Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 45 (1995), No. 2, 253–266

Persistent URL: `http://dml.cz/dmlcz/128515`

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Czechoslovak Mathematical Journal, 45 (120) 1995, Praha

ON SOME COMPLETENESS PROPERTIES FOR LATTICE ORDERED GROUPS

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(Received March 29, 1993)

G. J. M. H. Buskes [2] investigated a series of completeness properties for an archimedean Riesz space E. Each of these properties can be applied also in a more general setting, i.e., for the case when E is a lattice ordered group. If α is one of the properties under consideration, then we denote by $\mathcal{G}_\alpha$ the class of all lattice ordered groups G which have the property α.

The notion of radical class of lattice ordered groups was introduced in [8]; cf. also [4], [5], [12], [13], [16]. The relations between this notion and the classes $\mathcal{G}_\alpha$ will be dealt with in the present paper. We are mainly interested in the question whether $\mathcal{G}_\alpha$ (or some reasonably large subclass of $\mathcal{G}_\alpha$) is a radical class.

This question is related to the problem of existence of α-kernels. For some properties defined by means of sequences similar considerations were established in [10] and [11].

1. Preliminaries

The standard notation for lattice ordered groups will be applied (cf. [3] and [6]). The group operation will be written additively.

We denote by $\mathcal{G}$ the class of all lattice ordered groups. For $G \in \mathcal{G}$ let $c(G)$ be the system of all convex ℓ-subgroups of G; this system is partially ordered by inclusion. The lattice operations in $c(G)$ will be denoted by $\overset{c}{\bigwedge}$ and $\overset{c}{\bigvee}$. In fact, $\overset{c}{\bigwedge}$ coincides with the set-theoretical intersection. Let $\{A_i\}_{i\in I}$ be a nonempty subset of $c(G)$ and let $H = \overset{c}{\bigvee\limits_{i\in I}} H_i$. It is well-known that H is the set of all $g \in G$ having the property that there is a finite subset $\{i(1), i(2), \ldots, i(n)\}$ of I such that there exist elements $h_1 \in H_{i(1)}, \ldots, h_n \in H_{i(n)}$ with $g = h_1 + h_2 + \ldots + h_n$.

Supported by Grant GA-SAV 362/92

A nonempty subclass X of $\mathcal{G}$ is said to be a radical class if it is closed with respect to

a) convex ℓ-subgroups, and
b) joins of convex ℓ-subgroups.

A nonempty subset A of G^+ is called disjoint if $a_1 \wedge a_2 = 0$ whenever a_1 and a_2 are distinct elements of A. We write $a \perp b$ if $a \wedge b = 0$.

Let G be a lattice ordered group. We shall consider the following conditions for G (cf. [2]):

($\alpha(1)$) (G is boundedly laterally complete): each order bounded disjoint subset of G has a supremum.

($\alpha(2)$) (G is a disjoint order complete): for every disjoint sequence (f_n) in G such that $f_n \to 0$ in order, the element $\sup\{f_n\}$ exists.

($\alpha(3)$) (G is order complete): whenever (f_n) and (g_n) are sequences in G with $f_n \leqslant g_m$ for all m, n such that $\inf(g_n - f_n) = 0$, then there exists $h \in G$ such that $f_n \leqslant h \leqslant g_n$ for all n.

($\alpha(4)$) (G has the σ-interpolation property): whenever (f_n) and (g_n) are sequences in G such that $f_n \leqslant g_n$ for all m, n, there exists $h \in G$ such that $f_n \leqslant h \leqslant g_n$ for all n.

($\alpha(5)$) (G is uniformly complete): cf. Section 4 for a thorough definition.

($\alpha(6)$) (G is an A-group): for every disjoint set $\{f_\lambda\}$ in G which is order bounded there exists an element $g \in G^+$ such that $g - f_\lambda \perp f_\lambda$ for all λ.

For $i \in \{1, 2, \ldots, 6\}$ we denote by $\mathcal{G}_{\alpha(i)}$ the class of all lattice ordered groups which satisfy the condition $\alpha(i)$.

In Sections 1-3 it will be proved that if $i \in \{1, 2, 4\}$, then $\mathcal{G}_{\alpha(i)}$ is a radical class. The questions whether $\mathcal{G}_{\alpha(3)}, \mathcal{G}_{\alpha(5)}$ and $\mathcal{G}_{\alpha(6)}$ are radical classes remain open; some partial results in these directions will be established in Sections 3, 4 and 5. E.g., it will be shown that the class of all abelian lattice ordered groups belonging to $\mathcal{G}_{\alpha(3)}$ and the class of all abelian projectable lattice ordered groups belonging to $\mathcal{G}_{\alpha(6)}$ are radical classes.

2. The conditions ($\alpha(1)$) and ($\alpha(2)$)

The following lemma is easy to verify; the proof will be omitted. In what follows, G is a lattice ordered group.

2.1. Lemma. *Let $i \in \{1, 2, 3, 4\}$. Assume that $G \in \mathcal{G}_{\alpha(i)}$ and $H \in c(G)$. Then $H \in \mathcal{G}_{\alpha(i)}$.*

Let α be any property of lattice ordered groups. We denote by S_α the system of all elements of $c(G)$ which have the property α. If a convex ℓ-subgroup H of G is a largest element of S_α, then H is said to be the α-kernel of G.

