then Beem, Ehrlich and Powell pointed out that many exact solutions in Einstein's field equation can be expressed in terms of Lorentzian warped products [2].

In general, doubly warped product was studied by Btilent Unal in [12], can be considered as a generalization of singly warped product. A doubly warped product (M, g) is a product manifold which is of the form $M =_f B \times_b F$ with the metric $g = f^2_{g_B} \oplus b^2_{g_F}$ where $b : B \longrightarrow (0, \infty)$ and $f : F \longrightarrow (0, \infty)$ are smooth map.
So if (B, g_B) and (F, g_F) be pseudo-Riemannian manifolds and also $b : B \longrightarrow (0, \infty)$ and $f : F \longrightarrow (0, \infty)$ be smooth functions, then the doubly warped product is the product manifold $B \times F$ furnished with the metric tensor $f^2_{g_B} \oplus b^2_{g_F}$ defined by

$$g = (f \circ \sigma)^2 \pi^*(g_B) \oplus (b \circ \pi)^2 \sigma^*(g_F).$$

The functions $b : B \longrightarrow (0, \infty)$ and $f : F \longrightarrow (0, \infty)$ are called warping functions and $\pi : B \times F \longrightarrow B$ and $\sigma : B \times F \longrightarrow F$ are usual projections map.

If (F, g_F) and (B, g_B) are both Riemannian manifolds, then $({}_fB \times_b F, f^2_{g_B} \oplus b^2_{g_F})$ is also a Riemannian manifold. We call $({}_fB \times_b F, f^2_{g_B} \oplus b^2_{g_F})$ a Loretzian doubly warped product if (F, g_F) is Riemannian and either (B, g_B) is Lorentzian or else (B, g_B) is a one-dimensional manifold with a negative definite metric $-dt^2$. If neither b nor f is constant, then we have a proper doubly warped product.

Global hyperbolicity is the most important condition on Causality, which lies at the top of the so-called causal hierarchy of spacetimes and is involved in problems as Cosmic Censorship, predictability etc.

A connected Lorentzian manifold is called time-orientable iff it admits a nowhere-vanishing timelike vector field (defining future causal directions). A piecewise C^1 curve $c : I \longrightarrow M$ in a time-oriented manifold (M, g) is called future iff $c'(t)$ is future for every $t \in I$. For any point $p \in M$, the future of p (resp. past of p), denoted by $J^+(p)$ (resp. $J^-(p)$), is the set of all points q s.t.there is a future curve from p to q (resp. from q to p).

There are different alternative definitions of what global hyperbolicity means, but perhaps the most standard one is the following. A spacetime (M, g) is said globally hyperbolic if and only if it satisfies two conditions: (A) compactness of $J^+(p) \cap J^-(q)$ for all $p, q \in M$ (i.e. no "naked" singularity can exist) and (B) strong causality (no "almost closed" causal curve exists).

Global hyperbolicity is also discussed in the theorem (34), (36) and the corollary (36) in [13].

In this paper we find that a Riemannian manifold is a manifold of mixed super quasi constant curvature iff it is conformally flat $MS(QE)_n$. Also we have studied Ricci-pseudosymmetric $MS(QE)_n$. Next we have obtained some expressions for Riemannian curvature tensor when $MS(QE)_n$ satisfies the curvature conditions $C.S = 0$, $\tilde{C}.S = 0$ and $C_1.S = 0$, where C is the Weyl conformal curvature tensor, $\tilde{C}$ is the concircular curvature tensor and C_1 quasi-conformal curvature tensor. We have also proved that in a conformally flat $MS(QE)_n$ $(n \geq 3)$ with $R(X, Y).S = 0$, the vector fields U, V corresponding to 1-forms A, B are co-directional. Finally in the last two sections, we discuss about the doubly warped product on $MS(QE)_n$ and completeness of doubly warped products on $MS(QE)_4$ with examples.

2 Preliminaries

In this section we consider $MS(QE)_n$, $(n \geq 3)$ with associated scalars a, b, c, d, e, associated main and auxiliary 1-forms A, B, main and auxiliary generators U, V and associated symmetric $(0, 2)$ tensor D.

