ON SOME CLASSES OF ℝ-COMPLEX HERMITIAN FINSLER SPACES

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Abstract. In this paper, we investigate the ℝ-complex Hermitian Finsler spaces, emphasizing the differences that separate them from the complex Finsler spaces. The tools used in this study are the Chern-Finsler and Berwald connections. By means of these connections, some classes of the ℝ-complex Hermitian Finsler spaces are defined, (e.g. weakly Kähler, Kähler, strongly Kähler). Here the notions of Kähler and strongly Kähler do not coincide, unlike the complex Finsler case. Also, some kinds of Berwald notions for such spaces are introduced. A special approach is devoted to obtain the equivalence conditions for an ℝ-complex Hermitian Finsler space to become a weakly Berwald or Berwald. Finally, we obtain the conditions under which an ℝ-complex Hermitian Finsler space with Randers metric is Berwald. We get some clear examples which illustrate the interest for this work.

1. Introduction

Riemannian geometry is based on the fundamental concept of metric that depends on each point of the manifold, and which is defined by the metric tensor $g_{ij}(x)$. For instance, by means of this it is possible to calculate the length of the curve $c : t \to \left(x^i(t)\right)$, $t \in [a, b] \subset \mathbb{R}_+$, which is $l = \int_a^b F\left(x(t), \frac{dx(t)}{dt}\right) dt$, where $F = \sqrt{g_{ij}(x)y^i y^j}$ and $y^i = \frac{dx^i}{dt}$ is the tangent vector of the curve.

In 1918, P. Finsler was inspired to generalize Riemann's ideas, taking $F = \sqrt{g_{ij}(x,y)y^i y^j}$. Of course, the first surprise was to find that in the integral, which gives the arc length of the curve, the metric tensor depends on the parameter $t \in \mathbb{R}$. Therefore, an additional condition is required, namely the function F is homogeneous in y, meaning $F(x, \lambda y) = \lambda F(x, y)$ for any $\lambda > 0$. Subsequently, the development of the Finsler geometry has been remarkable, (see for instance [1, 9, 8, 10, 14, 20, 21], etc.).

In 1963, G. B. Rizza [17] extended the above mentioned results to the complex case, considering the function $F : T'M \to \mathbb{R}_+$ with the following homogeneity property $F(z, \lambda\eta) = |\lambda| F(z, \eta)$, for any $\lambda \in \mathbb{C}$, where (z, η) are the coordinates in holomorphic tangent bundle $T'M$ of a complex manifold M. But, this complex theory gathered momentum after that S. Kobayashi introduced the well-known Kobayashi metric which satisfies the mentioned homogeneity, (in 1967). During the last years, a complex Finsler geometry has been developed which contains many interesting results (see [1, 2, 3, 4, 5, 6, 12, 13, 15, 18, 22, 23], etc.).

If we account only the initial problem, in complex Finsler geometry the arc length is calculated for the curves which depend on a real parameter ($c : t \to (z^i(t))$ and $\eta^i = \frac{dz^i}{dt}$) and the invariance of the integral to the change of parameters is ensured only for the real parameters. This led to the idea of restricting the homogeneity of the function $F : T'M \to \mathbb{R}_+$ to the real scalars, i.e., $F(z, \lambda\eta) = |\lambda| F(z, \eta)$ for any $\lambda \in \mathbb{R}$, (see [16]). Although the restriction seems insignificant, the changes which arise are considerable. First of all, the indicatrix is a convex set, it is not strongly pseudo-convex. Also, $L := F^2$ satisfies the so called $\mathbb{R}$-homogeneity condition, (of second degree, see (2.2)), instead of $(1,1)$-homogeneity as in complex Finsler geometry and many others differences.

The purpose of this paper is to continue the study of the $\mathbb{R}$-complex Finsler spaces and to point out others differences from the complex Finsler spaces. The present paper appears as a necessary extension of [16], required by the already known results from real and complex Finsler geometry, (e.g. [1, 4, 5, 6, 8, 9, 10, 20, 21, 23]).

Subsequently, we will present an overview of the content of the paper.

In Section 2, some preliminary properties of the n-dimensional $\mathbb{R}$-complex Finsler spaces are presented. A $\mathbb{R}$-complex Finsler function L produces two tensors $g_{ij} := \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}$ and $g_{i\bar{j}} := \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$. The first is a symmetric one while the second is a Hermitian one. We find that the dependence of one of these tensors only on the base point of the complex manifold, implies the same property for the second (Proposition 2.1). Subsequently, we will focus only on the study of the $\mathbb{R}$-complex Hermitian Finsler spaces, (meaning $g_{i\bar{j}}$ is invertible). The instruments of our study are the Chern-Finsler and the canonical complex nonlinear connections which satisfy the $\mathbb{R}$ - homogeneity conditions. Also, we work with two complex linear connections: Chern-Finsler and Berwald which have specific properties.

In Section 3 we will introduce some classes of $\mathbb{R}$-complex Hermitian Finsler spaces. The main theme is that of finding the correspondences between them. Namely, using some ideas from the complex Finsler case, we speak about three kinds of Kähler metrics: strongly Kähler, Kähler and weakly Kähler. Here the notions of Kähler and strongly Kähler do not coincide. We determine the conditions under which such a Kähler metric becomes a strongly Kähler

metric, (Theorem 3.1). Also, we find when the Chern-Finsler and the canonical complex nonlinear connections coincide, (Theorem 3.2). By means of Chern-Finsler and Berwald connections we define for ℝ-complex Hermitian Finsler spaces the classes of Berwald and weakly Berwald spaces. We show that any purely Hermitian ℝ-complex Hermitian Finsler space is a Berwald space. The necessary and sufficient conditions under which a ℝ-complex Hermitian Finsler space is weakly Berwald or Berwald are established, (Theorems 3.4 and 3.5). The Berwald spaces with Kähler property are called strongly Berwald. We prove that any strongly Berwald space is strongly Kähler, (Corollary 3.1).

The general theory of the ℝ-complex Hermitian Berwald spaces applies to the class of Randers spaces. Theorem 4.2 offers sufficient conditions for a Randers metric $F = \alpha + \beta$ to be a Berwald metric. The existence of the ℝ-complex Hermitian Berwald spaces with Randers metrics is attested by some explicit examples.

2. Preliminaries

In this section, we give some preliminaries about n-dimensional ℝ-complex Finsler geometry with Chern-Finsler and Berwald complex linear connections. We set the basic notions (for more details, see [15, 16]), and we prove some important properties of these connections.

2.1. ℝ-complex Hermitian Finsler spaces

Let M be an n-dimensional complex manifold and $z = (z^k)_{k=\overline{1,n}}$ be the complex coordinates in a local chart. The complexification of the real tangent bundle $T_C M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is itself a complex manifold and the local coordinates in a local chart will be denoted by $u = (z^k, \eta^k)_{k=\overline{1,n}}$. These are changed into $(z'^k, \eta'^k)_{k=\overline{1,n}}$ by the rules $z'^k = z'^k(z)$ and $\eta'^k = \frac{\partial z'^k}{\partial z^l} \eta^l$.

