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Research Article

On Some Compound Random Variables Motivated by Bulk Queues

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We consider the distribution of the number of customers that arrive in an arbitrary bulk arrival queue system. Under certain conditions on the distributions of the time of arrival of an arriving group ($Y(t)$) and its size (X) with respect to the considered bulk queue, we derive a general expression for the probability mass function of the random variable $Q(t)$ which expresses the number of customers that arrive in this bulk queue during any considered period t. Notice that $Q(t)$ can be considered as a well-known compound random variable. Using this expression, without the use of generating function, we establish the expressions for probability mass function for some compound distributions $Q(t)$ concerning certain pairs $(Y(t), X)$ of discrete random variables which play an important role in application of batch arrival queues which have a wide range of applications in different forms of transportation. In particular, we consider the cases when $Y(t)$ and/or X are some of the following distributions: Poisson, shifted-Poisson, geometrical, or uniform random variable.

1. Introduction

As noticed by Feller [1], a substantial part of probability theory is connected with sums of independent random variables, and in many situations the number of terms in such sums is itself a random variable. In particular, in several situations the numbers of these sums are integer-valued random variables. In queueing theory, a discipline within the mathematical theory of probability, a bulk queue (sometimes called batch queue) is a general queueing model where customers arrive and/or are served in groups of random size. There is a large practical interest in investigating the behavior of general-arrival queueing systems (see, e.g., Dragović et al. [2], Gontijo et al. [3], Lee [4], Liu [5], Manfield and Tran-Gia [6], and Mezghiche and Tadj [7]). In Kendall notation [8] for single queueing nodes, the random variable denoting bulk arrivals or service is denoted by a superscript, for example, $M^X/M^S/1$ denotes an $M/M/1$ queue where the arrivals are in batches determined by the random variable X and the services in bulk are determined by the random variable S. Mathematically and also from practical point of view, the cases when the size of an arriving group is a random variable are more common and also more difficult to handle.

According to Kendall-Lee notation, an arbitrary bulk queueing system can be described as $Y(t)^X/S/c/K/N/D$. In this notation the first component $Y(t)^X$ means that customers arrive in system in groups following a random variable $Y(t)$ depending on the time t, and the group size X (i.e., the number of customers that arrive in the system at the same time) is a discrete random variable. The system has c servers whose service times are distributed in accordance to the random variable S. The capacity of the queue is equal to K, N is the size of the population of customers to be served, and D is the queueing discipline. In this paper we focus our attention on the study of the number of customers that arrive in an arbitrary previously described bulk queue under certain conditions on the distributions of random variables $Y(t)$ and X. Notice that this number is closely related to the notion of compound discrete distribution (see, e.g., Charalambides [9, Section 7.3], Feller [1, Chapter XII], Minkova [10], and Pek5z and Ross [11]) given as follows.

Let $X_1, X_2, \ldots$, be a sequence of mutually independent and identically distributed positive random variables that are independent of the nonnegative integer-valued random variable Y. Then the random variable S_Y defined as a sum

$$S_Y = X_1 + X_2 + \cdots + X_Y \tag{1}$$

is called a *compound random variable* (or a *compound distribution*). In this respect, the distribution of Y is a *compounded random variable* (or a *compounded distribution*), while the random variable X is called a *compounding random variable* (or a *compounding distribution*). If $f(s) = \sum f_i s^i$ is a generating function of X_i $(i = 1, 2, \ldots)$ and $g(s) = \sum g_i s^i$ is a generating function of Y, then it is well known (see, e.g., Feller [1, Chapter XII, Theorem on page 287]) that the generating function of the random variable S_Y given by (1) is the compound function $g(f(s))$. This fact was used in [5, Chapter XII] for determining the probability function for some compound discrete random variables. The term compound random variable (distribution) is used by Feller [1] in his classic book on discrete probability and subsequently by several other authors. These random variables are also known as *generalized random variables* (*distributions*), a term used by Feller [12] and Gurland [13] and others. In these two papers instead of mixture and compound distributions the terms compound and generalized distributions, respectively, were used (for the notion of the mixture distribution see [9, Section 7.3]).