So (11), (12) and (13) will hold. Since U and V are mutually orthogonal unit vector fields, we have

$$g(U, U) = 1, \quad g(V, V) = 1 \quad \text{and} \quad g(U, V) = 0, \tag{14}$$

$$\operatorname{trace} D = 0 \tag{15}$$

$$D(X, U) = 0, \quad \forall\, X \in \chi(M). \tag{16}$$

Also using (14) in (12), we get

$$A(V) = B(U) = 0. \tag{17}$$

Now setting $X = Y = e_i$, where $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold, in (11) and taking summation over i, $1 \leq i \leq n$, we obtain

$$r = na + b + c, \tag{18}$$

where r is the scalar curvature of the manifold.
Also, from (11), we have

$$S(U, U) = a + b \tag{19}$$

$$S(V, V) = a + c + eD(V, V) \tag{20}$$

$$S(U, V) = d. \tag{21}$$

If X is a unit vector field, then $S(X, X)$ is the Ricci-curvature in the direction of X. Hence from (19) and (20) we can state that $a + b$ and $a + c + eD(V, V)$ are the Ricci curvature in the directions of U and V respectively.
Let Q be the Ricci operator, i.e.,

$$g(QX, Y) = S(X, Y) \qquad \forall\ X, Y \in \chi(M). \tag{22}$$

Also we have

$$g(lX, Y) = D(X, Y). \tag{23}$$

Another notion of curvature called mixed super quasi constant curvature was introduced in [4]. A Riemannian manifold is said to be a manifold of mixed super quasi-constant curvature if it is conformally flat and the curvature tensor R of type $(0, 4)$ satisfies the condition

$$\begin{aligned}\tilde{R}(X, Y, Z, W) = {} & m[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + p[g(X, W)A(Y)\\ & A(Z) - g(Y, W)A(X)A(Z) + g(Y, Z)A(X)A(W) - g(X, Z)\\ & A(Y)A(W)] + q[g(X, W)B(Y)B(Z) - g(Y, W)B(X)B(Z)\\ & + g(Y, Z)B(X)B(W) - g(X, Z)B(Y)B(W)] + s[\{A(Y)B(Z)\\ & + B(Y)A(Z)\}g(X, W) - \{A(X)B(Z) + B(X)\\ & A(Z)\}g(Y, W) + \{A(X)B(W) + B(X)A(W)\}g(Y, Z)\\ & - \{A(Y)B(W) + B(Y)A(W)\}g(X, Z)] + t[g(Y, Z)D(X, W)\\ & - g(X, Z)D(Y, W) + g(X, W)D(Y, Z) - g(Y, W)D(X, Z)].\end{aligned} \tag{24}$$

An n-dimensional *Riemannian manifold* (M^n, g) is called *Ricci-pseudosymmetric* [9] if the tensors $R.S$ and $Q(g, S)$ are *linearly dependent*, where

$$(R(X, Y).S)(Z, W) = -S(R(X, Y)Z, W) - S(Z, R(X, Y)W), \tag{25}$$

$$Q(g, S)(Z, W; X, Y) = -S((X \wedge Y)Z, W) - S(Z, (X \wedge Y)W) \tag{26}$$

and

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$$

for vector fields X, Y, Z, W on M^n, R denotes the curvature tensor of M^n. The condition of *Ricci-pseudosymmetry* is equivalent to the relation

$$(R(X, Y).S)(Z, W) = L_S Q(g, S)(Z, W; X, Y) \tag{27}$$

holding on the set

$$U_S = \{x \in M : S \neq \frac{r}{n} g \text{ at } x\}, \tag{28}$$

where L_S is some function on U_S. If $R.S = 0$ then M^n is called *Ricci-semi-symmetric.* Every *Ricci-semisymmetric manifold is Ricci-pseudosymmetric but the converse is not true* [9].
The Weyl conformal curvature tensor C of type $(1,3)$ of an n-dimensional *Riemannian manifold* (M^n, g), $(n \geq 3)$ is defined by [15]

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ - g(X,Z)QY] + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{29}$$

The *concircular curvature tensor* $\tilde{C}$ of type $(1,3)$ of n-dimentional *Riemanian manifold* (M^n, g),$(n \geq 3)$ is defined by [15]

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \tag{30}$$

for any vector fields $X, Y, Z \in \chi(M)$.
The *quasi-conformal curvature tensor* was defined by Yano and Sawaki [14] as

$$\begin{aligned} C_1(X,Y)Z = \lambda R(X,Y)Z + \mu\{S(Y,Z)X + S(X,Z)Y + g(Y,Z)QX \\ - g(X,Z)QY\} - \frac{r}{n}\left[\frac{\lambda}{(n-1)} + 2\mu\right][g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{31}$$

where λ and μ are nonzero constants. If $\lambda = 1$ and $\mu = \frac{1}{n-2}$, then quasi-conformal curvature tensor is reduced to the conformal curvature tensor.