A ℝ-complex Finsler space is a pair (M, F), where F is a continuous function $F : T'M \longrightarrow \mathbb{R}_+$ satisfying the conditions:

i) $L := F^2$ is smooth on $\widetilde{T'M} := T'M \backslash \{0\}$;

ii) $F(z, \eta) \geq 0$ the equality holds if and only if $\eta = 0$;

iii) $F(z, \lambda\eta, \bar{z}, \lambda\bar{\eta}) = |\lambda| F(z, \eta, \bar{z}, \bar{\eta}), \ \forall \lambda \in \mathbb{R}$.

The fundamental function L of a ℝ-complex Finsler space, induces the following tensors

$$(2.1) \qquad g_{ij}(z, \eta) := \frac{\partial^2 L}{\partial \eta^i \partial \eta^j} \ ; \ g_{i\bar{j}}(z, \eta) := \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j} \ ; \ g_{\bar{i}\bar{j}}(z, \eta) := \frac{\partial^2 L}{\partial \bar{\eta}^i \partial \bar{\eta}^j},$$

which satisfy interesting properties, obtained as consequences of the homogeneity condition iii), [16],

$$(2.2) \qquad \frac{\partial L}{\partial \eta^i} \eta^i + \frac{\partial L}{\partial \overline{\eta}^i} \overline{\eta}^i = 2L \ ; \ g_{ij}\eta^i + g_{j\bar{i}}\overline{\eta}^i = \frac{\partial L}{\partial \eta^j};$$

$$2L = g_{ij}\eta^i\eta^j + 2g_{i\bar{j}}\eta^i\overline{\eta}^j + g_{\bar{i}\bar{j}}\overline{\eta}^i\overline{\eta}^j;$$

$$\frac{\partial g_{ik}}{\partial \eta^j}\eta^j + \frac{\partial g_{ik}}{\partial \overline{\eta}^j}\overline{\eta}^j = 0 \; ; \; \frac{\partial g_{i\overline{k}}}{\partial \eta^j}\eta^j + \frac{\partial g_{i\overline{k}}}{\partial \overline{\eta}^j}\overline{\eta}^j = 0.$$

We say that a function f on $\widetilde{T'M}$ is (p,q)-*homogeneous* with respect to the fibre variables η if and only if $f(z^k, \mu\eta^k) = \mu^p \overline{\mu}^q f(z^k, \eta^k)$, for any $\mu \in \mathbb{C}$. This leads to $\frac{\partial f}{\partial \eta^j}\eta^j = pf$ and $\frac{\partial f}{\partial \overline{\eta}^j}\overline{\eta}^j = qf$. Also, we say that f is $\mathbb{R}$-homogenous of degree p in the fibre variables η if and only if $\frac{\partial f}{\partial \eta^i}\eta^i + \frac{\partial f}{\partial \overline{\eta}^i}\overline{\eta}^i = pf$. For example, L is $\mathbb{R}$-homogenous of degree 2 in the fibre variables.

Moreover, we can prove:

Proposition 2.1. *Let (M, F) be an $\mathbb{R}$-complex Finsler space. Then, the tensor $g_{i\overline{j}}$ depends only on position z if and only if the tensor g_{ij} depends only on position z.*

Proof. We suppose that $g_{i\overline{j}}(z) = \frac{\partial^2 L}{\partial \eta^i \partial \overline{\eta}^j}$ depends only on z. This leads to $\frac{\partial g_{i\overline{j}}}{\partial \eta^k} = \frac{\partial g_{ik}}{\partial \overline{\eta}^j} = 0$, which substituted into (2.2) gives $\frac{\partial g_{ik}}{\partial \eta^j}\eta^j = 0$. These last two properties mean that g_{ik} is holomorphic and $(0,0)$-homogenous with respect to η. Based on strong maximum principle, we make a similar reasoning like in [12, Proposition 1.1], for g_{ij}. Thus, we obtain that g_{ij} depends only on z.

Conversely, if $g_{ij}(z) = \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}$ depends only on z, then its complex conjugate $g_{\overline{ij}}$ depends only on z. As above, it results that $g_{i\overline{j}}$ depends only on z. $\qquad \square$

Hereinafter, an $\mathbb{R}$-complex Finsler space with $g_{i\overline{j}}(z)$ (or $g_{ij}(z)$) will be called *purely Hermitian.*

Having an $\mathbb{R}$-complex Finsler space, if we suppose that F satisfies the regularity conditions: $g_{i\overline{j}}$ is nondegenerated, (i.e., $\det(g_{i\overline{j}}) \neq 0$, in any $u \in \widetilde{T'M}$), and it defines a positive definite Levi-form for all $z \in M$, then such a class of spaces is called an $\mathbb{R}$-*complex Hermitian Finsler space* [16].

Consider the sections of the complexified tangent bundle of $T'M$. Let $VT'M \subset T'(T'M)$ be the vertical bundle, locally spanned by $\{\frac{\partial}{\partial \eta^k}\}$. $VT''M$ is its complex conjugate. The idea of complex nonlinear connection, briefly (*c.n.c.*), is an instrument in linearization of the geometry of the manifold $T'M$. A (*c.n.c.*) is a supplementary complex subbundle to $VT'M$ in $T'(T'M)$, i.e., $T'(T'M) = HT'M \oplus VT'M$. The horizontal distribution $H_u T'M$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z, \eta)$ are the coefficients of the (*c.n.c.*), i.e., they transform by a certain rule

$$(2.3) \qquad\qquad N_j'^i \frac{\partial z'^j}{\partial z^k} = \frac{\partial z'^i}{\partial z^j}N_k^j - \frac{\partial^2 z'^i}{\partial z^j \partial z^k}\eta^j.$$

The pair $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$ will be called the *adapted frame* of the (*c.n.c.*) which obey to the change rules $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta_j'$ and $\dot{\partial}_k = \frac{\partial z'^j}{\partial z^k}\dot{\partial}_j'$. By conjugation everywhere we have obtained an adapted frame $\{\delta_{\overline{k}}, \dot{\partial}_{\overline{k}}\}$ on $T''_u(T'M)$. The dual adapted bases are $\{dz^k, \delta\eta^k\}$ and $\{d\overline{z}^k, \delta\overline{\eta}^k\}$.

Let us consider T the natural tangent structure which behaves on $T'(T'M)$ by $T(\frac{\partial}{\partial z^k}) = \dot\partial_k$, $T(\dot\partial_k) = 0$, and it is globally defined (see [15]).

Definition 2.1 ([15]). A vector field $S \in T'(T'M)$ is a complex spray if $T \circ S = \Gamma$, where $\Gamma = \eta^k \dot\partial_k$ is the complex Liouville vector field.

Locally, this property of a complex spray can be expressed as follows

$$(2.4) \qquad S = \eta^k \frac{\partial}{\partial z^k} - 2G^k(z, \eta) \dot\partial_k,$$

where the functions $G^k(z, \eta)$ are the local coefficients of the complex spray S.

Under the changes of complex coordinates on $T'M$, the coefficients G^k are transformed by the rule

$$(2.5) \qquad 2G'^i = 2G^k \frac{\partial z'^i}{\partial z^k} - \frac{\partial^2 z'^i}{\partial z^j \partial z^k} \eta^j \eta^k.$$

Between the notions of complex spray and $(c.n.c.)$ there exists an interdependence, one determining the other. Differentiating (2.5) with respect to η^j it follows that the functions $N_j^i := \dot\partial_j G^i$ satisfy the rule (2.3), and hence N_j^i define a nonlinear connection. Conversely, any $(c.n.c.)$ determines a complex spray. Indeed, a simple computation shows that if N_j^i are the coefficients of a $(c.n.c.)$, then $\frac{1}{2} N_j^i \eta^j$ satisfy (2.5) and hence, they define a complex spray.