For more information on the notion of *compound random variable* followed by several examples see Charalambides [9, Chapter 7]. Namely, in [9, Chapter 7], under the random occupancy model, with a random number of urns and the number of balls distributed into any specific urn obeying a discrete probability law, the compound discrete variables (distributions) of the total number of balls distributed into the urns are derived. Furthermore, several particular compound discrete distributions are examined in [9, Chapter 7]. Notice that in order to determine the probability function for many compound discrete random variables it was applied in [9, Sections 7.4–7.8] a technique involving the binomial moment generating function of related compound distribution. Furthermore, Peköz and Ross [11] considered the cases when the compounded distribution Y is Poisson, binomial, negative binomial random, hypergeometric, logarithmic, or negative hypergeometric random variable. Namely, in [11] were established the recursive formulas for the probability mass function of compound random variables involving any of the mentioned random variable. Furthermore, the notion of compound distributions in more general setting and several related examples are established by Minkova in [10].

The rest of the paper is organized as follows. In Section 2, we focus our attention on the study of the number of customers, $Q(t)$, that arrive during considered period t at the system modelled by a general $Y(t)^X/S/c/K/N/D$ bulk queue. Without the use of generating function, we derive the expression (2) of Theorem 2 for probability mass function of the compound random variable $Q(t)$. In view of the fact that $Q(t)$ may be considered as a compound random variable, in this setting is reformulated Theorem 2 by Theorem 4. Furthermore, in Section 2 are given some combinatorial notions (Definitions 5 and 6) and a related result (Lemma 7) which are used in the next section. Using Theorem 2 and Lemma 7, in Section 3, we derive the expressions concerning the following pairs $(Y(t), X)$: (1) $Y(t)$ is a Poisson distribution and X is a geometric distribution; (2) $Y(t)$ is a Poisson distribution and X is a shifted-Poisson distribution; (3) $Y(t)$ is a geometric distribution and X is also a geometric distribution; (4) $Y(t)$ is a uniform distribution and X is a shifted-Poisson distribution; and (5) $Y(t)$ is a uniform distribution and X is a geometric distribution. Concluding remarks are given in Section 4.

2. The Main Result and Auxiliary Results

2.1. The Main Result. Let us take into consideration a bulk queue $Y(t)^X/S/c/K/N/D$ described in Introduction. Namely, customers arrive in this queue in groups following a random variable $Y(t)$ depending on the time t. Assume that, for any fixed t, the probability mass function of $Y(t)$ is distributed as $b_k(t) = P(Y(t) = k)$ with $k = 0, 1, 2, \ldots$, and its mean is $E(Y(t)) = \overline{b(t)}$. Furthermore, the group size X (i.e., the number of customers that arrive in the system at the same time) is a discrete random variable whose distribution is given by $a_k = P(X = k)$ with $k = 1, 2, \ldots$, where $k \geq 1$ is a number of customers in a group, and its mean is $E(X) = \overline{a}$. We also suppose that the random variables X and $Y(t)$ are mutually independent for any fixed $t > 0$.

The notions and related notations, which will be used in the sequel, are given by the following definition.

Definition 1. The number of customers that arrive in the system modelled by a $Y(t)^X/S/c/K/N/D$ bulk queue during considered period t is a discrete random variable $Q(t)$ whose distribution is given by $q_k(t) = P(Q(t) = k)$, $k = 0, 1, 2, \ldots$, and whose mean is $E(Q(t))$.

The following result gives the expressions for the distributional values $q_k(t)$ of $Q(t)$ independent of the values $b_k(t)$ and a_i $(k = 0, 1, 2, \ldots, i = 1, 2, \ldots)$.

Theorem 2. *Under the notations of Definition 1, for any fixed $t > 0$, the following formula for the probability mass function of the random variable $Q(t)$ holds:*

$$q_k(t) = \sum_{l=1}^{k} b_l(t) \sum_{i_1+i_2+\cdots+i_l=k} a_{i_1} a_{i_2} \cdots a_{i_l}, \quad k = 1, 2, \ldots, \tag{2}$$

where the summation ranges over all l-tuples $(i_1, i_2, \ldots, i_l)$ of positive integers satisfying the condition $i_1 + i_2 + \cdots + i_l = k$. Furthermore, the following holds:

$$q_0(t) = b_0(t). \tag{3}$$

Proof. Let $t > 0$ be any fixed positive real number. Let A_k $(k = 1, 2, \ldots)$ denote the event that exactly k customers arrive in the system during a time t. Furthermore, let $B_l(t) = B_l$ $(l = 0, 1, 2, \ldots)$ denote the event that exactly l groups of customers arrive in the system during a time t; that is,

