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ON SOME CONVERGENCE PROPERTIES OF THE INTERPOLATION POLYNOMIALS

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It is well known that there exist continuous functions whose Lagrange interpolation polynomials taken at the roots of the Tchebycheff polynomials $T_n(x)$ diverge everywhere in $(-1, +1)$.[1] On the other hand a few years ago S. Bernstein proved the following result[2]: Let $f(x)$ be any continuous function; then to every $c > 0$ there exists a sequence of polynomials $\varphi_n(x)$ where $\varphi_n(x)$ is of degree $n - 1$ and it coincides with $f(x)$ at, at least $n - cn$ roots of $T_n(x)$ and $\varphi_n(x) \to f(x)$ uniformly in $(-1, +1)$.

Fejér proved the following theorem[3]: Let the fundamental points of the interpolation be a normal[4] point group

$$\begin{matrix} x_1^{(1)} \\ x_1^{(2)}, \quad x_2^{(2)} \\ \cdots\cdots\cdots\cdots; \end{matrix}$$

then for every continuous $f(x)$ there exists a sequence of polynomials $\varphi_n(x)$ of degree $\leqq 2n - 1$ such that $\varphi_n(x_i^{(n)}) = f(x_i^{(n)})$, $i = 1, 2, \cdots n$ and $\varphi_n(x) \to f(x)$ uniformly in $(-1, +1)$. In the present paper we are going to prove the following more general

THEOREM 1. *Let the point group be such that the fundamental functions $l_k^{(n)}(x)$ are uniformly bounded in $(-1, +1)$. Then to every continuous function $f(x)$ and $c > 0$ there exists a sequence of polynomials $\varphi_n(x)$, such that, 1) the degree of $\varphi_n(x)$ is $\leqq n(1 + c)$, 2) $\varphi_n(x_i^{(n)}) = f(x_i^{(n)})$, $i = 1, 2 \cdots n$, 3) $\varphi_n(x) \to f(x)$ uniformly in $(-1, +1)$.*

Theorem 1 generalizes the result of Fejér in two directions; first the point group is more general since it can be shown[5] that the fundamental functions are uniformly bounded for normal point groups, and secondly the degree of $\varphi_n(x)$ is lowered from $2n - 1$ to $n(1 + c)$.

Theorem 1 does not directly generalize the result of S. Bernstein, but we can prove the following

THEOREM 2. *Let the $x_i^{(n)}$ be such that the fundamental functions are uniformly bounded in $(-1, +1)$; then to every continuous function $f(x)$ there exists a sequence of polynomials $\varphi_n(x)$ of degree $\leqq n - 1$ which coincides with $f(x)$ at, at least n-cn points $x_i^{(n)}$ and $\varphi_n(x) \to f(x)$.*

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[1] G. Grünwald, Annals of Math. Vol. 37, (1936), p. 908–918.

[2] S. Bernstein, Comptes Rendus de l'Acad. des Sciences Vol.

[3] L. Fejér, Amer. Math. Monthly Vol. 41 p. 12.

[4] Ibid.

[5] Fejér proves this only for the so called strongly normal point groups (ibid). The proof for normal point groups is much more complicated and we do not give it here.

We are not going to give a proof of Theorem 2.

The following problem is due to Fejér: Let the $x_i^{(n)}$ be the equidistant abscissae that is $x_i^{(n)} = -1 + \frac{2i-1}{n}$, $i = 1, 2, \cdots n$. The question is, does there exist to every continuous $f(x)$ a sequence of polynomials $\varphi_n(x)$ of degree $< 2n$ such that $\varphi_n(x_i^{(n)}) = f(x_i^{(n)})$ and $\varphi_n(x) \to f(x)$. In other words, does his result proved for the normal point groups also hold for the equidistant point group. We prove the following

THEOREM 3. *To every continuous function $f(x)$ and to every c there exists a sequence of polynomials $\varphi_n(x)$ of degree $\leqq \frac{\pi}{2} n(1+c)$ such that $\varphi_n(x_i^{(n)}) = f(x_i^{(n)})$ and $\varphi_n(x) \to f(x)$ uniformly in $(-1, 1)$, and it can be shown that the constant $\frac{\pi}{2}$ is the best possible.*

Throughout this paper the c's denote absolute constants not necessarily the same. If there is no danger of confusion we will omit the upper index n in $x_i^{(n)}$, $l_k^{(n)}(x)$ etc.

To prove Theorem 1 we need two lemmas.