The above lemma implies that for $i \in \{1,2,3,4\}$ the $\alpha(i)$-kernel exists for each $G \in \mathcal{G}$ iff $\mathcal{G}_{\alpha(i)}$ is a radical class.

2.2. Lemma. *Whenever $a_1, a_2, \ldots, a_n$ are elements of G^+ there exists a system $S(a_1, a_2, \ldots, a_n)$ of mappings*

$$\psi_i\colon [0, a_1 + \ldots + a_n] \to [0, a_i] \quad (i = 1, 2, \ldots, n)$$

such that

(i) *each ψ_i is isotone;*

(ii) *for each $x \in [0, a_1 + \ldots + a_n]$ the relation $x = \psi_1(x) + \ldots + \psi_n(x)$ is valid.*

Proof. We proceed by induction with respect to n. For $n = 1$ we put $S(a_1) = \{\psi_1\}$, where ψ_1 is the identity on $[0, a_1]$.

Let $n > 1$ and assume that the assertion is valid for $n-1$. We put $\psi_1(x) = a_1 \wedge x$ for each $x \in [0, a_1 + \ldots + a_n]$. Next, let us consider the pairs

$$(1) \qquad (x, -x + a_1 + \ldots + a_n),\ (a_1, a_2'),$$

where $a_2' = a_2 + \ldots + a_n$.

We apply the facts demonstrated in the proof of Theorem 1.2.16, [1] (Riesz theorem) concerning the case $m = n = 2$ (instead of the pairs (a_1, a_2), (b_1, b_2) from the mentioned proof we take now the pairs (1)). In our case we get

$$(2) \qquad 0 \leqslant -\psi_1(x) + x \leqslant a_2'.$$

By the induction hypothesis there exists a system $S(a_2, \ldots, a_n) = \{\psi_i'\}$ $(i = 2, \ldots, n)$, where ψ_i' is a mapping of $[0, a_2 + \ldots + a_n]$ into $[0, a_i]$ $(i = 2, \ldots, n)$ such that the conditions (i) and (ii) above are satisfied for the elements which are now under consideration.

Hence all ψ_i' are isotone and

$$(3) \qquad t = \psi_2'(t) + \ldots + \psi_n'(t) \quad \text{for each} \quad t \in [0, a_2 + \ldots + a_n].$$

Denote $\psi_i(t) = \psi_i'(-\psi_1(t) + t)$ for each $t \in [0, a_1 + \ldots + a_n]$ and $i = 2, 3, \ldots, n$. Hence (ii) holds.

It remains to verify that all ψ_i are isotone. For $i = 1$ this is obvious. Let $x, y \in [0, a_1 + \ldots + a_n]$, $x \geqslant y$. Since all ψ_i' are isotone we have to show that

$$-\psi_1(y) + y \leqslant -\psi_1(x) + x,$$

i.e., that

$$(4) \qquad -(a_1 \wedge y) + y \leqslant -(a_1 \wedge x) + x.$$

An easy computation shows that the interval $[a_1 \wedge y, y]$ is transposed to a subinterval of the interval $[a_1 \wedge x, x]$. Thus the relation (4) is valid, completing the proof. $\square$

2.3. Lemma. *Let $\{G_i\}_{i\in I}$ be a nonempty subset of $c(G)$ such that $G_i \in \mathcal{G}_{\alpha(1)}$ for each $i \in I$. Then $\bigvee\limits_{i\in I}^{c} G_i$ belongs to $\mathcal{G}_{\alpha(1)}$.*

Proof. Put $\bigvee\limits_{i\in I}^{c} G_i = H$. Let A be an order bounded disjoint subset of H. Thus there is $h \in H$ such that $0 \leqslant a \leqslant h$ is valid for each $a \in A$.

There exist $i(1), i(2), \ldots, i(n)$ in I such that $h \in G_{i(1)} + G_{i(2)} + \ldots + G_{i(n)}$. Thus there are $g_1 \in G_{i(1)}, \ldots, g_n \in G_{i(n)}$ with $h = g_1 + g_2 + \ldots + g_n$. Hence $h \leqslant |g_1| + |g_2| + \ldots + |g_n|$.

Now let us apply Lemma 2.2, where the elements a_i from 2.2 are replaced by $|g_j|$ ($j = 1, 2, \ldots, n$), and let ψ_j have analogous meaning as in 2.2. For each $a \in A$ we have $a \leqslant |g_1| + \ldots + |g_n|$. Put $\psi_j(a) = a_j$. Thus

$$(1) \qquad a = a_1 + a_2 + \ldots + a_n, \; 0 \leqslant a_j \leqslant |g_j| \in G_j \quad (j = 1, 2, \ldots, n).$$

Let $j \in \{1, 2, \ldots, n\}$ be fixed. Since A is disjoint, the set $\{a_j\}_{a\in A}$ is disjoint as well. Because G_j belongs to $\mathcal{G}_{\alpha(1)}$ we conclude that $\bigvee\limits_{a\in A} a_j = b_j$ does exist in G_j.