2.2. Connections on an $\mathbb{R}$-complex Hermitian Finsler space

A $(c.n.c.)$ related only to the fundamental function of the $\mathbb{R}$-complex Hermitian Finsler space (M, F), (called Chern-Finsler $(c.n.c.)$), has the following local coefficients

$$(2.6) \qquad N_k^i := g^{\bar{m}i} \frac{\partial^2 L}{\partial z^k \partial \bar\eta^m} = g^{\bar{m}i} (\frac{\partial g_{\bar{r}\bar{m}}}{\partial z^k} \bar\eta^r + \frac{\partial g_{s\bar{m}}}{\partial z^k} \eta^s).$$

Note that the Chern-Finsler $(c.n.c.)$, locally given in (2.6), is not obtained from a complex spray. We see in the next section that it comes from a complex spray, only in a particular context. But, it induces a complex spray by

$$(2.7) \qquad G^i := \frac{1}{2} N_k^i \eta^k = \frac{1}{2} g^{\bar{m}i} (\frac{\partial g_{\bar{r}\bar{m}}}{\partial z^k} \bar\eta^r + \frac{\partial g_{s\bar{m}}}{\partial z^k} \eta^s) \eta^k.$$

It follows from the fact that L is $\mathbb{R}$-homogenous of degree 2 in the fibre variables that Chern-Finsler $(c.n.c.)$ (2.6) and its induced complex spray (2.7) satisfy the conditions,

$$(2.8) \qquad (\dot\partial_j G^i) \eta^j + (\dot\partial_{\bar{r}} G^i) \bar\eta^r = 2G^i \ ; \ (\dot\partial_j N_k^i) \eta^j + (\dot\partial_{\bar{r}} N_k^i) \bar\eta^r = N_k^i.$$

This means that G^i and N_k^i are $\mathbb{R}$-homogeneous of degree 2, respectively 1, with respect to η.

Further, the complex spray (2.7) induces a $(c.n.c.)$ by $\overset{c}{N_j^i} := \dot\partial_j G^i$, which we call the *canonical* $(c.n.c.)$. A simpler computation gives that $\overset{c}{N_j^i}$ are $\mathbb{R}$-homogeneous of degree 1 and, the complex spray induced of the canonical

($c.n.c.$) is this

$$(2.9) \qquad \overset{c}{G}{}^i := \frac{1}{2}\overset{c}{N}{}^i_k\eta^k = G^i - \frac{1}{2}(\dot\partial_{\bar r}G^i)\bar\eta^r.$$

It is obvious that the Chern-Finsler ($c.n.c.$) and the canonical ($c.n.c.$) induce the same complex spray (2.7), ($\overset{c}{G}{}^i = G^i$), if and only if the coefficients G^i given in (2.7) are $(2,0)$-homogeneous with respect to η.

Also, in an $\mathbb{R}$-complex Hermitian Finsler space, we have recovered the *Chern-Finsler connection*, ([16]), which is metrical, of $(1,0)$-type, and it is given by

$$(2.10) \qquad L^i_{jk} = g^{\bar m i}(\delta_j g_{k\bar m}) \ ; \ C^i_{jk} = g^{\bar m i}(\dot\partial_j g_{k\bar m}) \ ; \ L^i_{j\bar k} = C^i_{j\bar k} = 0,$$

where here and further on δ_k is the frame corresponding to the Chern-Finsler ($c.n.c.$). Moreover, we have the properties

$$(2.11) \qquad L^i_{jk} = \dot\partial_j N^i_k \ ; \ N^i_k = L^i_{jk}\eta^j + (\dot\partial_{\bar r}N^i_k)\bar\eta^r,$$

which together with (2.8) lead to

$$(2.12) \qquad (\dot\partial_j L^i_{hk})\eta^j + (\dot\partial_{\bar r}L^i_{hk})\bar\eta^r = 0,$$

i.e., the horizontal coefficients L^i_{jk} are $\mathbb{R}$-homogeneous of degree 0 with respect to η. Also, an elementary calculations gives $\delta_k(\dot\partial_{\bar r}L) = 0$.

Now, we associate to the canonical ($c.n.c.$), a complex linear connection of Berwald type

$$B\Gamma := \left(\overset{c}{N}{}^i_j, B^i_{jk} := \dot\partial_k\overset{c}{N}{}^i_j, B^i_{j\bar k} := \dot\partial_{\bar k}\overset{c}{N}{}^i_j, 0, 0\right),$$

where $\overset{c}{\delta}_k$ is with respect to $\overset{c}{N}{}^i_j$. $B\Gamma$ is neither h- nor v-metrical. Moreover, it satisfies the following properties

$$(2.13) \qquad B^i_{jk}\eta^j = \overset{c}{N}{}^i_k - (\dot\partial_{\bar r}\overset{c}{N}{}^i_k)\bar\eta^r \ ; \ B^i_{jk} = B^i_{kj}.$$

Subsequently, by means of this complex linear connection we characterize a lot of classes of $\mathbb{R}$-complex Hermitian Finsler spaces.

3. Classes of $\mathbb{R}$-complex Hermitian Finsler spaces

We set the connection form of the Chern-Finsler connection,

$$(3.1) \qquad \omega^i_j(z,\eta) = L^i_{jk}dz^k + C^i_{jk}\delta\eta^k,$$

which satisfies the following structure equations

$$(3.2) \qquad d(dz^i) - dz^k \wedge \omega^i_k = \theta^i \ ; \ d(\delta\eta^i) - \delta\eta^k \wedge \omega^i_k = \tau^i,$$

and their complex conjugates, where d is exterior differential with respect to the Chern-Finsler ($c.n.c.$). Since

$$d(\delta\eta^i) = -(\delta_{\bar h}N^i_k)dz^k \wedge d\bar z^h - (\dot\partial_h N^i_k)dz^k \wedge \delta\eta^h - (\dot\partial_{\bar h}N^i_k)dz^k \wedge \delta\bar\eta^h,$$

the torsion forms are

$$\theta^i = -\frac{1}{2}T^i_{jk}dz^j \wedge dz^k - C^i_{jk}dz^j \wedge \delta\eta^k \; ;$$

$$\tau^i = -(\delta_{\bar{h}}N^i_k)dz^k \wedge d\bar{z}^h - (\dot{\partial}_{\bar{h}}N^i_k)dz^k \wedge \delta\bar{\eta}^h,$$

where $T^i_{jk} := L^i_{jk} - L^i_{kj}$. Further on, it is easy to check that

$$(3.3) \qquad \overset{c}{N}{}^i_k = N^i_k - \frac{1}{2}[T^i_{jk}\eta^j + (\dot{\partial}_{\bar{r}}N^i_k)\bar{\eta}^r].$$

Its differentiation with respect to η gives the link between the local coefficients L^i_{jk} and B^i_{jk},

$$(3.4) \qquad B^i_{jk} = L^i_{jk} - \frac{1}{2}[\dot{\partial}_j(T^i_{lk}\eta^l) + (\dot{\partial}_{\bar{r}}L^i_{jk})\bar{\eta}^r].$$

The mixed part of the torsion form θ^i vanishes in the purely Hermitian case, (i.e., $C^i_{jk} = 0$), and the condition $\theta^i = 0$ is equivalent to the Kähler property of such a metric, i.e., $\frac{\partial g_{j\bar{m}}}{\partial z^i} = \frac{\partial g_{i\bar{m}}}{\partial z^j}$. But in the non-purely Hermitian case, the situation is a bit subtler because the torsion form θ^i has horizontal and mixed parts.