$$P(B_l) = b_l(t). \tag{4}$$

Then the conditional probability $P(A_k \mid B_l)$ is obviously equal to the sum of products

$$P(A_k \mid B_l) = \sum_{a_{i_1}+a_{i_2}+\cdots+a_{i_l}=k} a_{i_1} a_{i_2} \cdots a_{i_l}, \tag{5}$$

where the summation ranges over all l-tuples $(a_{i_1}, a_{i_2}, \ldots, a_{i_l})$ of positive integers satisfying the condition $a_{i_1}+a_{i_2}+\cdots+a_{i_l}=k$. Then using (4), (5), and the assumption that the random variables X and $Y(t)$ are mutually independent, applying the formula for the total probability, we find that for each $k \geq 1$ there holds

$$\begin{aligned} P(A_k) &= \sum_{l=1}^{k} P(B_l) P(A_k \mid B_l) \\ &= \sum_{l=1}^{k} b_l(t) \sum_{a_{i_1}+a_{i_2}+\cdots+a_{i_l}=k} a_{i_1} a_{i_2} \cdots a_{i_l}, \end{aligned} \tag{6}$$

which proves (2). Finally, (3) trivially holds in view of the fact that no customer arrives in the system if and only if no group arrives in the system. □

Remark 3. Notice that in accordance to the notion of a compound random variable given by (1) and assuming that each term of a sequence $X_1, X_2, \ldots$, of mutually independent and identically distributed positive random variables coincides with the random variable X from Definition 1 (with $a_k = P(X = k)$ being the number of customers that arrive in the queue $Y(t)^X/S/c/K/N/D$ at the same time), then for any $t > 0$ the random variable $Q(t)$ from Definition 1 may be written as a sum

$$Q(t) = X_1 + X_2 + \cdots + X_{Y(t)}. \tag{7}$$

This means that $Q(t)$ may be considered as a compound random variable $S_{Y(t)}$ defined by (2) and (3). Furthermore, under these notations and notations of Theorem 2, there hold $B_l = \{Y(t) = l\}$, $P(B_l) = b_l(t) = P(Y(t) = l)$ for all $l = 0, 1, 2, \ldots$, and $A_k = \{S_{Y(t)} = k\}$ for all $k = 1, 2, \ldots$. Accordingly, the conditional probability $P(A_k \mid B_l)$ is equal to $P(S_{Y(t)} = k \mid Y(t) = l)$, and, therefore, Theorem 2 can be reformulated in the following form.

Theorem 4. *Let $X_1, X_2, \ldots$, be a sequence of mutually independent and identically distributed positive integer-valued random variables that are independent of the nonnegative integer-valued random variable Y. Assume that for any $i = 1, 2, \ldots$, $a_k = P(X_i = k)$ with $k = 1, 2, \ldots$, and $b_l = P(Y = l)$ with $l = 0, 1, 2, \ldots$, then the probability mass function of compound random variable*

$$S_Y = X_1 + X_2 + \cdots + X_Y \tag{8}$$

is given by the following double sum:

$$P(S_Y = k) = \sum_{l=1}^{k} b_l \sum_{i_1+i_2+\cdots+i_l=k} a_{i_1} a_{i_2} \cdots a_{i_l}, \quad k = 1, 2, \ldots, \tag{9}$$

where the summation ranges over all l-tuples $(i_1, i_2, \ldots, i_l)$ of positive integers satisfying the condition $i_1 + i_2 + \cdots + i_l = k$. Furthermore, the following holds:

$$P(S_Y = 0) = b_0. \tag{10}$$

2.2. Auxiliary Results. The formulae (2) and (3) (i.e., the formulae (9) and (10)) are suitable for deriving related expressions for the probability mass function of a random variable $Q(t)$ (i.e., of a compound random variable S_Y) concerning several pairs of important discrete random variables $(Y(t), X)$. Related examples are presented in the next section. Moreover, for a simplification of some formulae concerning some pairs of random variables considered in our examples, it is necessary to use some results involving the notion of a composition in combinatorics.

Definition 5 (see, e.g., [14, Section 4.2, pp. 54-55]). A *composition* of a positive integer k is any l-tuple $(a_1, a_2, \ldots, a_l)$ of positive integers $(1 \leq l \leq k)$ such that

$$a_1 + a_2 + \cdots + a_l = k. \tag{11}$$

Furthermore, l is said to be the *number of parts* or *length* of the above composition. A composition with l parts is said to be a *l-composition*.