Put $b = b_1 + b_2 + \ldots + b_n$. Then clearly $b \in H$ and $a \leqslant b$ for each $a \in A$. Let $x \in G$ be such that $a \leqslant x$ for each $a \in A$. Denote $x \wedge h = y$. Hence $a \leqslant y$ for each $a \in A$. We set $y_j = \psi_j(y)$ for $j = 1, 2, \ldots, n$. Then $a_j \leqslant y_j$ for each $a \in A$ and hence $b_j \leqslant y_j$. Because of $y = y_1 + y_2 + \ldots + y_n$ we obtain that $b \leqslant y$. Hence $b \leqslant x$. This shows that $b = \sup A$, completing the proof. $\square$

Now, Lemmas 2.1 and 2.3 yield

2.4. Theorem. *$\mathcal{G}_{\alpha(1)}$ is a radical class.*

A radical class which is closed with respect to homomorphic images is said to be a torsion class [15]. Now we shall deal with the question whether $\mathcal{G}_{\alpha(1)}$ is a torsion class.

Let M be an infinite set and let F be the set of all integer valued functions defined on M. The operation $+$ in F has the natural meaning and the partial order on F is defined componentwise. Then $F \in \mathcal{G}_{\alpha(1)}$.

Let H be the system of all $f \in F$ such that the set $\{x \in M : f(x) \neq 0\}$ is finite. Then H is an ℓ-ideal in F. Denote $G = F/H$.

Now let f_1 be the element of F with $f_1(x) = 1$ for each $x \in M$. The interval $B = [0, f_1]$ of F is a Boolean algebra. Put $\Delta = B \cap H$. Hence Δ is an ideal of the Boolean algebra B.

Consider the quotient Boolean algebra B/Δ. The following lemma is easy to verify.

2.5. Lemma. *Let $f, g \in B$. Then f and g belong to the same element of B/Δ if and only if they belong to the same element of F/H.*

As a consequence of 2.5 we obtain

2.6. Lemma. *For each element A of B/Δ let $\varphi(A) = a + H$, where $a \in A$. Then φ is an isomorphism of B/Δ onto the interval $[H, f_1 + H]$ of F/H.*

Now, Theorem 21.8 of [17] implies that the Boolean algebra B is not complete. Thus according to 20.1, [17] there exists a subset $\{A_i\}_{i\in I}$ of B/Δ such that (i) $A_i \neq \Delta$ for each $i \in I$, (ii) $A_{i(1)} \wedge A_{i(2)} = \Delta$ whenever $i(1)$ and $i(2)$ are distinct elements of I, and (iii) the join $\bigvee_{i\in I} A_i$ does not exist in B/Δ. Hence by applying the isomorphism φ we infer that $\{\varphi(A_i)\}_{i\in I}$ is a disjoint subset of $[H, f_1 + H]$ such that the join of this subset does not exist in the interval $[H, f_1 + H]$. But then the join of this subset does not exist in F/H and hence F/H fails to belong to the class $\mathcal{G}_{\alpha(1)}$. Therefore we have

2.7. Proposition. *$\mathcal{G}_{\alpha(1)}$ fails to be a torsion class.*

The condition $\alpha(1)$ can be weakened as follows:

($\alpha(1\sigma)$) (G is σ-laterally complete): each countable order bounded disjoint subset of G has a supremum.

By the same method as in the proof of 2.3 we obtain that Lemma 2.3 remains valid if $\alpha(1)$ is replaced by $\alpha(1\sigma)$. A similar situation occurs for Lemma 2.1. Therefore we can replace $\alpha(1)$ by $\alpha(1\sigma)$ in 2.4 as well.

Next, let us consider the condition ($\alpha(2)$). We can denote by 2.3' the assertion which we obtain from 2.3 if $\alpha(1)$ is replaced by $\alpha(2)$. To prove 2.3' we have to work (instead of A as in 2.3) with a disjoint sequence (f_m) in G^+. We apply the same procedure as in the proof of 2.3 with the distinction that instead of (1) we write

$$f_m = a_{m1} + a_{m2} + \ldots + a_{mn} \tag{1'}$$

with the obvious further modifications of notation. It suffices to observe that whenever $f_n \to 0$ in order, then for each $j \in \{1, 2, \ldots, n\}$ the relation $a_{mj} \to 0$ in order is valid. Therefore we obtain

2.8. Theorem. *$\mathcal{G}_{\alpha(2)}$ is a radical class.*

For investigating the question whether $\mathcal{G}_{\alpha(1\sigma)}$ (or $\mathcal{G}_{\alpha(2)}$) is a torsion class the above consideration which was applied for $\mathcal{G}_{\alpha(1)}$ does not suffice.

2.9. Example. Let F and H be as above. There exists a system $\{M_n\}_{n\in\mathbb{N}}$ of infinite subsets of M such that $M_{n(1)} \cap M_{n(2)} = \emptyset$ whenever $n(1)$ and $n(2)$ are distinct positive integers. For each $n \in \mathbb{N}$ let $f_n \in F$ be such that $f_n(x) = 1$ whenever $x \in M_n$ and $f_n(x) = 0$ otherwise. Then $\{f_n + H\}_{n\in\mathbb{N}}$ is a disjoint subset of F/H.