3 Relation between manifold of mixed super quasi constant curvature and $MS(QE)_n$

Let M be a Riemannian manifold with mixed super quasi constant curvature and $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold. Taking $X = W = \{e_i\}$ and summing over i, $1 \leq i \leq n$ in (24) and using (23), we obtain

$$\begin{aligned} S(Y,Z) = m(n-2)g(Y,Z) + p(n-2)A(Y)A(Z) + q(n-2)B(Y)B(Z) \\ + s(n-2)[A(Y)B(Z) + A(Z)B(Y)] + t(n-2)D(Y,Z), \end{aligned} \tag{32}$$

which imply

$$\begin{aligned} S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) + d[A(X)B(Y) \\ + A(Y)B(X)] + eD(X,Y), \end{aligned} \tag{33}$$

where $a = m(n-2), b = p(n-2), c = q(n-2), d = s(n-2), e = t(n-2)$.
So, (M^n, g) is a $MS(QE)_n$.
Conversely, suppose (M^n, g) is conformally flat $MS(QE)_n$. Then

$$\begin{aligned} R(X,Y)Z = &\frac{1}{n-2}\{g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y\} \\ &- \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{34}$$

Now using (11), (18) and (19), we get

$$\begin{aligned} \tilde{R}(X,Y,Z,W) = &[g(Y,Z)g(X,W) \\ &- g(X,Z)g(Y,W)]\left\{\frac{2a}{n-2} - \frac{na+b+c}{(n-1)(n-2)}\right\} \\ &+ [g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\ &+ g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)]\left\{\frac{b}{n-2}\right\} \\ &+ [g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z) \\ &+ g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)]\left\{\frac{c}{n-2}\right\} \\ &+ [\{A(Y)B(Z) + B(Y)A(Z)\}g(X,W) - \{A(X)B(Z) \\ &+ B(X)A(Z)\}g(Y,W) + \{A(X)B(W) + B(X)A(W)\}g(Y,Z) \\ &- \{A(Y)B(W) + B(Y)A(W)\}g(X,Z)]\left\{\frac{d}{n-2}\right\} \\ &+ [g(Y,Z)D(X,W) - g(X,Z)D(Y,W) \\ &+ g(X,W)D(Y,Z) - g(Y,W)D(X,Z)]\left\{\frac{e}{n-2}\right\}. \end{aligned} \tag{35}$$

So,

$$\begin{aligned} \tilde{R}(X,Y,Z,W) = &m_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ p_1[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\ &+ g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)] \\ &+ q_1[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z) \\ &+ g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)] \\ &+ s_1[\{A(Y)B(Z) + B(Y)A(Z)\}g(X,W) \\ &- \{A(X)B(Z) + B(X)A(Z)\}g(Y,W) + \{A(X)B(W) \\ &+ B(X)A(W)\}g(Y,Z) - \{A(Y)B(W) \end{aligned} \tag{36}$$

$$+ B(Y)A(W)\}g(X,Z)] + t[g(Y,Z)D(X,W)$$
$$- g(X,Z)D(Y,W) + g(X,W)D(Y,Z) - g(Y,W)D(X,Z)],$$

where, $m = \frac{a(n-2)-b-c}{(n-1)(n-2)}$, $p = \frac{b}{n-2}$, $q = \frac{c}{n-2}$, $s = \frac{d}{n-2}$, $t = \frac{e}{n-2}$.
So, (M^n, g) is a manifold of mixed super quasi constant curvature.
Then we have the following theorem:

Theorem 1 *A Riemannian manifold is a manifold of mixed super quasi constant curvature iff it is conformally flat* $MS(QE)_n$.

4 Ricci-pseudosymmetric $MS(QE)_n$

In this section we consider a *Ricci-pseudosymmetric* $MS(QE)_n$ and prove the following theorem:

Theorem 2 *Let* (M^n, g), $(n \geq 3)$, *be a* $MS(QE)_n$. *If* M^n *is Ricci-pseudosym metric then the following conditions hold on* M^n :

$$R(X,Y,U,V) = L_S\{A(Y)B(X) - A(X)B(Y)\} \tag{37}$$
$$D(R(X,Y)U,V) = L_S\{A(Y)D(X,V) - A(X)D(Y,V)\} \tag{38}$$
$$D(R(X,Y)V,V) = L_S\{B(Y)D(X,V) - B(X)D(Y,V)\} \tag{39}$$

for all vector fields X, Y *on* M^n, *where* U, V *are the generators of the manifold* M^n.

Proof. Assume that M^n is *Ricci-pseudosymmetric*. Then by the use of (25) to (28), we can obtain

$$\begin{aligned} S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = L_S\{g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ + g(Y,W)S(X,Z) - g(X,W)S(Y,Z)\}. \end{aligned} \tag{40}$$