Definition 6 (see, e.g., [14, Section 4.2, pp. 54-55]). A *composition* of a positive integer k with l parts is any l-tuple $(a_1, a_2, \ldots, a_l)$ of positive integers such that

$$a_1 + a_2 + \cdots + a_l = k. \tag{12}$$

Lemma 7 (see, e.g., [14, Section 4.2, pp. 54-55]). *For every positive integers k and l the number of l-compositions of a positive integer k with l parts, $C_l(k)$, is equal to*

$$C_l(k) = \binom{k-1}{l-1}. \tag{13}$$

3. Applications of Theorem 2

In this section we will apply Theorem 2 to different pairs of discrete random variables $(Y(t), X)$. We focus our attention on the random variables that are involved in several queueing systems which have numerous applications in transportation. In particular, this concerns the batch Poisson arrivals processes.

3.1. The Case When $Y(t)$ Is a Poisson Distribution and X Is a Geometric Distribution. Suppose that $Y(t)$ is the *Poisson distribution* with the associated parameter λt; that is,

$$P(Y(t) = l) = b_l(t) = e^{-\lambda t} \frac{(\lambda t)^l}{l!}, \quad l = 0, 1, 2, \ldots; \ \lambda > 0, \tag{14}$$

with the mean λt and the variance λt, and let X be the *geometric distribution* with the parameter a; that is,

$$P(X = i) = (1 - a) a^{i-1}, \quad i = 1, 2, \ldots, \tag{15}$$

whose mean is $1/(1 - a)$ and variance is $a/(1 - a)^2$.

In order to find the values of a distribution $Q(t)$ concerning the pair $(Y(t), X)$ given by Definition 1, the product of the second sum on the right-hand side of equality (2) of Theorem 2 with $a_i = a^{i-1}(1-a)$ is equal to

$$\begin{aligned} a_{i_1} a_{i_2} \cdots a_{i_l} &= a^{i_1 - 1}(1-a)\, a^{i_2 - 1}(1-a) \cdots a^{i_l - 1}(1-a) \\ &= a^{i_1 + i_2 + \cdots + i_l - l}(1-a)^l = a^{k-l}(1-a)^l. \end{aligned} \tag{16}$$

Then, substituting equality (16) into (2), we get

$$\begin{aligned} P(Q(t) = k) &= q_k(t) \\ &= \sum_{l=1}^{k} b_l(t)\, a^{k-l}(1-a)^l \sum_{i_1 + i_2 + \cdots + i_l = k} 1 \\ &\qquad \text{for each } k = 1, 2, \ldots, \end{aligned} \tag{17}$$

where the summation ranges over all l-tuples $(i_1, i_2, \ldots, i_l)$ of positive integers satisfying the condition $i_1 + i_2 + \cdots + i_l = k$. Observing that $\sum_{i_1+i_2+\cdots+i_l=k} 1$ is in fact equal to the number of compositions of the integer k with l parts, which is by (13) of Lemma 7 equal to $\binom{k-1}{l-1}$, (17) becomes

$$\begin{aligned} P(Q(t) = k) = q_k(t) = \sum_{l=1}^{k} b_l(t) \binom{k-1}{l-1} a^{k-l}(1-a)^l \\ \text{for each } k = 1, 2, \ldots. \end{aligned} \tag{18}$$

Finally, substituting (14) into (18), we find that

$$\begin{aligned} P(Q(t) = k) &= q_k(t) \\ &= e^{-\lambda t} \sum_{l=1}^{k} \frac{(\lambda t)^l}{l!} \binom{k-1}{l-1} a^{k-l}(1-a)^l \\ &\qquad \text{for each } k = 1, 2, \ldots. \end{aligned} \tag{19}$$

Furthermore, inserting (14) with $l = 0$ into (3), we have

$$P(Q(t) = 0) = q_0(t) = e^{-\lambda t}. \tag{20}$$

Remark 8. Notice that the distribution $Q(t)$ defined by (19) and (20) is in fact a compound Poisson distribution with respect to the geometric distribution (with the mean $\lambda t/(1-a)$ and the variance $\lambda t(1+a)/(1-a)^2$), and it is sometimes called the *Polya-Aeppli distribution* (see [9, Subsection 7.5.3]; also see [10]). Minkova [10, Remark 1] noticed that the Polya-Aeppli distribution coincides with the inflated-parameter Poisson distribution (see Johnson et al. [15, Section 2]). In [16, Section 2] Haydn and Vaienti proved a very general theorem that can be used to establish the distribution in many other settings. Moreover, in [16], a result is proved on the approximation of the compound Poisson distribution. For more general compound Poisson distributions see Feller's book [1, Chapter XII].