Next, for $n \in \mathbb{N}$ let $g_n \in F$ be such that $g_n(x) = 1$ if $x \in \bigcup\limits_{i\geqslant n} M_i$, and $g_n(x) = 0$ otherwise. Hence $g_n + H > g_{n+1} + H > H$ is valid in F/H for each $n \in \mathbb{N}$. Moreover, $\bigwedge\limits_{n\in\mathbb{N}} (g_n + H) = H$. Also, $g_n + H > f_n + H$ for each $n \in \mathbb{N}$. Thus $f_n + H \to H$ is order.

Let $f \in F$ such that $f_n + H \leqslant f + H$ for each $n \in \mathbb{N}$. Put $X_n = \{x \in M_n : f_n(x) \leqslant f(x)\}$. Hence the set X_n must be infinite. For each $n \in \mathbb{N}$ we choose an element $x_n \in X_n$ and put $Y = \{x_n\}_{n\in\mathbb{N}}$. Let $f' \in F$ be such that $f'(x) = 0$ if $x \in Y$ and $f'(x) = f(x)$ otherwise. Then $f_n + H \leqslant f' + H$ for each $n \in \mathbb{N}$, and $f' + H < f + H$. Hence the set $\{f_n + H\}$ does not possess a supremum in F/H.

This example implies that the following result is valid (in fact, it also gives an alternative proof of 2.7):

2.10. Proposition. *Neither $\mathcal{G}_{\alpha(1\sigma)}$ nor $\mathcal{G}_{\alpha(2)}$ is a torsion class.*

3. The conditions $(\alpha(3))$ and $(\alpha(4))$

Let us first consider the following condition which we obtain by modifying $(\alpha(3))$:

$(\alpha'(3))$ Whenever (f_n) and (g_n) are bounded sequences in G^+ with $f_n \leqslant g_m$ for all m, n and such that $\inf(g_n - f_n) = 0$, then there exists $h \in G$ such that $f_n \leqslant h \leqslant g_n$ for all n.

3.1. Lemma. *The conditions $(\alpha(3))$ and $(\alpha'(3))$ are equivalent.*

Proof. It is obvious that $(\alpha(3)) \Rightarrow (\alpha'(3))$. Assume that $(\alpha'(3))$ is valid and let (f_n) and (g_n) be as in $(\alpha(3))$. Denote

$$f'_n = (f_n \vee f_1) - f_1, \quad g'_n = (g_n \wedge g_1) - f_1$$

for each $n \in \mathbb{N}$. Then $f'_n \leqslant g'_m$ for all m, n. Next we have

$$g'_n - f'_n \leqslant g_n - f_n \quad \text{for each } n \in \mathbb{N},$$

whence $\inf(g'_n - f'_n) = 0$. Thus there is $h' \in G$ such that $f'_n \leqslant h' \leqslant g'_n$ for all n. Put $h = h' + f_1$. Then $f_n \leqslant f_n \vee f_1 \leqslant h \leqslant g_n \wedge g_1 \leqslant g_n$ for each n. □

3.2. Lemma. *Let G be abelian. Let us apply the same assumptions and notation as in 2.2. Let $x, y \in [0, a_1 + \ldots + a_n]$, $x \geqslant y$. Then $x - y \geqslant \psi_i(x) - \psi_i(y)$ for $i = 1, 2, \ldots, n$.*

Proof. By induction on n. For $n = 1$ the assertion obviously holds. Let $n > 1$. Denote $a'_2(x) = -\psi_1(y) + y$. Next let $z = \psi_1(x) \vee y$. The intervals $[a \wedge y, y]$ and $[a \wedge x, z]$ are transposed, whence

$$(1) \qquad -y + (a \wedge y) = -z + (a \wedge x).$$

Thus we have

$$a'_2(x) = (x - z) + (z - \psi_1(x)) = (x - z) + a'_2(y),$$
$$a'_2(x) - a'_2(y) = x - z \leqslant x - y.$$

Now, by the induction hypothesis and by the definition of $\psi_2, \ldots, \psi_n$ we infer that $\psi_i(x) - \psi_i(y) \leqslant x - y$ for $i = 2, \ldots, n$. Clearly $\psi_1(x) - \psi_1(y) \leqslant x - y$. □

3.3. Lemma. *Let $\{G_i\}_{i \in I}$ be a nonempty subset of $c(G)$ such that $G_i \in \mathcal{G}_{\alpha'(3)}$ for each $i \in I$. Then $\bigvee^{c}_{i \in I} G_i$ belongs to $\mathcal{G}_{\alpha'(3))}$.*

Proof. Put $\bigvee^{c}_{i \in I} G_i = H$. Assume that (f_n) and (g_n) are bounded sequences in H^+ with $f_n \leqslant g_m$ for all n, m and such that $\inf(g_n - f_n) = 0$. Hence there is $h \in H^+$ such that $g_m \leqslant h$ for each m.