Since M_n is also $MS(QE)_n$, using the well-known properties of the curvature tensor R we get

$$\begin{aligned} &b[A(R(X,Y)Z)A(W) + A(Z)A(R(X,Y)W)] + c[B(R(X,Y)Z)B(W) \\ &\quad + B(Z)B(R(X,Y)W)] + d[A(R(X,Y)Z)B(W) + A(W)B(R(X,Y)Z) \\ &\quad + A(Z)B(R(X,Y)W) + A(R(X,Y)W)B(Z)] + e[D(R(X,Y)Z,W) \\ &\quad + D(Z,R(X,Y)W)] = L_S\{b[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) \\ &\quad + g(Y,W)A(X)A(Z) - g(X,W)A(Y)A(Z)] + c[g(Y,Z)B(X)B(W) \\ &\quad - g(X,Z)B(Y)B(W) + g(Y,W)B(X)B(Z) - g(X,W)B(Y)B(Z)] \end{aligned} \tag{41}$$

$$+ d[g(Y,Z)A(X)B(W) + g(Y,Z)A(W)B(X) - g(X,Z)A(Y)B(W) - g(X,Z)A(W)B(Y) + g(Y,W)A(X)B(Z) + g(Y,W)A(Z)B(X) - g(X,W)A(Y)B(Z) - g(X,W)A(Z)B(Y)] + e[g(Y,Z)D(X,W) - g(X,Z)D(Y,W) + g(Y,W)D(X,Z) - g(X,W)D(Y,Z)]\}.$$

Putting $Z = U$ and $W = V$ in (41), we get

$$\begin{aligned} b[R(X,Y,V,U) - L_S\{A(X)B(Y) - A(Y)B(X)\}] + c[R(X,Y,U,V) \\ - L_S\{A(Y)B(X) - A(X)B(Y)\}] + e[D(R(X,Y)U,V) \\ - L_S\{A(Y)D(X,V) - A(X)D(Y,V)\}] = 0. \end{aligned} \tag{42}$$

Taking $Z = W = U$ in (41), we get

$$d[R(X,Y,U,V) - L_S\{A(Y)B(X) - A(X)B(Y)\}] = 0.$$

Since $d \neq 0$, we get

$$R(X,Y,U,V) - L_S\{A(Y)B(X) - A(X)B(Y)\} = 0. \tag{43}$$

Which gives (37). Similarly, if we take $Z = W = V$ in (41), we get

$$\begin{aligned} d[R(X,Y,V,U) - L_S\{A(X)B(Y) - A(Y)B(X)\}] + e[D(R(X,Y)V,V) \\ - L_S\{B(Y)D(X,V) - B(X)D(Y,V)\}] = 0. \end{aligned} \tag{44}$$

From (42) and (43), we get

$$e[D(R(X,Y)U,V) - L_S\{A(Y)D(X,V) - A(X)D(Y,V)\}] = 0.$$

Since $e \neq 0$,

$$D(R(X,Y)U,V) - L_S\{A(Y)D(X,V) - A(X)D(Y,V)\} = 0.$$

Which gives (38).
Again from (43) and (44), we obtain (39). So our theorem is proved. □

5 $MS(QE)_n$ satisfying the condition $C.S = 0$

In this section we consider a $MS(QE)_n)(n \geq 3)$ satisfying the condition $C.S = 0$. Then we have

$$S(C(X,Y)Z,W) + S(Z,C(X,Y)W) = 0. \tag{45}$$

Now using (11) in (45), we get,

$$\begin{aligned} & ag(C(X,Y)Z,W) + bA(C(X,Y)Z)A(W) + cB(C(X,Y)Z)B(W) \\ & d[A(C(X,Y)Z)B(W) + B(C(X,Y)Z)A(W)] + eD(C(X,Y)Z,W) \\ & ag(Z,C(X,Y)W) + bA(Z)A(C(X,Y)W) + cB(Z)B(C(X,Y)W) \\ & d[A(Z)B(C(X,Y)W) + B(Z)A(C(X,Y)W)] + eD(Z,C(X,Y)W) = 0. \end{aligned} \tag{46}$$

From (46),

$$\begin{aligned} & b[A(C(X,Y)Z)A(W) + A(Z)A(C(X,Y)W)] + c[B(C(X,Y)Z)B(W) \\ & \quad + B(Z)B(C(X,Y)W)] + d[A(C(X,Y)Z)B(W) + B(C(X,Y)Z)A(W) \\ & \quad + A(Z)B(C(X,Y)W) + B(Z)A(C(X,Y)W)] + e[D(C(X,Y)Z,W) \\ & \quad + D(Z,C(X,Y)W)] = 0. \end{aligned} \tag{47}$$

Putting $Z = W = U$ in (47), we get

$$2b[A(C(X,Y)U] + 2d[B(C(X,Y)U] = 0. \tag{48}$$

So, we obtain

$$2d[B(C(X,Y)U] = 0$$

As $d \neq 0$, we get,

$$B(C(X,Y)U = 0. \tag{49}$$

That is

$$C(X,Y,U,V) = 0. \tag{50}$$

So, from (29), we obtain

$$\begin{aligned} R(X,Y,U,V) = & \frac{1}{n-2}[A(QY)B(X) - A(X)B(QY) + A(Y)B(QX) \\ & - A(QX)B(Y)] - \frac{r}{(n-1)(n-2)}\{A(Y)B(X) - A(X)B(Y)\} \end{aligned} \tag{51}$$