3.2. The Case When $Y(t)$ Is a Poisson Distribution and X Is a Shifted-Poisson Distribution. Let $Y(t)$ be the Poisson distribution given by (14) with the mean λt and the variance λt.

Let X be the *shifted-Poisson distribution* with the parameter a; that is,

$$P(X = i) = a_i = e^{-a} \frac{a^{i-1}}{(i-1)!}, \quad i = 1, 2, \ldots;\ a > 0, \tag{21}$$

with the mean $a + 1$ and the variance a.

In order to find the values of related distribution $Q(t)$ given by Definition 1, the product which appears in the second sum on the right-hand side of equality (2) of Theorem 2 with $a_i = e^{-a} a^{i-1}/(i-1)!$ is equal to

$$\begin{aligned} a_{i_1} a_{i_2} \cdots a_{i_l} &= e^{-la} \frac{a^{i_1 - 1} a^{i_2 - 1} \cdots a^{i_l - 1}}{(i_1 - 1)!(i_2 - 1)! \cdots (i_l - 1)!} \\ &= e^{-la} \frac{a^{k-l}}{(i_1 - 1)!(i_2 - 1)! \cdots (i_l - 1)!}. \end{aligned} \tag{22}$$

Observe that by the multinomial formula, we have

$$\begin{aligned} \left(\sum_{i=1}^{l} 1\right)^{k-l} &= \underbrace{(1 + 1 + \cdots + 1)}_{l}{}^{k-l} \\ &= \sum_{j_1 + j_2 + \cdots + j_l = k-l} \frac{(k-l)!}{j_1! j_2! \cdots j_l!}, \end{aligned} \tag{23}$$

where the summation ranges over all l-tuples $(j_1, j_2, \ldots, j_l)$ of nonnegative integers satisfying the condition $j_1 + j_2 + \cdots + j_l = k - l$. The identity (23) immediately yields

$$l^{k-l} = \sum_{i_1 + i_2 + \cdots + i_l = k} \frac{(k-l)!}{(i_1 - 1)!(i_2 - 1)! \cdots (i_l - 1)!}, \tag{24}$$

whence it follows that

$$\sum_{i_1 + i_2 + \cdots + i_l = k} \frac{1}{(i_1 - 1)!(i_2 - 1)! \cdots (i_l - 1)!} = \frac{l^{k-l}}{(k-l)!}, \tag{25}$$

where the summation ranges over all l-tuples $(i_1, i_2, \ldots, i_l)$ of positive integers satisfying the condition $i_1 + i_2 + \cdots + i_l = k$.

Then, substituting equalities (25) and (22) into (2) of Theorem 2, we immediately obtain

$$q_k(t) = \sum_{l=1}^{k} b_l(t) \frac{(al)^{k-l}}{(k-l)!} e^{-la} \quad \text{for each } k = 1, 2, \ldots. \tag{26}$$

Finally, substituting (14) into (26), we get

$$P(Q(t) = k) = q_k(t) = \sum_{l=1}^{k} e^{-\lambda t} \frac{(\lambda t)^l}{l!} \cdot \frac{(al)^{k-l}}{(k-l)!} e^{-la}, \tag{27}$$

whence it follows that

$$\begin{aligned} P(Q(t) = k) = q_k(t) = \sum_{l=1}^{k} e^{-(\lambda t + la)} \frac{(\lambda t)^l (al)^{k-l}}{l!(k-l)!} \\ \text{for each } k = 1, 2, \ldots. \end{aligned} \tag{28}$$