We proceed by applying an analogous argument as in the proof of 2.3. There exist indices $i(1), \ldots, i(k)$ in I and elements $g_1 \in G_{i(1)}, \ldots, g_k \in G_{i(k)}$ such that $h = g_1 + g_2 + \ldots + g_k$. Hence

$$(1) \qquad h \leqslant |g_1| + |g_2| + \ldots + |g_k|.$$

Thus in view of 2.2 for each positive integer n there are elements $a_{nj} \in G_{i(j)}$ ($j = 1, 2, \ldots, k$) with

$$(2) \qquad f_n = a_{n1} + \ldots + a_{nk};$$

similarly, for each positive integer m there are $b_{mj} \in G_{i(j)}$ ($j = 1, 2, \ldots, k$) such that

$$(3) \qquad g_m = b_{m1} + \ldots + b_{mk},$$

and, moreover, $a_{nj} \leqslant b_{mj}$ for each $j \in \{1, \ldots, k\}$ and for each m, n.

Next, according to 3.2 the relation $g_{nj} - f_{nj} \leqslant g_n - f_n$ is valid for each $j \in \{1, 2, \ldots, k\}$ and each n. Hence $\inf(g_{nj} - f_{nj}) = 0$ holds for $j = 1, 2, \ldots, k$. Because of $G_{i(j)} \in \mathcal{G}_{\alpha'(3)}$ we infer that there is $h_j \in G_{i(j)}$ such that $f_{nj} \leqslant h_j \leqslant g_{nj}$ for all n. Denote $h_1 + \ldots + h_k = h$. Then (2) and (3) yield that $f_n \leqslant h \leqslant g_n$ for all n. Therefore H belongs to $\mathcal{G}_{\alpha'(3)}$. □

We denote by $\mathcal{G}_a$ the class of all abelian lattice ordered groups. Then $\mathcal{G}_a$ is a radical class. This can be easily proved directly, but it is also a particular case of a more general result proved in [7].

3.4. Theorem. *$\mathcal{G}_{\alpha(3)} \cap \mathcal{G}_a$ is a radical class.*

Proof. This is a consequence of 2.1, 3.1, 3.3 and of the above mentioned result concerning $\mathcal{G}_a$. □

The method of proving the following result is analogous to that which was used in proving 3.4 (with the distinction that we need not apply 3.2); the detailed proof will be omitted.

3.5. Theorem. *$\mathcal{G}_{\alpha(4)}$ is a radical class.*

The question whether $\mathcal{G}_{\alpha(3)} \cap \mathcal{G}_a$ (or $\mathcal{G}_{\alpha(4)}$) is a torsion class remains open.

4. Uniform completeness

First we recall the basic definitions concerning uniform completeness of Riesz spaces (cf., e.g., [14]).

Let L be a Riesz space.

4.1. Definition. Given an element $e \geqslant 0$ in L, we say that a *sequence (f_n) in L converges e-uniformly to the element $f \in L$* whenever, for every real $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that $|f - f_n| \leqslant \varepsilon e$ holds for all $n \geqslant n_0(\varepsilon)$.

4.2. Definition. Let $e \in L$, $e \geqslant 0$. A sequence (f_n) in L is called an *e-uniform Cauchy sequence* whenever, for every real $\varepsilon > 0$, there exists a positive integer $n_1(\varepsilon)$ such that $|f_m - f_n| \leqslant \varepsilon e$ holds for all $m, n \geqslant n_1(\varepsilon)$.

Again, let $e \in L$, $e \geqslant 0$. It is easy to verify that the condition expressed in 4.1 is equivalent to the following one (for a given sequence (f_n) in L and an element $f \in L$):

(i) For every positive integer k there exists a positive integer $n_0(k)$ such that $k|f - f_n| \leqslant e$ holds for all $n \geqslant n_0(k)$.

Analogously, the following condition is equivalent to that from 4.2:

(ii) For every positive integer k there exists a positive integer $n_1(\varepsilon)$ such that $k|f_m - f_n| \leqslant e$ holds for $m, n \geqslant n_1(k)$.

Moreover, the conditions (i) and (ii) can be applied also in the case when L is a lattice ordered group. Thus if (i) holds, then we say that (f_n) converges e-uniformly to the element f. If (ii) is valid, then (f_n) is called an e-uniform Cauchy sequence.

Next, analogously to the definition 42.1 in [14] we introduce

4.3. Definition. A lattice ordered group G is said to be *uniformly complete* whenever, for every $e \in G^+$, each e-uniform Cauchy sequence has an e-uniform limit.

4.4. Lemma. *Let H be a convex ℓ-subgroup of a lattice ordered group G. Let $0 \leqslant e \in H$, $f \in G$ and let (f_n) be a sequence in H. Suppose that (f_n) converges e-uniformly to the element f (in G). Then $f \in H$.*

Proof. Under the notation as in (i) above let $n \geqslant n_0(1)$. Then $|f - f_n| \leqslant e$, hence $-e \leqslant f - f_n \leqslant e$. Since H is convex in G we infer that f belongs to H. □

4.5. Corollary. *The class $\mathcal{G}_{\alpha(5)}$ is closed with respect to convex ℓ-subgroups.*

Let us consider the following condition for a lattice ordered group G:

(iii) Whenever $0 \leqslant e \in G$ and (g_n) is an e-uniform Cauchy sequence in G with $0 \leqslant g_n \leqslant 2e$ for all n, then there exists $g \in G$ such that (g_n) converges e-uniformly to the element g.