So, we can state that

Theorem 3 *In a* $MS(QE)_n$ $(n \geq 3)$ *with* $C.S = 0$*, the curvature tensor* R *of the manifold satisfies the relation* (51)*.*

6 $MS(QE)_n$ satisfying the condition $\tilde{C}.S = 0$

In this section we consider a $MS(QE)_n$ ($n \geq 3$) satisfying the condition $\tilde{C}.S = 0$.Then we have,

$$S(\tilde{C}(X,Y)Z,W) + S(Z,\tilde{C}(X,Y)W) = 0. \tag{52}$$

From (11) in (52), we get,

$$\begin{aligned}
&ag(\tilde{C}(X,Y)Z,W) + bA(\tilde{C}(X,Y)Z)A(W) + cB(\tilde{C}(X,Y)Z)B(W)\\
&d[A(\tilde{C}(X,Y)Z)B(W) + B(\tilde{C}(X,Y)Z)A(W)] + eD(\tilde{C}(X,Y)Z,W)\\
&ag(Z,\tilde{C}(X,Y)W) + bA(Z)A(\tilde{C}(X,Y)W) + cB(Z)B(\tilde{C}(X,Y)W)\\
&d[A(Z)B(\tilde{C}(X,Y)W) + B(Z)A(\tilde{C}(X,Y)W)] + eD(Z,\tilde{C}(X,Y)W) = 0.
\end{aligned} \tag{53}$$

Putting $Z = W = U$ in (53), we get

$$d[B(C(X,Y)U] = 0.$$

As $d \neq 0$,

$$B(\tilde{C}(X,Y)U = 0. \tag{54}$$

That is,

$$\tilde{C}(X,Y,U,V) = 0. \tag{55}$$

So, from (30), we get

$$R(X,Y,U,V) = \frac{r}{n(n-1)}[A(Y)B(X) - A(X)B(Y)]. \tag{56}$$

Thus, we have

Theorem 4 *In a* $MS(QE)_n$ $(n \geq 3)$ *with* $\tilde{C}.S = 0$*, the curvature tensor* R *of the manifold satisfies the relation* (56)*.*

7 $MS(QE)_n$ satisfying the condition $C_1.S = 0$

In this section we consider a $MS(QE)_n$ ($n \geq 3$) satisfying the condition $C_1.S = 0$. Then we have,

$$S(C_1(X,Y)Z,W) + S(Z,C_1(X,Y)W) = 0 \tag{57}$$

for any vector fields $X, Y, Z, W \in \chi(M)$. Then we have the following theorem:

Theorem 5 *Let* (M^n, g) $(n \geq 3)$ *be a* $MS(QE)_n$*. If the condition* $C_1.S = 0$ *holds on* M^n *then the curvature tensor R of* M^n *satisfies the following property:*

$$\lambda R(X, Y, U, V) = \left[\frac{na+b+c}{n}\left(\frac{\lambda}{n-1} + 2\mu\right) - \mu(2a+b+c)\right]\{A(Y)B(X) - A(X)B(X)\} - \mu e\{D(X, V)A(Y) - D(Y, V)A(X)\} \tag{58}$$

for all vector fields X, Y *on* M^n*, where* U, V *are the generators of the manifold* M^n*.*

Proof. Since, $C_1.S = 0$ holds on M^n we have,

$$S(C_1(X, Y)Z, W) + S(Z, C_1(X, Y)W) = 0.$$

Since M^n be a $MS(QE)_n$, using (11) in (57), we obtain

$$\begin{aligned} &ag(C_1(X, Y)Z, W) + bA(C_1(X, Y)Z)A(W) + cB(C_1(X, Y)Z)B(W) \\ &d[A(C_1(X, Y)Z)B(W) + B(C_1(X, Y)Z)A(W)] + eD(C_1(X, Y)Z, W) \\ &ag(Z, C_1(X, Y)W) + bA(Z)A(C_1(X, Y)W) + cB(Z)B(C_1(X, Y)W) \\ &d[A(Z)B(C_1(X, Y)W) + B(Z)A(C_1(X, Y)W)] + eD(Z, C_1(X, Y)W) = 0. \end{aligned} \tag{59}$$

From (59),

$$\begin{aligned} &b[A(C_1(X, Y)Z)A(W) + A(Z)A(C_1(X, Y)W)] + c[B(C_1(X, Y)Z)B(W) \\ &+ B(Z)B(C_1(X, Y)W)] + d[A(C_1(X, Y)Z)B(W) + B(C_1(X, Y)Z)A(W) \\ &+ A(Z)B(C_1(X, Y)W) + B(Z)A(C_1(X, Y)W)] + e[D(C_1(X, Y)Z, W) \\ &+ D(Z, C_1(X, Y)W)] = 0. \end{aligned} \tag{60}$$