Furthermore, inserting (14) with $l = 0$ into (3), we have

$$P(Q(t) = 0) = q_0(t) = e^{-\lambda t}. \tag{29}$$

If $a = \lambda t$, or equivalently, at the time $t = a/\lambda$, then equality (28) clearly becomes

$$P(Q(t) = k) = q_k(t) = e^{-a}\frac{a^k}{k!}\sum_{l=1}^{k} e^{-la} l^{k-l}\frac{k!}{l!(k-l)!}, \tag{30}$$

which by the identity $\binom{k}{l} = k!/l!(k-l)!$ can be written as

$$P(Q(t) = k) = q_k = e^{-a}\frac{a^k}{k!}\sum_{l=1}^{k}\binom{k}{l} e^{-la} l^{k-l} \quad \text{for each } k = 1, 2, \ldots. \tag{31}$$

Notice that the factor $e^{-a}a^k/k!$ preceding the sum on the right-hand side of (31) is in fact the $(k+1)$th probability of considered shifted-Poisson distribution X given by (21). Recall that the distribution $Q(t)$ was discovered by Thomas [17], and hence it is often called *Thomas distribution*.

3.3. The Case When Both $Y(t)$ and X Are Geometric Distributions. Let X be the geometric distribution with the parameter a defined by (15), and let $Y(t)$ be also the geometric distribution with the parameter $b = b(t)$ depending on t; that is,

$$b_i(t) = P(Y(t) = i) = (1-b)b^{i-1}, \quad i = 1, 2, \ldots. \tag{32}$$

Then substituting (32) into (18) and using the binomial formula, for each $k = 1, 2, \ldots$, we get

$$\begin{aligned} P(Q(t) = k) &= q_k(t) \\ &= \sum_{l=1}^{k}\binom{k-1}{l-1} a^{k-l}(1-a)^l(1-b)b^{l-1} \\ &= a^{k-1}(1-a)(1-b)\sum_{l=1}^{k}\binom{k-1}{l-1}\left(\frac{(1-a)b}{a}\right)^{l-1} \\ &= a^{k-1}(1-a)(1-b)\left(1+\frac{(1-a)b}{a}\right)^{k-1}. \end{aligned} \tag{33}$$

3.4. The Case When $Y(t)$ Is a Uniform Distribution and X Is a Shifted-Poisson Distribution. Let $Y(t)$ be the uniform distribution whose probability mass function is given by

$$b_l = P(Y(t) = l) = \frac{1}{s(t)} \quad \text{for } l = 0, 1, \ldots, s(t)-1, \tag{34}$$

where $s(t)$ is a positive integer depending on t. Assume that X is the shifted-Poisson distribution with the parameter a defined by (21). Then substituting (34) into (26), we obtain

$$P(Q(t) = k) = q_k(t) = \frac{1}{s(t)}\sum_{l=1}^{\min\{s(t)-1,k\}}\frac{(al)^{k-l}}{(k-l)!}e^{-la} \quad \text{for each } k = 1, 2, \ldots. \tag{35}$$

Furthermore, inserting (34) with $l = 0$ into (3), we have

$$q_0(t) = \frac{1}{s(t)}. \tag{36}$$

3.5. The Case When $Y(t)$ Is a Uniform Distribution and X Is a Geometric Distribution. Let $Y(t)$ be the uniform distribution whose probability mass function is given by (34), and let X be the geometric distribution defined by (15). Then substituting (34) into (18), we obtain

$$\begin{aligned} P(Q(t) = k) &= q_k(t) \\ &= \frac{1}{s(t)}\sum_{l=1}^{\min\{s(t)-1,k\}}\binom{k-1}{l-1} a^{k-l}(1-a)^l \quad \text{for each } k = 1, 2, \ldots. \end{aligned} \tag{37}$$

By the well-known identity we have

$$\sum_{l=1}^{k} l\binom{k}{l} a^{k-l}(1-a)^l = k(1-a). \tag{38}$$

The above identity is, for example, equivalent to the fact that the mean of the binomial distribution with parameters n and $1-a$ is equal to $n(1-a)$. Then using the identities $\binom{k-1}{l-1} = (l/k)\binom{k}{l}$, (38), and the binomial formula, we obtain

$$\begin{aligned} \sum_{l=1}^{k}\binom{k-1}{l-1} a^{k-l}(1-a)^l &= \frac{1}{k}\sum_{l=1}^{k} l\binom{k}{l} a^{k-l}(1-a)^l \\ &= \frac{k(1-a)}{k} = 1-a. \end{aligned} \tag{39}$$

Substituting (39) into (37) under the condition that $k \le s(t)-1$, we get

$$P(Q(t) = k) = q_k(t) = \frac{1-a}{s(t)} \quad \text{for each } k = 1, 2, \ldots, s(t)-1. \tag{40}$$