In complex Finsler geometry, accordingly to [1], there are three kinds of Kähler properties. This fact suggest us to introduce the similar notions on the ℝ-complex Hermitian Finsler spaces. Therefore, we have:

Definition 3.1. Let (M, F) be an ℝ-complex Hermitian Finsler space. (M, F) is called

i) strongly Kähler if $T^i_{jk} = 0$;

ii) Kähler if $T^i_{jk}\eta^j = 0$;

iii) weakly Kähler if $g_{i\bar{m}}T^i_{jk}\eta^j\bar{\eta}^m = 0$.

We notice that any strongly Kähler metric is a Kähler metric and any Kähler metric is a weakly Kähler metric. Moreover, in the purely Hermitian case all those nuances of Kähler are same with $\frac{\partial g_{j\bar{m}}}{\partial z^i} = \frac{\partial g_{i\bar{m}}}{\partial z^j}$.

Although, in complex Finsler geometry, the notions of strongly Kähler and Kähler coincide (see [11]), here this does not happen.

Theorem 3.1. *Let (M, F) be an ℝ-complex Hermitian Finsler space. Then, F is Kähler and the coefficients L^i_{jk} are $(0,0)$-homogeneous with respect to η if and only if F is strongly Kähler and $B^i_{jk} = L^i_{jk}$.*

Proof. We first prove the necessity. Under the Kähler assumption and using (3.4), we obtain $B^i_{jk} = L^i_{jk} - \frac{1}{2}(\dot{\partial}_{\bar{r}}L^i_{jk})\bar{\eta}^r$. But, the $(0,0)$-homogeneity with respect to η of L^i_{jk} implies $B^i_{jk} = L^i_{jk}$. The symmetry property of B^i_{jk} gives $L^i_{jk} = L^i_{kj}$, which means that F is strongly Kähler.

Using again (3.4), the sufficiency is immediate. $\square$

Theorem 3.2. *Let (M, F) be an $\mathbb{R}$-complex Hermitian Finsler space. Then, the Chern-Finsler (c.n.c.) comes from the complex spray (2.7) if and only if its local coefficients (2.7) are $(1, 0)$-homogeneous with respect to η and F is Kähler. Moreover, in this case F is strongly Kähler and $N_k^i = \overset{c}{N}_k^i$.*

Proof. If the Chern-Finsler (c.n.c.) comes from the complex spray (2.7), then $N_k^i = \overset{c}{N}_k^i$. By (3.3), it results $T_{jk}^i \eta^j + (\dot{\partial}_{\bar{r}} N_k^i) \bar{\eta}^r = 0$, which contracted with η^k, implies $(\dot{\partial}_{\bar{r}} G^i) \bar{\eta}^r = 0$. Now, differentiating the last relation with respect to η^k, we obtain $(\dot{\partial}_{\bar{r}} \overset{c}{N}_k^i) \bar{\eta}^r = 0$ and so, $(\dot{\partial}_{\bar{r}} N_k^i) \bar{\eta}^r = 0$, i.e., N_k^i are $(1, 0)$-homogeneous with respect to η. Also, the differentiation of the relation $N_k^i = \overset{c}{N}_k^i$ gives $L_{jk}^i = B_{jk}^i$ and so, $L_{jk}^i = L_{kj}^i$, which means that F is strongly Kähler.

The converse immediately results by (3.3). $\qquad\square$

Note that the $(1, 0)$-homogeneity with respect to η of the local coefficients of Chern-Finsler (c.n.c.) implies the $(0, 0)$-homogeneity with respect to η of the coefficients L_{jk}^i, but the converse is not true.

A simpler computation gives

$$(3.5) \qquad \overset{c}{\delta}_k = \delta_k - (\overset{c}{N}_k^i - N_k^i) \dot{\partial}_i.$$

Thus, by Theorem 3.2 and (3.5), $\overset{c}{\delta}_k = \delta_k$ if and only if the Chern-Finsler (c.n.c.) comes from the complex spray (2.7).

Theorem 3.3. *Let (M, F) be an $\mathbb{R}$-complex Hermitian Finsler space. Then, F is weakly Kähler if and only if*

$$[2\overset{c}{\delta}_k(\dot{\partial}_{\bar{r}} L) + g_{i\bar{m}} \bar{\eta}^m (\dot{\partial}_{\bar{r}} N_k^i)] \bar{\eta}^r = 0.$$

Proof. We contract the relation (3.3) with $g_{i\bar{m}} \bar{\eta}^m$ and then, we obtain

$$(3.6) \qquad (\overset{c}{N}_k^i - N_k^i) g_{i\bar{m}} \bar{\eta}^m = -\frac{1}{2} g_{i\bar{m}} \bar{\eta}^m [T_{jk}^i \eta^j + (\dot{\partial}_{\bar{r}} N_k^i) \bar{\eta}^r].$$

Using (3.5), we have $\overset{c}{\delta}_k(\dot{\partial}_{\bar{r}} L) \bar{\eta}^r = (\overset{c}{N}_k^i - N_k^i) g_{i\bar{r}} \bar{\eta}^r$, which substituted into (3.6) leads to

$$(3.7) \qquad 2\overset{c}{\delta}_k(\dot{\partial}_{\bar{r}} L) \bar{\eta}^r = -g_{i\bar{m}} \bar{\eta}^m T_{jk}^i \eta^j - g_{i\bar{m}} \bar{\eta}^m (\dot{\partial}_{\bar{r}} N_k^i) \bar{\eta}^r.$$

So, by (3.7), $g_{i\bar{m}} \bar{\eta}^m T_{jk}^i \eta^j = 0$ if and only if $[2\overset{c}{\delta}_k(\dot{\partial}_{\bar{r}} L) + g_{i\bar{m}} \bar{\eta}^m (\dot{\partial}_{\bar{r}} N_k^i)] \bar{\eta}^r = 0$. $\qquad\square$

Definition 3.2. Let (M, F) be an $\mathbb{R}$-complex Hermitian Finsler space.

   i) (M, F) is weakly Berwald if the local coefficients B_{jk}^i depend only on the position z.
   ii) (M, F) is Berwald if the local coefficients L_{jk}^i depend only on the position z.

Taking into account (3.4), we can say that any ℝ-complex Hermitian Finsler space which is Berwald is a weakly Berwald space. The converse is not true. Indeed, if the space is Berwald, then the relation (3.4) becomes $B^i_{jk}(z) = \frac{1}{2}(L^i_{jk} + L^i_{kj})$.

The necessary and sufficient circumstances for the weakly Berwald and Berwald properties of an ℝ-complex Hermitian Finsler space are given by the following theorems.