Putting $Z = W = U$ in (60), we get

$$2b[A(C_1(X, Y)U] + 2d[B(C_1(X, Y)U] = 0. \tag{61}$$

So, we obtain

$$2d[B(C_1(X, Y)U] = 0.$$

As $d \neq 0$, we get

$$B(C_1(X, Y)U = 0. \tag{62}$$

That is

$$C_1(X, Y, U, V) = 0.$$

Now using (31), we obtain

$$\begin{aligned} \lambda R(X,Y,U,V) = \mu\{&S(X,U)g(Y,V) - S(Y,U)g(X,V) - g(Y,U)S(X,V) \\ &+ g(X,U)S(Y,V)\} + \frac{r}{n}\left[\frac{\lambda}{(n-1)} + 2\mu\right][g(Y,U)g(X,V) \\ &- g(X,U)g(Y,V)]. \end{aligned} \tag{63}$$

Using (11) and (18) in (63), we get,

$$\begin{aligned} \lambda R(X,Y,U,V) = &\left[\frac{na+b+c}{n}\left(\frac{\lambda}{n-1} + 2\mu\right) - \mu(2a+b+c)\right]\{A(Y)B(X) \\ &- A(X)B(X)\} - \mu e\{D(X,V)A(Y) - D(Y,V)A(X)\}. \end{aligned}$$

Hence the proof. □

8 Conformally flat $MS(QE)_n$ $(n \geq 3)$ with $R(X,Y).S = 0$

Let us consider a *conformally flat* $MS(QE)_n$ $(n \geq 3)$. Then, from (29), we get

$$\begin{aligned} R(X,Y)Z = &\frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &+ \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{64}$$

Since the manifold satisfies $R(X,Y).S = 0$, we get

$$S(R(X,Y)Z,W) + S(Z,C(X,Y)W) = 0. \tag{65}$$

Using (64) in (65) we obtain

$$\begin{aligned} g(Y,Z)S(QX,W) &- g(X,Z)S(QY,W) + g(Y,W)S(QX,Z) \\ &- g(X,W)S(QY,Z) = \frac{r}{n-1}[g(Y,Z)S(X,W) \\ &- g(X,Z)S(Y,W) + g(Y,W)S(X,Z) \\ &- g(X,W)S(Y,Z)]. \end{aligned} \tag{66}$$

Let λ be the eigen value of Q corresponding to the eigen vector X. Then $QX = \lambda X$, i.e., $S(X,W) = \lambda g(X,W)$ (where the manifold is not Einstein) and hence

$$S(QX,W) = \lambda S(X,W). \tag{67}$$

Now using (67) in (66) we get,

$$\left(\lambda - \frac{r}{n-1}\right)[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + g(Y,W)S(X,Z) - g(X,W)S(Y,Z) = 0.$$

Which gives

$$g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + g(Y,W)S(X,Z) - g(X,W)S(Y,Z) = 0, \quad (68)$$

provided $\lambda - \frac{r}{n-1} \neq 0$. Now using (11) in (68) we get

$$\begin{aligned} &g(Y,Z)[ag(X,W) + bA(X)A(W) + cB(X)B(W) + d\{A(X)B(W) \\ &\quad + B(X)A(W)\} + eD(X,W)] - g(X,Z)[ag(Y,W) + bA(Y)A(W) \\ &\quad + cB(Y)B(W) + d\{A(Y)B(W) + B(Y)A(W)\} + eD(Y,W)] + g(Y,W) \\ &\quad [ag(X,Z) + bA(X)A(Z) + cB(X)B(Z) + d\{A(X)B(Z) + B(X)A(Z)\} \\ &\quad + eD(X,Z)] - g(X,W)[ag(Y,Z) + bA(Y)A(Z) + cB(Y)B(Z) \\ &\quad + d\{A(Y)B(Z) + B(Y)A(Z)\} + eD(Y,Z)] = 0. \end{aligned} \quad (69)$$

Now putting $Z = W = U$ in (69), we obtain,

$$2d[A(Y)B(X) - B(Y)A(X)] = 0.$$

As $d \neq 0$, so

$$A(Y)B(X) - B(Y)A(X) = 0, \quad (70)$$

that is, the vector fields U and V are co-directional. Thus we can state the following:

Theorem 6 *If, in a conformally flat Ricci-semisymmetric $MS(QE)_n$ $(n \geq 3)$ $\frac{r}{n-1}$ is not an eigenvalue of the Ricci-operator Q, the vector fields U and V corresponding to the 1-forms A and B respectively are co-directional.*

9 Example of doubly warped product on $MS(QE)_n$

In [10], B. Pal, A. Bhattacharyya and M. Tarafdar defined warped product on $MS(QE)_4$. Here, we define doubly warped product on four dimensional $MS(QE)_n$. Let (M^4, g) be a 4-dimensional Lorentzian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2], \quad (71)$$

where $p > 0$ is a smooth function and $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4.
In [11], A. A. Shaikh and S. K. Hui have shown that (71) becomes $G(QE)_n$. As it is non-Einstein metric, so one can easily show that (71) is $MS(QE)_n$.