4. Conclusion

Motivated by the notion of bulk queue, in this paper we focus our attention on the distribution of the number of customers that arrive in an arbitrary bulk arrival queue system with some conditions on the distributions of the time of arriving group $(Y(t))$ and its size (X). For such a bulk queue model, we derive a general expression for the probability mass function of the random variable $Q(t)$ which expresses the number of customers that arrive in this bulk queue during any considered period of time t. Using this expression and some auxiliary combinatorial results, without the use of generating function, we derive the related expressions concerning some pairs $(X, Y(t))$ of discrete random variables which have a wide range of applications in transportation, computer networks, telecommunications, and so forth. We believe that this expression can be used for the same purposes with respect to some other pairs $(X, Y(t))$ of discrete random variables. Since there are no expressions in closed form for the basic performance measures related to many investigated types of bulk queues, our future research could be directed to estimating some of these performance measures. In particular, we hope that the obtained results in this paper should be applied for finding efficient simulation techniques to estimate significant performance characteristics of some bulk arrival queueing systems.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. 1, John Wiley & Sons, New York, NY, USA, 3rd edition, 1968.

[2] B. Dragović, N.-K. Park, N. Đ. Zrnić, and R. Meštrović, "Mathematical models of multiserver queuing system for dynamic performance evaluation in port," *Mathematical Problems in Engineering*, vol. 20112, Article ID 710834, 19 pages, 2012.

[3] G. M. Gontijo, G. S. Atuncar, F. R. B. Cruz, and L. Kerbache, "Performance evaluation and dimensioning of GI^X/M/c/N systems through kernel estimation," *Mathematical Problems in Engineering*, vol. 2011, Article ID 348262, 20 pages, 2011.

[4] Y. Lee, "Discrete-Time bulk queueing system with variable service capacity depending on previous service time," *Mathematical Problems in Engineering*, vol. 2015, Article ID 482179, 6 pages, 2015.

[5] R. Liu, "Probabilistic decomposition method on the server indices of an $M^{\xi}/G/1$ vacation queue," *Journal of Applied Mathematics*, vol. 2014, Article ID 241636, 9 pages, 2014.

[6] D. R. Manfield and P. Tran-Gia, "Analysis of a finite storage system with batch input arising out of message packetization," *IEEE Transactions on Communications*, vol. 30, no. 3, pp. 456–463, 1982.

[7] A. Mezghiche and L. Tadj, "Note on a binomial schedule for an M^X/G/1 queueing system with an unreliable server," *ISRN Probability and Statistics*, vol. 2013, Article ID 508045, 6 pages, 2013.

[8] D. G. Kendall, "Stochastic processes occurring in the theory of queues and their analysis by the method of the imbedded Markov chain," *The Annals of Mathematical Statistics*, vol. 24, no. 3, pp. 338–354, 1953.

[9] C. A. Charalambides, *Combinatorial Methods in Discrete Distributions*, John Wiley & Sons, Hoboken, NJ, USA, 2005.

[10] L. D. Minkova, "The Pólya-Aeppli process and ruin problems," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 2004, no. 3, pp. 221–234, 2004.

[11] E. Peköz and S. M. Ross, "Compound random variables," *Probability in the Engineering and Informational Sciences*, vol. 18, no. 4, pp. 473–484, 2004.

[12] W. Feller, "On a general class of 'contagious' distributions," *The Annals of Mathematical Statistics*, vol. 14, no. 4, pp. 389–400, 1943.

[13] J. Gurland, "A generalized class of contagious distributions," *Biometrics*, vol. 14, no. 2, pp. 229–249, 1957.

[14] G. E. Andrews, *The Theory of Partitions, Encyclopedia of Mathematics and Its Applications*, Addison-Wesley, Reading, Mass, USA, 1976.

[15] N. L. Johnson, S. Kotz, and A. W. Kemp, *Univariate Discrete Distributions*, Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics, John Wiley & Sons, New York, NY, USA, 3rd edition, 2005.

[16] N. Haydn and S. Vaienti, "The compound Poisson distribution and return times in dynamical systems," *Probability Theory and Related Fields*, vol. 144, no. 3-4, pp. 517–542, 2009.

[17] M. Thomas, "A generalization of Poisson's binomial limit for use in ecology," *Biometrika*, vol. 36, no. 1-2, pp. 18–25, 1949.