4.6. Lemma. *Let G be a lattice ordered group satisfying the condition* (iii). *Then G is uniformly complete.*

Proof. Let (f_n) be an e-uniform Cauchy sequence in G. Denote $n_1(1) = t$. Let $m \geqslant t$. Thus

$$-e \leqslant f_m - f_t \leqslant e.$$

Put $g_m = f_m - f_t + e$. Hence

$$0 \leqslant g_m \leqslant 2e.$$

Next, let j be a positivie integer, $j \geqslant t$. Then

$$g_m - g_j = f_m - f_j.$$

Thus (g_n) is an e-uniform Cauchy sequence. Since G satisfies the condition (iii) there is $g \in G$ such that (g_n) converges e-uniformly to the element g. Put $f = g - e + f_t$. Then (f_n) converges e-uniformly to the element f. □

Let us consider the following condition for a lattice ordered group G:

(A) If $H \in c(G), 0 \leqslant e \in G$ and if (f_n) is a sequence in H such that (f_n) is e-uniform Cauchy (in G), then there is $0 \leqslant e_1 \in H$ such that (f_n) is e_1-uniform Cauchy in H.

It is easy to verify that if G fails to be archimedean, then it does not satisfy the condition (A). It is an open question whether each archimedean lattice ordered group must satisfy the condition (A).

4.7. Lemma. *Let G be an abelian lattice ordered group satisfying the condition* (A). *Let G_1 and G_2 be convex ℓ-subgroups of G such that $G = G_1 \vee G_2$. Assume that both G_1 and G_2 are uniformly complete. Then G is uniformly complete as well.*

Proof. In view of 4.6 it suffices to verify that G satisfies the condition (iii). Let e and (g_n) be as in (iii).

Since G is abelian we have $G = G_1 + G_2$. Hence there are $a_1 \in G_1^+$ and $a_2 \in G_2^+$ such that $2e = a_1 + a_2$. For each g_n let us denote

$$g_{n1} = g_n \wedge a_1, \quad g_{n2} = g_n - g_{n1}.$$

Then we have (cf. [6], p. 77, the property O)

$$|g_{m1} - g_{n1}| \leqslant |g_m - g_n|.$$

Then (g_{n1}) is a sequence in G_1 and in view of (A) there is $e_1 \in G_1^+$ such that (g_{n1}) is e_1-uniformly Cauchy (in G_1). Hence there is $g^1 \in G_1$ such that (g_{n1}) converges e_1-uniformly to the element g^1.

Next, we have $g_{n2} \in [0, a_2]$ for each positive integer n (cf. the proof of 2.2), hence (g_{n2}) is a sequence in G_2. Let m, n be positive integers. Then

$$|g_{m2} - g_{n2}| = |(g_m - g_{m1}) - (g_n - g_{n1})| = |(g_m - g_n) + (g_{n1} - g_{m1})| \leqslant$$

$$\leqslant |g_m - g_n| + |g_{m1} - g_{n1}| \leqslant 2|g_n - g_n|.$$

Hence (g_{n2}) is e-uniform Cauchy in G. In view of (A) and since G_2 is uniformly complete, there are $g^2 \in G_2$ and $e_2 \in G_2^+$ such that (g_{n2}) converges e_2-uniformly to the element g^2.

Put $g = g^1 + g^2$. The above results yield that (g_n) converges $(e_1 + e_2)$-uniformly to the element g, completing the proof. □

By obvious induction we can verify that 4.7 remains valid when the two-element system $\{G_1, G_2\}$ is replaced by a finite system $\{G_1, G_2, \ldots, G_n\} \in c(G)$ such that $\bigvee^{c}_{i=1,2,\ldots,n} G_i = G$ and all G_i are uniformly complete.

4.8. Lemma. *Let G be an abelian lattice ordered group satisfying the condition* (A). *Let $G_i \in c(G)$, $i \in I$ such that $G = \bigvee^{c} G_i$ and all G_i are uniformly complete. Then G is uniformly complete.*

Proof. Again, according to 4.6 it suffices to show that G satisfies the condition (iii). Let e and (g_n) be as in (iii). There exist $i(1), i(2), \ldots, i(n)$ in I such that $e \in H$, where $H = G_{i(1)} \vee \ldots \vee G_{i(n)}$. Let t be as in the proof of 4.6. Then $f_n \in H$ for each $n \geqslant t$. Now we can apply to H the above mentioned generalization of Lemma 4.7. □

From 4.8 we obtain

4.9. Theorem. *Let G be an abelian lattice ordered group satisfying the condition* (A). *Then the uniform complete kernel of G does exist.*

Let $\mathcal{G}_A$ be the class of all abelian lattice ordered groups which satisfy the condition (A).

4.10. Lemma. *$\mathcal{G}_A$ is a radical class.*

Proof. It is easy to verify that $\mathcal{G}_A$ is closed with respect to convex ℓ-subgroups. Let G be an abelian lattice ordered group and let G_i $(i = 1, 2)$ be elements of $c(G)$ satisfying the condition (A). By a similar consideration as in the proofs of 4.6 and 4.7 we can show that $G_1 \vee G_2$ belongs to $\mathcal{G}_A$; the details will be omitted. Hence by applying obvious induction and by the same method as in 4.8 we obtain that $\mathcal{G}_A$ is closed with respect to joins of convex ℓ-subgroups. Therefore $\mathcal{G}_A$ is a radical class. □

4.11. Theorem. *$\mathcal{G}_{\alpha(5)} \cap \mathcal{G}_A$ is a radical class.*

Proof. This is a consequence of 4.5, 4.9 and 4.10. □

The question whether $\mathcal{G}_{\alpha(5)}$ is a radical class remains open.