We know that $({}_fB \times_b F, f^2_{g_B} \oplus b^2_{g_F})$ is a Lorentzian doubly warped product if (F, g_F) is Riemannian and either (B, g_B) is Lorentzian or else (B, g_B) is a one-dimensional manifold with a negative definite metric $-dt^2$. To define Lorentzian doubly warped product on$MS(QE)_n$, we take the line element on $R \times R^3$ where we consider R is the B and R^3 is the F. If we consider the above example, we have the metric g_F , where (F, g_F) is Riemannian and the metric g_B, where (B, g_B) is a one-dimensional manifold with a negative definite metric $ds^2_B = -(dx^4)^2$. Here, the metric g_F on R^3 is

$$ds^2_F = \frac{1}{1+2p}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]$$

and the warping function

$$f : R^3 \longrightarrow (0, \infty)$$

is defined by

$$f(x^1, x^2, x^3) = \sqrt{(1 + 2p)}$$

and the other warping function is

$$b : R \longrightarrow (0, \infty),$$

which is defined by

$$b(x^4) = (1 + 2p).$$

Here, we see that the warping functions $f = \sqrt{(1 + 2p)} > 0$ and $b = (1+2p) > 0$, both are also smooth functions. Therefore the metric

$$ds^2_M = f^2 ds^2_B + b^2 ds^2_F$$

which is

$$ds^2 = g_{ij} dx^i dx^j = -(1 + 2p)(dx^4)^2 + (1 + 2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2].$$

This is the example of Lorentzian doubly warped product on $MS(QE)_4$.

Next we consider the another example. Let (M^4, g) be a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij} dx^i dx^j = e^{2x^1}(dx^1)^2 + \sin^2 x^1[(dx^2)^2 + (dx^3)^2 + (dx^4)^2], \tag{72}$$

where $0 < x^1 < \frac{\pi}{2}$ but $x^1 \neq \frac{\pi}{4}$ and $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4. Then it can be easily shown that it is a mixed super quasi-Einstein manifold with non-vanishing scalar curvature.

We know that $({}_f B \times_b F, f^2_{g_B} \oplus b^2_{g_F})$ is a Riemannian doubly warped product if (F, g_F) and (B, g_B) are both Riemannian manifolds. To define Riemannian doubly warped product on $MS(QE)_4$, we take the line element on $L^2 \times L^2$, where $B = F = L^2 = R \times R$. If we consider the example (72), we have the metric g_B, where (B, g_B) is Riemannian and the metric g_F , where (F, g_F) is also Riemannian with metrices

$$ds_B^2 = (dx^1)^2 + \frac{1}{e^{2x^1}} \sin^2 x^1 (dx^2)^2,$$

$$ds_F^2 = \frac{1}{\sin^2 x^1}[(dx^3)^2 + (dx^4)^2]$$

and the warping function

$$f : L^2 \longrightarrow (0, \infty)$$

is defined by

$$f(x^1, x^2) = \sqrt{e^{2x^1}}$$

and the other warping function is

$$b : L^2 \longrightarrow (0, \infty),$$

which is defined by

$$b(x^3, x^4) = \sin^2 x^1.$$

Here, we see that the warping functions $f = \sqrt{e^{2x^1}} > 0$ and $b = \sin^2 x^1 > 0$ both are also smooth functions. Therefore the metric

$$ds_M^2 = f^2 ds_B^2 + b^2 ds_F^2$$

which is

$$ds_M^2 = e^{2x^1}\left[(dx^1)^2 + \frac{1}{\sin^2 x^1}(dx^2)^2\right] + \sin^4 x^1 \left[\frac{1}{\sin^2 x^1}(dx^3)^2 + \frac{1}{\sin^2 x^1}(dx^4)^2\right],$$

is the example of Riemannian doubly warped product on $MS(QE)_4$.

10 Completeness of doubly warped products on $MS(QE)_4$

In this section, we obtain some results on completeness properties of Riemannian doubly warped products and Lorentzian doubly warped products on $MS(QE)_4$.

The Riemannian case

In this subsection, we state some results about completeness of Riemannian doubly warped products. Here we want to investigate about the completeness properties of Riemannian doubly warped products with respect to the example (72), which is $MS(QE)_4$. Now it is clear that $\inf(f) > 0$ and $\inf(b) > 0$ and $B = F = L^2 = R \times R$. Therefore (B, g_B) and (F, g_F) are complete Riemannian manifolds. Hence by proposition (32) of [12], we can state that