Theorem 3.4. *Let (M, F) be an ℝ-complex Hermitian Finsler space. Then, (M, F) is a weakly Berwald space if and only if $B^i_{j\bar{k}}$ depends only on z. Moreover, in this case,*

$$(3.8) \qquad \overset{c}{N^i_k} = B^i_{jk}(z)\eta^j + B^i_{k\bar{h}}(z)\bar{\eta}^h.$$

Proof. We first suppose that $B^i_{jk} = B^i_{jk}(z)$. Then $0 = \dot{\partial}_{\bar{r}}B^i_{jk} = \dot{\partial}_{\bar{r}}(\dot{\partial}_k \overset{c}{N^i_j}) = \dot{\partial}_k(\dot{\partial}_{\bar{r}}\overset{c}{N^i_j}) = \dot{\partial}_k B^i_{j\bar{r}}$ and, by complex conjugation we have $\dot{\partial}_{\bar{k}}B^{\bar{\imath}}_{\bar{j}r} = 0$, which means that $B^{\bar{\imath}}_{\bar{j}r}$ are holomorphic and $(0,0)$-homogeneous in the fibre variables η. Due to the strong maximum principle, we obtain that $B^{\bar{\imath}}_{\bar{j}r}$ depends only on z. So, the complex conjugation of the coefficients $B^{\bar{\imath}}_{\bar{j}r}$ also depend only on z.

Conversely, if $B^i_{j\bar{k}} = B^i_{j\bar{k}}(z)$, then $0 = \dot{\partial}_l B^i_{j\bar{k}} = \dot{\partial}_l(\dot{\partial}_{\bar{k}}\overset{c}{N^i_j}) = \dot{\partial}_{\bar{k}}(\dot{\partial}_l \overset{c}{N^i_j}) = \dot{\partial}_{\bar{k}}B^i_{lj}$.

This implies that the coefficients B^i_{lj} are holomorphic and $(0,0)$-homogeneous with respect to η, and further on it results $B^i_{jk}(z)$.

Now, using (2.13), we obtain (3.8). $\qquad \qquad \square$

Taking into account (2.11), (3.3) and (3.4), by a similar reasoning as the above theorem, we prove:

Theorem 3.5. *Let (M, F) be an ℝ-complex Hermitian Finsler space. Then, (M, F) is a Berwald space if and only if $\dot{\partial}_{\bar{r}}N^i_k$ depends only on z. Moreover, in this case,*

$$(3.9) \qquad N^i_k = L^i_{jk}(z)\eta^j + (\dot{\partial}_{\bar{h}}N^i_j)(z)\bar{\eta}^h;$$

$$\dot{\partial}_{\bar{k}}\overset{c}{N^i_j} = \frac{1}{2}\dot{\partial}_{\bar{k}}N^i_j.$$

We call *strongly Berwald* space, an ℝ-complex Hermitian Finsler spaces which is at the same time Berwald and Kähler. Owing to Theorem 3.1, we have proved the result.

Corollary 3.1. *Let (M, F) be an ℝ-complex Hermitian Finsler space. If F is strongly Berwald, then F is strongly Kähler.*

Examples of Berwald ℝ-complex Hermitian Finsler spaces are provided firstly by the class of purely Hermitian spaces. Indeed, considering a purely Hermitian

metric (i.e., $g_{i\bar{j}} = g_{i\bar{j}}(z)$ and $g_{\bar{r}\bar{m}}(z)$) with $g_{i\bar{j}}$ invertible, we have

$$N_k^i = g^{\bar{m}i}\left(\frac{\partial g_{\bar{r}\bar{m}}}{\partial z^k}\bar{\eta}^r + \frac{\partial g_{s\bar{m}}}{\partial z^k}\eta^s\right),$$

which implies that $L_{sk}^i = g^{\bar{m}i}\frac{\partial g_{s\bar{m}}}{\partial z^k}$ depends only on z. So, all purely Hermitian spaces, with $g_{i\bar{j}}$ invertible, are Berwald.

Another example of Berwald space is given by the function

$$(3.10) \qquad F^2 = L(z,w;\eta,\theta) = e^{2\sigma}\sqrt{(\eta+\bar{\eta})^4 + (\theta+\bar{\theta})^4}, \text{ with } \eta, \theta \neq 0,$$

on $\mathbb{C}^2$, where $\sigma(z,w)$ is a real valued function. In (3.10) we relabeled the usual local coordinates z^1, z^2, η^1, η^2 as z, w, η, θ, respectively. Direct computation leads to $N_i^1 = 2(\eta + \bar{\eta})\frac{\partial\sigma}{\partial z^i}$; $N_i^2 = 2(\theta + \bar{\theta})\frac{\partial\sigma}{\partial z^i}$, $i = 1, 2$, and then we find that the horizontal coefficients of the Chern-Finsler connection:

$$L_{1i}^1 = L_{2i}^2 = 2\frac{\partial\sigma}{\partial z^i}; \ L_{2i}^1 = L_{1i}^2 = 0, \ i = 1, 2,$$

depend only on z^i, $i = 1, 2$.

4. $\mathbb{R}$-complex Hermitian Berwald spaces with Randers metrics

We consider $z \in M$, $\eta \in T_z'M$, $\eta = \eta^i\frac{\partial}{\partial z^i}$. An $\mathbb{R}$-complex Finsler space (M, F) is called Randers if

$$(4.1) \qquad\qquad\qquad F = \alpha + \beta,$$

where

$$(4.2) \qquad\qquad \alpha^2(z, \eta, \bar{z}, \bar{\eta}) := Re\{a_{ij}\eta^i\eta^j\} + a_{i\bar{j}}\eta^i\bar{\eta}^j;$$
$$\beta(z, \eta, \bar{z}, \bar{\eta}) := Re\{b_i\eta^i\},$$

with $a_{ij} = a_{ij}(z)$, $a_{i\bar{j}} = a_{i\bar{j}}(z)$, and $b = b_i(z)dz^i$ is a $(1,0)$-differential form. The Randers function (4.1) produces two tensor fields g_{ij} and $g_{i\bar{j}}$.

In order to study the $\mathbb{R}$-complex Hermitian Finsler spaces with Randers metrics, we suppose that $a_{ij} = 0$. Thus, only the tensor field $g_{i\bar{j}}$ is invertible and it is characterized by the following properties:

Proposition 4.1 ([7]). *For the $\mathbb{R}$-complex Hermitian Randers space, with $a_{ij} = 0$, we have*

i) $g_{i\bar{j}} = \frac{F}{\alpha}a_{i\bar{j}} - \frac{\beta}{2\alpha^3}l_i l_{\bar{j}} + \frac{1}{2}b_i b_{\bar{j}} + \frac{1}{2\alpha}(b_{\bar{j}}l_i + b_i l_{\bar{j}})$ *and* $g_{ij} = -\frac{\beta}{2\alpha^3}l_i l_j + \frac{1}{2}b_i b_j + \frac{1}{2\alpha}(b_j l_i + b_i l_j)$;

ii) $g^{\bar{j}i} = \frac{\alpha}{F}a^{\bar{j}i} + \frac{2\beta+\alpha\omega}{FH}\eta^i\bar{\eta}^j - \frac{\alpha^3}{FH}b^i\bar{b}^j - \frac{\alpha}{FH}\left[(\bar{\varepsilon} + 2\alpha)\eta^i\bar{b}^j + (\varepsilon + 2\alpha)b^i\bar{\eta}^j\right]$;

iii) $\det(g_{i\bar{j}}) = \left(\frac{F}{\alpha}\right)^n\frac{H}{4\alpha F}\det(a_{i\bar{j}})$, *where*

$$(4.3) \qquad \alpha^2 = a_{j\bar{k}}\eta^j\bar{\eta}^k; \ l_i = a_{i\bar{j}}\bar{\eta}^j; \ b^k = a^{\bar{j}k}b_{\bar{j}}; \ b_l = b^{\bar{k}}a_{l\bar{k}}; \ b^{\bar{k}} := \bar{b}^k;$$
$$\varepsilon := b_j\eta^j; \ \omega := b_j b^j = \bar{\omega}; \ \varepsilon + \bar{\varepsilon} = 2\beta;$$
$$H := \alpha(4F + 2\beta + \alpha\omega) + \varepsilon\bar{\varepsilon} > 0.$$

Once obtained the metric tensor of an ℝ-complex Hermitian Randers space, it is a technical computation to give the expressions of Chern-Finsler (*c.n.c.*) from (2.6). Certainly, it involves some trivial calculus which lead to

$$(4.4) \qquad N_j^i = \overset{a}{N}_j^i + \frac{2}{H}[(2F - \varepsilon)\eta^i + \alpha^2 b^i](\overset{a}{\delta}_j\beta) + Fg^{\bar{r}i}\frac{\partial b_{\bar{r}}}{\partial z^j},$$

where

$$\overset{a}{N}_j^k := a^{\bar{m}i}\frac{\partial^2 \alpha^2}{\partial z^k \partial \bar{\eta}^m} = a^{\bar{m}i}\frac{\partial a_{s\bar{m}}}{\partial z^k}\eta^s;$$

$$2(\overset{a}{\delta}_j\beta) := \frac{\partial \beta}{\partial z^j} - \overset{a}{N}_j^k(\dot{\partial}_k\beta) = \frac{\partial \bar{b}^r}{\partial z^j}l_{\bar{r}} + \frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r.$$

Lemma 4.1. *Let* (M, F) *be an* ℝ-*complex Hermitian Randers space, with* $a_{ij} = 0$. *If* (M, F) *is Berwald, then*
$$(4.5)$$
$$\begin{cases} 3\beta(N_j^i - \overset{a}{N}_j^i)b_i = 2\varepsilon(\overset{a}{\delta}_j\beta) + 2\alpha^2 b^{\bar{r}}\frac{\partial b_{\bar{r}}}{\partial z^j}; \\ 3\beta(N_j^i - \overset{a}{N}_j^i)l_i = 2\alpha^2(\overset{a}{\delta}_j\beta) + \alpha^2\left[(2 + \omega)\bar{\eta}^r - 2\beta b^{\bar{r}}\right]\frac{\partial b_{\bar{r}}}{\partial z^j}; \\ \left(4\alpha^2 + \alpha^2\omega + \varepsilon\bar{\varepsilon}\right)(N_j^i - \overset{a}{N}_j^i)l_i = 4\alpha^2\bar{\varepsilon}(\overset{a}{\delta}_j\beta) + 2\alpha^2\left[(2\beta + \varepsilon)\bar{\eta}^r - \alpha^2 b^{\bar{r}}\right]\frac{\partial b_{\bar{r}}}{\partial z^j}; \\ \left(4\alpha^2 + \alpha^2\omega + \varepsilon\bar{\varepsilon}\right)(N_j^i - \overset{a}{N}_j^i)b_i = 2\left(\varepsilon\bar{\varepsilon} + \alpha^2\omega\right)(\overset{a}{\delta}_j\beta) \\ \qquad\qquad\qquad + 2\left[\left(\beta\varepsilon - \alpha^2\omega\right)\bar{\eta}^r + \alpha^2\left(\beta + \bar{\varepsilon}\right)b^{\bar{r}}\right]\frac{\partial b_{\bar{r}}}{\partial z^j}. \end{cases}$$

Proof. If (M, F) is Berwald, then $N_k^i = L_{jk}^i(z)\eta^j + (\dot{\partial}_{\bar{h}}N_j^i)(z)\bar{\eta}^h$, which means that N_k^i are ℝ-homogeneous polynomials in η and $\bar{\eta}$ of degree 1. Thus, using (4.4) we have

$$\{(4\alpha^2 + \alpha^2\omega + \varepsilon\bar{\varepsilon})(N_j^i - \overset{a}{N}_j^i) - 2(\bar{\varepsilon}\eta^i + \alpha^2 b^i)(\overset{a}{\delta}_j\beta)$$

$$- 2[3\alpha^2\beta a^{\bar{r}i} + \beta\bar{\eta}^r\eta^i - \alpha^2\eta^i\bar{b}^r - \alpha^2 b^i\bar{\eta}^r]\frac{\partial b_{\bar{r}}}{\partial z^j}\} + \alpha\{6\beta(N_j^i - \overset{a}{N}_j^i) - 4\eta^i(\overset{a}{\delta}_j\beta)$$

$$- [(4\alpha^2 + \alpha^2\omega + \varepsilon\bar{\varepsilon})a^{\bar{r}i} + \omega\bar{\eta}^r\eta^i - \alpha^2 b^i\bar{b}^r - \bar{\varepsilon}\eta^i\bar{b}^r - \varepsilon b^i\bar{\eta}^r]\frac{\partial b_{\bar{r}}}{\partial z^j}\} = 0,$$

which contains a rational part and an irrational part. Thus, we obtain

$$(4\alpha^2 + \alpha^2\omega + \varepsilon\bar{\varepsilon})(N_j^i - \overset{a}{N}_j^i)$$

$$= 2(\bar{\varepsilon}\eta^i + \alpha^2 b^i)(\overset{a}{\delta}_j\beta) + 2[3\alpha^2\beta a^{\bar{r}i} + \beta\bar{\eta}^r\eta^i - \alpha^2\eta^i\bar{b}^r - \alpha^2 b^i\bar{\eta}^r]\frac{\partial b_{\bar{r}}}{\partial z^j}$$

and

$$6\beta(N_j^i - \overset{a}{N}_j^i)$$

$$= 4\eta^i(\overset{a}{\delta}_j\beta) + [(4\alpha^2 + \alpha^2\omega + \varepsilon\bar{\varepsilon})a^{\bar{r}i} + \omega\bar{\eta}^r\eta^i - \alpha^2 b^i\bar{b}^r - \bar{\varepsilon}\eta^i\bar{b}^r - \varepsilon b^i\bar{\eta}^r]\frac{\partial b_{\bar{r}}}{\partial z^j}.$$

Contracting with b_i and l_i, they yield (4.5). □

Lemma 4.2. *The functions b^i and b_i are holomorphic if and only if $\overset{a}{\delta}_j\beta = 0$.*