5. The condition $\alpha(6)$

Let us recall the following notions and notation. Let G be a lattice ordered group. If $X \subseteq G$, then we set

$$X^{\perp} = \{g \in G \colon |g| \wedge |x| = 0 \quad \text{for each } x \in X\};$$

$X^{\perp}$ is said to be a polar of G. If $\operatorname{card} X = 1$, then $X^{\perp\perp}$ is a *principal* polar.

G is called projectable if each its principal polar is a direct factor, i.e. if $G = X^{\perp} \times X^{\perp\perp}$ whenever $\operatorname{card} X = 1$.

If we have a direct product decomposition $G = A \times B$ and $g \in G$, then the component of the element g in the direct factor A will be denoted by $g(A)$.

5.1. Lemma. *The class $\mathcal{G}_{\alpha(6)}$ is closed with respect to convex ℓ-subgroups.*

Proof. Let $G \in \mathcal{G}_{\alpha(6)}$ and $H \in c(G)$. Assume that $\{f_\lambda\}$ is a disjoint subset of H which is order bounded in H. Thus there is $h \in H$ such that h is an upper bound of $\{f_\lambda\}$. Since $G \in \mathcal{G}_{\alpha(6)}$ there exists $g \in G^+$ such that $g - f_\lambda \perp f_\lambda$ for all λ. Put $g' = g \wedge h$. Then $g' \in H$ and $g' - f_\lambda \perp f_\lambda$ for all λ. Therefore $H \in \mathcal{G}_{\alpha(6)}$. □

5.2. Lemma. *Let G be abelian and projectable, $G_i \in c(G)$ $(i = 1, 2)$, $a_1 \in G_1^+$, $a_2 \in G_2^+$. Then there are $a_1' \in G_2^+$ such that $a_1' \perp a_2'$ and $a_1 + a_2 \leqslant a_1' + a_2'$.*

Proof. Put $(a_1 - a_2)^+ = c_1$, $(a_1 - a_2)^- = c_2$ and denote

$$A = \{c_1\}^{\perp\perp}, \quad B = \{c_2\}^{\perp\perp}, \quad C = \{c_1 \vee c_2\}^{\perp}.$$

The projectability of G yields that

$$G = A \times B \times C.$$

We set $a_1' = 2a_1(A) + 2a_1(C)$ and $a_2' = 2a_2(B)$. Then $a_1(A)$ and $a_1(C)$ belong to the interval $[0, a_1]$, whence $a_1' \in G_1$. Similarly, $a_2' \in [0, a_2] \subseteq G_2$ and thus $a_2' \in C$.

In virtue of the definitions of A, B and C the relations

$$a_1(A) \geqslant a_2(A), \quad a_1(B) \leqslant a_2(B), \quad a_1(C) = a_2(C)$$

are valid. Since $a_1 = a_1(A) + a_1(B) + a_1(C)$ and similarly for a_2, we infer that $a_1 + a_2 \leqslant a_2' + a_2'$. □

5.3. Lemma. *Assume that G is abelian and projectable. Let $G_i \in c(G) \cap \mathcal{G}_{\alpha(6)}$ $(i = 1, 2)$. Then $G_1 \vee G_2 \in \mathcal{G}_{\alpha(6)}$.*

Proof. Put $H = G_1 \vee G_2$; thus $H = G_1 + G_2$. Let $h \in H$ and let $\{f_\lambda\}$ be a disjoint subset of H such that $f_\lambda \leqslant h$ for each λ. Then $h \in H^+$. □

There exist $a_i \in G_i^+$ $(i = 1, 2)$ such that $h = a_1 + a_2$. Let a_1' and a_2' be as in 5.2. For each f_λ there exist elements $f_{\lambda 1}$ and $f_{\lambda 2}$ in G^+ such that

$$f_\lambda = f_{\lambda 1} + f_{\lambda 2}, \quad f_{\lambda 1} \leqslant a_1', \quad f_{\lambda 2} \leqslant a_2'.$$

Then $f_{\lambda 1} \perp f_{\lambda 2}$ for each λ. Next, the system $\{f_{\lambda 1}\}$ is disjoint. Since G_1 satisfies the condition $\alpha(6)$ there is $g_1 \in G_1$ such that $g_1 - f_{\lambda 1} \perp f_{\lambda 1}$ for each λ. Analogously, there is $g_2 \in G_2$ such that $g_2 - f_{\lambda 2} \perp f_{\lambda 2}$ for each λ. Let A, B and C be as in the proof of 5.2.