Example 1 *Let* $M = B \times F$ *be a Riemannian doubly warped product on* $MS(QE)_4$ *endowed with the metric given by*

$$ds_M^2 = e^{2x^1}\left[(dx^1)^2 + \frac{1}{\sin^2 x^1}(dx^2)^2\right] + \sin^4 x^1 \left[\frac{1}{\sin^2 x^1}(dx^3)^2 + \frac{1}{\sin^2 x^1}(dx^4)^2\right],$$

where, x^1, x^2, x^3, x^4 *are the standard coordinates of* M^4*. Then* (M^4, g) *is a complete Riemannian manifold.*

Here we want to discuss about global hyperbolicity of mixed super quasi-Einstein space-time with doubly warped product fibers by using [1]. Let us consider the example. Let (M_4, g) be a Riemannian manifold endowed with the metric given by

$$ds_M^2 = -(dx^4)^2 + x^1\left[(x^3)^4\{(dx^1)^2\} + \frac{2d}{(x^3)^4}\left\{(dx^2)^2 + \frac{(x^3)^4}{2dx^1}(dx^3)^2\right\}\right],$$

where, x^1, x^2, x^3, x^4 are the standard coordinates of M^4. Then it can be easily shown that it is a mixed super quasi-Einstein manifold with non vanishing scalar curvature. Now this manifold is of the form

$$M = (c, d) \times_h (B_f \times_b F),$$

a Lorentzian singly warped product with the metric

$$g = -(dx^4)^2 \oplus h^2(f^2 g_B + b^2 g_F),$$

where $-\infty \le c \le d \le \infty$,

$$h : (c, d) \longrightarrow (0, \infty)$$

is defined by $h = \sqrt{x^1}$, which is strictly positive and smooth. Also (B, g_B) and (F, g_F) are complete Riemannian manifolds and $\inf(b)$ that is $\inf(\frac{\sqrt{2d}}{(x^3)^2}) > 0$ or $\inf(f)$ that is $\inf((x^3)^2) > 0$. Then we have the following:

Example 2 *Let* $M = (c, d) \times_h (B_f \times_b F)$, *be a Lorentzian singly warped product on* $MS(QE)_4$ *endowed with the metric given by*

$$ds^2_M = -(dx^4)^2 + x^1\left[(x^3)^4\{(dx^1)^2\} + \frac{2d}{(x^3)^4}\left\{(dx^2)^2 + \frac{(x^3)^4}{2dx^1}(dx^3)^2\right\}\right],$$

where, x^1, x^2, x^3, x^4 *are the standard coordinates of* M^4. *Then* (M^4, g) *is globally hyperbolic.*

Lorentzian case

We now consider the nonspacelike geodesic completeness of Lorentzian warped products of the form

$$M =_f (c, d) \times_b F$$

with the metric

$$g = f^2 dt^2 \oplus b^2 g_F,$$

where $-\infty \leq c \leq d \leq \infty$. Here a space-time is said to be null (respectively, timelike) geodesically incomplete if some future directed null (respectively, timelike) geodesic can not be extended for arbitrary negative and positive values of an affine parameter. Let us consider (M^4, g) be a 4-dimensional Lorentzian manifold endowed with the metric given by

$$ds^2 = e^{2x^1}(dx^1)^2 + (\sin^2)x^1[(dx^2)^2 + (dx^3)^2 - (dx^4)^2],$$

where, x^1, x^2, x^3, x^4 are the standard coordinates of M^4. Then it is clear that it is mixed super quasi-Einstein manifold with non vanishing scalar curvature. Now, this metric can be written as

$$ds^2 = e^{2x^1}\left[(dx^1)^2 + \frac{1}{\sin^2 x^1}(dx^2)^2\right] + \sin^4 x^1\left[\frac{1}{\sin^2 x^1}(dx^3)^2 - \frac{1}{\sin^2 x^1}(dx^4)^2\right].$$

Take $B = F = L^2 = R \times R$ and define

$$f : L^2 \longrightarrow (0, \infty)$$

is defined by

$$f(x^1, x^2) = \sqrt{e^{2x^1}}$$

and the another function

$$b : L^2 \longrightarrow (0, \infty)$$

is defined by
$$b(x^3, x^4) = \sin^2 x^1.$$
Let us define
$$\alpha : (-\infty, \infty) \longrightarrow B$$
is defined by
$$\alpha(t) = (t, t)$$
and
$$\beta : (-\infty, \infty) \longrightarrow B$$
is defined by
$$\beta(t) = (t, t).$$
Clearly, α and β are complete null geodesics of B and F. Also, if $\gamma = (\alpha, \beta)$ then it is a null pre-geodesic in M and $\gamma'' = \gamma$ by equation in proposition 2.3 in [12]. Now using the example (3.8) in [12], we get γ is incomplete. Then we can state

Example 3 *If* (B, g_B) *and* (F, g_F) *are null complete pseudo-Riemannian manifolds then* $M =_f B \times_b f$ *is not a null complete pseudo-Riemannian doubly warped product with the metric* $g_M = f^2 g_B \oplus b^2 g_F$ *on* $MS(QE)_4$.

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