# 134. *On Some Classes of Operators. I*

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**Introduction.** Generalizing the concept of normality, several authors have introduced classes of non-normal operators.

Thus, one of these classes is the class of hyponormal operators of *P*. Halmos [1]. In the paper [2] it was introduced a new class of operators generalizing hyponormal operators.

It is the aim of this Note to introduce a new class of operators which generalizes the class of operators of class $(N)$ of [2] and give some properties. The definition and some properties has sense also for operators on Banach spaces, however we consider only Hilbert spaces operators.

**1.** Let $T$ be a bounded linear operator on a Hilbert space $H$.

The operator is of class $(N)$ if

$$x \in H, \qquad \|x\|=1, \qquad \|Tx\|^2 \leq \|T^2x\|.$$

This definition suggests the following

**Definition 1.** The operator $T$ is of class $(N)$ and order $k$ if

$$x \in H, \qquad \|x\|=1 \qquad \|Tx\|^k \leq \|T^kx\|.$$

We write this as $T \in \mathcal{C}(N, k)$. It is clear that the operators of class $(N)$ is $\mathcal{C}(N, 2)$.

**Theorem 1.** *If $T \in \mathcal{C}(N, k)$, then the spectral radius of $T$, $\gamma_T$ is equal to $\|T\|$.*

**Proof.** By definition there exists a sequence $\{x_n\}$, $\|x_n\|=1$ such that

$$\|Tx_n\| \to \|T\|=1.$$

(We may suppose, without loss of generality that $\|T\|=1$.) Since, for every $x$, $\|x\|=1$,

$$\|Tx\|^k \leq \|T^kx\|$$

we have

$$\lim \|T^kx_n\|=1$$

This leads, also, to

$$\lim \|T^jx_n\|=1 \qquad 1 \leqslant J \leqslant K.$$

Since

$$\|T^{k+1}x\|=\left\|T^k\frac{Tx}{\|Tx\|}\right\| \|Tx\| \geqslant \|T^2x\|^k \frac{1}{\|Tx\|^{k-1}}$$

If we put in this inequality, $x=x_n$ we obtain

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$$\lim \| T^{k+1}x_n \| = 1.$$

Also, if

$$\lim \| T^p x_n \| = 1 \qquad p \leqslant l$$

then

$$\lim \| T^{l+1}_{x_n} \| = 1.$$

This is the consequence of the inequality

$$\| T^{l+1}x \| = \| T^k T^{l+1-k}x \| = \left\| T^k \frac{T^{l+1-k}}{\| T^{l+1-k}x \|} \right\| \, \| T^{l+1-k}x \|$$
$$\geqslant \| T^{l+2-k}x \|^k \| T^{l+1-k}x \|^{k-1}.$$

By an induction argument, we obtain that for every $j$

$$\lim \| T^j x_n \| = 1.$$

This proves the theorem.

**Corollary 1.** *Every operator in $C(N, k)$ is normaloid in the sense of A. Wintner* [6].

**Theorem 2.** *If $T$ is of class $(N)$ then $T \in C(N, k)$ with $k \geqslant 2$.*

**Proof.** For $k=2$ this is trivial. Suppose now, that the assertion is true for all $1, 2, \cdots, J$ and then we prove that it is true for $J+1$.

Since $T$ is of class $(N)$ we have [5]

$$\| TJ^{+1}x \|^2 \geqslant \| T^j x \|^2 \| T^2 x \| \geqslant \| Tx \|^{2j} \| T^2 x \| \geqslant \| Tx \|^{2(j+1)}$$

and an induction argument completes the proof of the theorem. (In [5] the class of operators of class $(N)$ is called paranormal operators).

In [2] [3] were given some structure theorems about operators of class $(N)$ or hyponormal.

In the same way, as in [3] we prove the following

**Theorem 3.** *Let $T$ be an operator in $C(N, k)$ such that $T^{*p_1}T^{q_1} \cdots T^{*p_m}T^{q_m}$ is completely continuous for some non-negative integers $p_1, q_1, \cdots p_m, q_m$. Then $T$ is necessarily a normal operator.*

In the paper that will follow we shall give new properties of these classes of operators.

## References

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