Denote $g_1' = g_1(A + C)$, $g_2' = g_2(B)$. Since $f_{\lambda 1} \in A + C$ and $g_1 \geqslant f_{\lambda 1}$, we obtain that $g_1(A + C) \geqslant f_{\lambda 1}(A + C) = f_{\lambda 1}$, thus $g_1' - f_{\lambda 1} \geqslant 0$. Next, since $g_1' \leqslant g_1$ we get $g_1' - f_{\lambda 1} \perp f_{\lambda 1}$ for each λ. Similarly, $g_2' - f_{\lambda 2} \perp f_{\lambda 2}$ for each λ. Moreover, $g_1' \perp g_2'$. Therefore $0 \leqslant f_{\lambda 1} + f_{\lambda 2} \leqslant g_1' + g_2'$ and the element $g = g_1' + g_2'$ satisfies the relations

$$\begin{aligned} g - f_\lambda = (g_1' + g_2') - (f_{\lambda 1} + f_{\lambda 2}) = (g_1' - f_{\lambda 1}) + (g_2' - f_{\lambda 2}) = \\ = (g_1' - f_{\lambda 1}) \vee (g_2' - f_{\lambda 2}), \end{aligned}$$

$$(g - f_\lambda) \wedge f_\lambda = ((g'_1 - f_{\lambda 1}) \vee (g'_2 - f_{\lambda 2})) \wedge (f_{\lambda 1} \vee f_{\lambda 2}) =$$
$$= ((g'_1 - f_{\lambda 1}) \wedge f_{\lambda 1}) \vee ((g'_2 - f_{\lambda 2}) \wedge f_{\lambda 2}) = 0.$$

Hence $H \in \mathcal{G}_{\lambda(6)}$.

By obvious induction we can generalize the assertion of 5.3 to the case of n convex ℓ-subgroups of G. Next by the same method as in the proof of 4.8 we conclude that the following result is valid:

5.4. Lemma. *Assume that G is abelian and projectable. Let $G_i \in c(G) \cap \mathcal{G}_{\alpha(6)}$ $(i \in I)$. Then $\bigvee\limits_{i \in I}^{c} G_i \in \mathcal{G}_{\alpha(6)}$.*

Let $\mathcal{G}_a$ and $\mathcal{G}_p$ be the class of all abelian or all projectable lattice ordered groups, respectively. It has been already remarked above that $\mathcal{G}_a$ is a radical class. Next, $\mathcal{G}_p$ is a radical class (cf. [9]). Therefore in virtue of 5.1 and 5.4 we arrive at the following result:

5.5. Theorem. *$\mathcal{G}_a \cap \mathcal{G}_p \cap \mathcal{G}_{\alpha(6)}$ is a radical class.*

Some open questions have been already proposed above. Let us add the following ones:

Are $\mathcal{G}_a \cap \mathcal{G}_{\alpha(6)}$ or $\mathcal{G}_p \cap \mathcal{G}_{\alpha(6)}$ radical classes?

Is $\mathcal{G}_a \cap \mathcal{G}_p \cap \mathcal{G}_{\alpha(6)}$ a torsion class?

References

[1] *A. Bigard, K. Keimel, S. Wolfenstein*: Groupes et anneaux réticulés, Lecture Notes in Mathematics 608. Springer Verlag, Berlin, 1977.

[2] *G. J. M. H. Buskes*: Disjoint sequences and completeness properties. Indag. Math (Proc. Netherl. Acad. Sci. A 88) *47* (1985), 11–19.

[3] *P. Conrad*: Lattice ordered groups. Tulane University, 1970.

[4] *P. Conrad*: K-radical classes of lattice ordered groups. Algebra, Proc. Conf. Carbondale (1980), Lecture Notes in Mathematics 848. 1981, pp. 186–207.

[5] *M. Darnel*: Closure operations on radicals of lattice ordered groups. Czechoslov. Math. J. *37* (1987), 51–64.

[6] *L. Fuchs*: Partially ordered algebraic systems. Oxford, 1963.

[7] *W. C. Holland*: Varieties of ℓ-groups are torsion classes. Czechoslov. Math. J. *29* (1979), 11–12.

[8] *J. Jakubík*: Radical mappings and radical classes of lattice ordered groups. Symposia Math. 31. Academic Press, New York-London, 1977, pp. 451–477.

[9] *J. Jakubík*: Projectable kernel of a lattice ordered group. Universal algebra and applications. Banach Center Publ. Vol. 9, 1982, pp. 105–112.

[10] *J. Jakubík*: Kernels of lattice ordered groups defined by properties of sequences. Časopis pěst. matem. *109* (1984), 290–298.

[11] *J. Jakubík*: On some types of kernels of a convergence ℓ-group. Czechoslov. Math. J. *39* (1989), 239–247.

[12] *J. Jakubík*: Closure operators on the lattice of radical classes of lattice ordered groups. Czechoslov. Math. J. *38* (1988), 71–77.
[13] *J. Jakubík*: On a radical class of lattice ordered groups. Czechoslov. Math. J. *39* (1989), 641–643.
[14] *W. A. J. Luxemburg, A. C. Zaanen*: Riesz spaces, Volume I. Amsterdam-London, 1971.
[15] *J. Martinez*: Torsion theory for lattice groups. Czechoslov. Math. J. *25* (1975), 284–292; *26* (1976), 93–100.
[16] *N. Ja. Medvedev*: On the lattice of radicals of a finitely generated ℓ-group. Math. Slovaca *33* (1983), 185–188. (In Russian.)
[17] *R. Sikorski*: Boolean algebras, Second edition. Berlin, 1964.

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