Proof. Since $2(\overset{a}{\delta}_j\beta) = \frac{\partial \bar{b}^r}{\partial z^j}l_{\bar{r}} + \frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r$, the direct implication is immediate. Conversely, the condition $\overset{a}{\delta}_j\beta = 0$ can be rewritten as

$$(4.6) \qquad \frac{\partial b_i}{\partial z^j}\eta^i - b_i \overset{a}{N}{}^i_j + \frac{\partial b_{\bar{m}}}{\partial z^j}\bar{\eta}^m = 0.$$

Its derivation with respect to $\bar{\eta}$ gives $\frac{\partial b_{\bar{m}}}{\partial z^j} = 0$ and so, (4.6) implies

$$(4.7) \qquad \frac{\partial b_i}{\partial z^j}\eta^i - b_i \overset{a}{N}{}^i_j = 0,$$

which, by derivation with respect to η, it leads to $\frac{\partial b_i}{\partial z^j} = b^{\bar{m}}\frac{\partial a_{i\bar{m}}}{\partial z^j}$. The last relation is equivalently to $a_{i\bar{m}}\frac{\partial b^{\bar{m}}}{\partial z^j} = 0$, because $b^{\bar{m}}a_{i\bar{m}} = b_i$. This implies $\frac{\partial b^{\bar{m}}}{\partial z^j} = 0$, which proves our claim. $\qquad\square$

Theorem 4.1. *Let (M, F) be an $\mathbb{R}$-complex Hermitian Randers space, with $a_{ij} = 0$. If (M, F) is a Berwald space and $(N^i_j - \overset{a}{N}{}^i_j)b_i = 0$, then $\overset{a}{\delta}_j\beta = 0$ and $N^i_j = \overset{a}{N}{}^i_j$.*

Proof. Under our assumptions, the conditions (4.5) become

$$(4.8)$$
$$\begin{cases} 0 = \varepsilon(\overset{a}{\delta}_j\beta) + \alpha^2 b^{\bar{r}}\frac{\partial b_{\bar{r}}}{\partial z^j}; \\ 3\beta(N^i_j - \overset{a}{N}{}^i_j)l_i = 2\alpha^2(\overset{a}{\delta}_j\beta) + \alpha^2\left[(2 + \omega)\bar{\eta}^r - 2\beta b^{\bar{r}}\right]\frac{\partial b_{\bar{r}}}{\partial z^j}; \\ \left(4\alpha^2 + \alpha^2\omega + \varepsilon\bar{\varepsilon}\right)(N^i_j - \overset{a}{N}{}^i_j)l_i = 4\alpha^2\bar{\varepsilon}(\overset{a}{\delta}_j\beta) + 2\alpha^2\left[(2\beta + \varepsilon)\bar{\eta}^r - \alpha^2 b^{\bar{r}}\right]\frac{\partial b_{\bar{r}}}{\partial z^j}; \\ 0 = \left(\varepsilon\bar{\varepsilon} + \alpha^2\omega\right)(\overset{a}{\delta}_j\beta) + \left[\left(\beta\varepsilon - \alpha^2\omega\right)\bar{\eta}^r + \alpha^2\left(\beta + \bar{\varepsilon}\right)b^{\bar{r}}\right]\frac{\partial b_{\bar{r}}}{\partial z^j}. \end{cases}$$

The first and the fourth relations from (4.8) give

$$(4.9) \qquad (\beta\varepsilon - \alpha^2\omega)(\varepsilon\bar{\eta}^r + \alpha^2 b^{\bar{r}})\frac{\partial b_{\bar{r}}}{\partial z^j} = 0.$$

We have $\beta\varepsilon \neq \alpha^2\omega$ (the equality is not possible because it implies $\beta = \alpha = 0$). Then, it results

$$\varepsilon(\overset{a}{\delta}_j\beta) = \varepsilon\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r = -\alpha^2\frac{\partial b_{\bar{r}}}{\partial z^j}b^{\bar{r}},$$

which substituted into the second and the third relations from (4.8) lead to $(\overset{a}{\delta}_j\beta) = 0$. Now, by Lemma 4.2 and (4.4), we obtain $N^i_j = \overset{a}{N}{}^i_j$. $\qquad\square$

The next theorem provides the sufficient conditions for an $\mathbb{R}$-complex Hermitian Randers space $F := \alpha + \beta$, with $a_{ij} = 0$ to be Berwald.

Theorem 4.2. *Let (M, F) be an $\mathbb{R}$-complex Hermitian Randers space, with $a_{ij} = 0$. If $\overset{a}{\delta}_j\beta = 0$, then it is a Berwald space and $N^i_j = \overset{a}{N}{}^i_j$. Moreover, if α is Kähler, then F is strongly Kähler.*

Proof. Using again Lemma 4.2 and (4.4), it results $N_j^i = \overset{a}{N}_j^i$. Since α is a purely Hermitian metric, it is Berwald, and by the last relation we obtain that $F := \alpha + \beta$ is also Berwald and $L_{kj}^i(z) = \overset{a}{L}_{kj}^i(z)$, where $\overset{a}{L}_{kj}^i := \dot{\partial}_k \overset{a}{N}_j^i$.

Now, if we suppose that α is Kähler, we have $T_{jk}^i = \overset{a}{L}_{jk}^i - \overset{a}{L}_{kj}^i = 0$, which proves our claim. $\square$

Finally, we give some explicit examples of such ℝ-complex Hermitian Randers metrics which are Berwald or strongly Berwald.

Example 4.1. On $M = \mathbb{C}^2$ we consider the metric

$$(4.10) \qquad \alpha^2 = e^{z^1 + \bar{z}^1} \left|\eta^1\right|^2 + e^{z^2 + \bar{z}^2} \left|\eta^2\right|^2$$

and we choose $\varepsilon = e^{z^2} \eta^2$. These imply $a_{ij} = 0$, $(i, j = 1, 2)$, $2\beta = e^{z^2} \eta^2 + e^{\bar{z}^2} \bar{\eta}^2$, $b_1 = b^1 = 0$, $b_2 = e^{z^2}$, $b^2 = e^{-z^2}$ and $\omega = 1$.

With the above tools we construct a complex ℝ-complex Randers function

$$(4.11) \qquad F = \sqrt{e^{z^1 + \bar{z}^1} \left|\eta^1\right|^2 + e^{z^2 + \bar{z}^2} \left|\eta^2\right|^2} + \frac{1}{2}(e^{z^2} \eta^2 + e^{\bar{z}^2} \bar{\eta}^2),$$

which is a Hermitian Randers metric having $\det(g_{i\bar{j}}) = \left(\frac{F}{\alpha}\right)^2 \frac{H}{4\alpha F} \det(a_{i\bar{j}}) = \frac{HF}{4\alpha^3} e^{z^1 + \bar{z}^1 + z^2 + \bar{z}^2} > 0$, $(i, j = 1, 2)$, and $H = \alpha(5F + \beta) + \varepsilon\bar{\varepsilon} > 0$. A direct computation gives $2(\overset{a}{\delta}_j \beta) = \frac{\partial \bar{b}^2}{\partial z^j} l_{\bar{2}} + \frac{\partial b_{\bar{2}}}{\partial z^j} \bar{\eta}^2 = 0$, and $\frac{\partial b_{\bar{m}}}{\partial z^j} = 0$, $(j, m = 1, 2)$. Substituting these relations into (4.4), we obtain

$$(4.12) \qquad N_1^1 = \overset{a}{N}_1^1 = \eta^1 \; ; \; N_2^1 = \overset{a}{N}_2^1 = N_1^2 = \overset{a}{N}_1^2 = 0 \; ; \; N_2^2 = \overset{a}{N}_2^2 = \eta^2,$$

and so, the metric (4.11) is Berwald one. Also, due to (4.12) it is obvious that the metric (4.10) is Kähler. Thus, by Theorem 4.2, the metric (4.11) is strongly Berwald, and so by Corollary 3.1, it is strongly Kähler.

The above example can be generalized to a class of strongly Berwald metrics, taking on $M = \mathbb{C}^n$,

$$\alpha^2 = \sum_{k=1}^{n} e^{z^k + \bar{z}^k} \left|\eta^k\right|^2 .$$

Example 4.2. On $M = \mathbb{C}^3$ we set the metric

$$(4.13) \qquad \alpha^2 = e^{z^1 + \bar{z}^1} \left|\eta^1\right|^2 + e^{z^2 + \bar{z}^2} \left|\eta^2\right|^2 + e^{z^1 + \bar{z}^1 + z^3 + \bar{z}^3} \left|\eta^3\right|^2$$

and we choose the $(1, 0)$-differential form ε as

$$(4.14) \qquad \varepsilon = e^{z^2} \eta^2.$$

Then, $2\beta = e^{z^2} \eta^2 + e^{\bar{z}^2} \bar{\eta}^2$ and so, $a_{ij} = 0$, $b_i = b^i = 0$, $(i, j = 1, 3)$, $b_2 = e^{z^2}$, $b^2 = e^{-z^2}$ and $\omega = 1$.

Using (4.13) and (4.14), we construct
(4.15)
$$F = \sqrt{e^{z^1+\bar{z}^1}\left|\eta^1\right|^2 + e^{z^2+\bar{z}^2}\left|\eta^2\right|^2 + e^{z^1+\bar{z}^1+z^3+\bar{z}^3}\left|\eta^3\right|^2} + \frac{1}{2}(e^{z^2}\eta^2 + e^{\bar{z}^2}\bar{\eta}^2),$$

which is a Hermitian Randers metric, with $\det(g_{i\bar{j}}) = \left(\frac{F}{\alpha}\right)^3 \frac{H}{4\alpha F} \det\left(a_{i\bar{j}}\right) = \frac{HF^2}{4\alpha^4}e^{2z^1+2\bar{z}^1+z^2+\bar{z}^2+z^3+\bar{z}^3} > 0$, $(i,j = 1,2,3)$, and $H = \alpha(5F + \beta) + \varepsilon\bar{\varepsilon} > 0$.

Some computations give that the metric (4.15) has $N_j^i = \overset{a}{N}_j^i$, $(i,j = 1,2,3)$, and so, it is Berwald.

Since $\frac{\partial a_{3\bar{1}}}{\partial z^1} = e^{z^1+\bar{z}^1+z^3+\bar{z}^3} \neq \frac{\partial a_{1\bar{1}}}{\partial z^3} = 0$, the metric (4.13) is not Kähler. Thus, the metric (4.15) is not strongly Berwald.

Moreover, this example can be generalized to a class of $\mathbb{R}$-complex Hermitian Berwald spaces with Randers metrics, taking on $M = \mathbb{C}^n$,

$$\alpha^2 = \sum_{\substack{k=1 \\ k\neq 3}}^n e^{z^k+\bar{z}^k}\left|\eta^k\right|^2 + e^{z^1+\bar{z}^1+z^3+\bar{z}^3}\left|\eta^3\right|^2.$$

For ε, we can choose one of the following possibilities $\varepsilon = e^{z^k}\eta^k$, where $k = \overline{1,n}$, excepting $k = 1$ and 3.

References

[1] M. Abate and G. Patrizio, *Finsler Metrics - A Global Approach*, Lecture Notes in Math., **1591**, Springer-Verlag, 1994.

[2] T. Aikou, *Finsler geometry on complex vector bundles*, A sampler of Riemann-Finsler geometry, 83–105, Math. Sci. Res. Inst. Publ., 50, Cambridge Univ. Press, Cambridge, 2004.

[3] N. Aldea and G. Munteanu, *On complex Finsler spaces with Randers metric*, J. Korean Math. Soc. **46** (2009), no. 5, 949–966.

[4] ———, *On complex Landsberg and Berwald spaces*, J. Geom. Phys. **62** (2012), no. 2, 368–380.

[5] ———, *On complex Douglas spaces*, J. Geom. Phys. **66** (2013), 80–93.

[6] ———, *New candidates for a Hermitian approach of Gravity*, Int. J. Geom. Methods Mod. Phys. **10** (2013), no. 9, 1350041, 22 pp.

[7] N. Aldea and M. Purcaru, $\mathbb{R}$-*complex Finsler spaces with* (α,β)-*metric*, Novi Sad J. Math. **38** (2008), no. 1, 1–9.

[8] D. Bao, S. S. Chern, and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Graduate Texts in Mathematics, 200. Springer-Verlag, New York, 2000.

[9] A. Bejancu and H. R. Faran, *Geometry of pseudo-Finsler submanifolds*, Kluwer Academic Publishers, Dordrecht, 2000.

[10] L. Berwald, *Über Finslersche und Cartansche Geometrie. IV*, Ann. of Math. **48** (1947), 755–781.

[11] B. Chen and Y. Shen, *Kähler Finsler metrics are actually strongly Kähler*, Chin. Ann. Math. Ser. B **30** (2009), no. 2, 173–178.

[12] ———, *On complex Randers metrics*, Internat. J. Math. **21** (2010), no. 8, 971–986.

[13] S. Kobayashi, *Complex Finsler vector bundles*, Finsler geometry (Seattle, WA, 1995), 145–153, Contemp. Math., 196, Amer. Math. Soc., Providence, RI, 1996.

[14] M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Saikawa, Otsu, 1986.

[15] G. Munteanu, *Complex Spaces in Finsler, Lagrange and Hamilton Geometries*, Fundamental Theories of Physics, 141. Kluwer Academic Publishers, Dordrecht, 2004.

[16] G. Munteanu and M. Purcaru, *On ℝ-complex Finsler spaces*, Balkan J. Geom. Appl. **14** (2009), no. 1, 52–59.

[17] G. B. Rizza, *Stutture di Finsler di tipo quasi Hermitiano*, Riv. Mat. Univ. Parma **4** (1963), 83–106.

[18] H. L. Royden, *Complex Finsler metrics*, Complex differential geometry and nonlinear differential equations (Brunswick, Maine, 1984), 119–124, Contemp. Math., 49, Amer. Math. Soc., Providence, RI, 1986.

[19] H. Rund, *The differential geometry of Finsler spaces*, Springer-Verlag, Berlin, 1959.

[20] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.

[21] Z. Szabo, *Positive definite Berwald spaces*, Tensor (N.S.) **35** (1981), no. 1, 25–39.

[22] P. M. Wong, *Theory of Complex Finsler Geometry and Geometry of Intrinsic Metrics*, Imperial College Press, 2011.

[23] P. M. Wong and C. Zhong, *On weighted complex Randers metrics*, Osaka J. Math. **48** (2011), no. 3, 589–612.

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