LEMMA 1. *Let the point group be such that the fundamental functions are uniformly bounded in $(-1, 1)$ and put $\cos \vartheta_i = x_i$, $x_1 < x_2 < \cdots < x_n$, $\vartheta_1 > \vartheta_2 > \cdots > \vartheta_n$; then*

$$\vartheta_i - \vartheta_{i+1} > \frac{c}{n}.^{6}$$

PROOF. Let $| l_i(x) | < D$, $i = 1, 2, \cdots n$. By a well known theorem of S. Bernstein[7] $| d/d\vartheta\, l_i(\cos \vartheta) | \leqq nD$ and since $l_i(x_i) = 1$, $l_i(x_{i\pm 1}) = 0$, we have finally

$$\vartheta_i - \vartheta_{i+1} \geqq \frac{1}{Dn}.$$

LEMMA 2. *Let $-1 \leqq y \leqq 1$, $\cos \theta = y$. Then there exists a polynomial $h_y^{(m)}(x)$ of degree $\leqq 2m$ such that $h_y^{(m)}(y) = 1$, $| h_y(x) | \leqq c$, $-1 \leqq x \leqq 1$ and for $\theta - \theta_0 > \frac{A}{m}$*

$$| h_y^{(m)}(\cos \theta_0) | < c_1 \min\left(1, \frac{1}{m^2(\theta - \theta_0)^2}\right).$$

Denote by $X_i^{(m)}$ and X_{i+1}^{m} the roots of $T_m(x)$ for which $X_i^{(m)} \leqq y \leqq X_{i+1}^{(m)}$. It is easy to see that

[6] If the fundamental functions are uniformly bounded we have

$$\frac{c_1}{n} < \vartheta_i - \vartheta_{i+1} < \frac{c_2}{n}.$$

But the upper estimate is not needed here. (Erdös-Turán, Annals of Math. Vol. 39 (1940) p. 706–707.)

[7] S. Bernstein, Belg. Mém. 1912 p. 19.

$$L_i^{(m)}(y) + L_{i+1}^{(m)}(y) \geqq 1,$$ [8]

where $L_i^{(m)}(y)$ denotes the fundamental polynomials belonging to the roots of $T_m(x)$. Without loss of generality we may assume $L_i(y) \geqq \frac{1}{2}$. It is well known that $| L_i^{(m)}(x) | \leqq \sqrt{2}, -1 \leqq x \leqq 1$.[9] Thus since $\theta_i - \theta_{i+1} = \frac{\pi}{m}$ ($\cos \theta_i = X_i$) our lemma will be proved if we can show that for $| \theta_i - \theta_0 | > \frac{A}{m}$

$$| h_y^{(m)}(\cos \theta_0) | = | L_i^{(m)}(x_0)/L_i^{(m)}(y) |^2 < \frac{c}{A^2}.$$

But

$$| h_y^{(m)}(\cos \theta_0) | \leqq \frac{4}{m^2(\theta_i - \theta_0)^2} < \frac{c}{A^2}.$$

Proof of Theorem 1. Let $\psi_{n-1}(x)$ be a polynomial of degree n-1 such that

$$| f(x) - \psi_{n-1}(x) | < \epsilon, \ -1 \leqq x \leqq 1.$$

Put $f(x_i) - \psi_{n-1}(x_i) = \epsilon_i$. Consider the polynomial of degree $\leqq n(1 + c)$ such that

$$\varphi_{n-1}(x) = \psi_{n-1}(x) + \sum_{i=1}^{n} \epsilon_i l_i(x) h_{x_i}^{(m)}(x), \qquad m = \left[\frac{cn}{2}\right].$$

Clearly $\varphi_{n-1}(x_i) = f(x_i)$, $i = 1, 2 \cdots n$. We shall prove that $\varphi_{n-1}(x) \to f(x)$ uniformly in $(-1, 1)$. It suffices to show that

$$| g(x) | = \left| \sum_{i=1}^{n} \epsilon_i l_i(x) h_{x_i}^{(m)}(x) \right| < c\epsilon, \qquad -1 \leqq x \leqq 1.$$

Now

$$| g(x) | < c\epsilon \sum_{i=1}^{n} | h_{x_i}^{(m)}(x) | = c\epsilon \sum_{x_i \geqq x} | h_{x_i}^{(m)}(x) | + c\epsilon \sum_{x_i \leqq x} | h_{x_i}^{(m)}(x) | = c\epsilon \left(\sum\nolimits_1 + \sum\nolimits_2\right)$$

Thus we only have to show that $\sum_1 + \sum_2 < c_1$. By Lemma 1,

$$\sum\nolimits_1 < \sum_r | h_{x+k_r}^{m}(x) |,$$

where $| \text{arc} \cos(x + k_r) - \text{arc} \cos x | > (rc)/n$. Thus by Lemma 2

$$\sum\nolimits_1 < \sum_r \frac{c_3}{r^2c^2} < c_4.$$

Similarly we obtain $\sum_2 < c_2$, which completes the proof of Theorem 1

[8] Erdös-Turán, Annals of Math. Vol. 41, (1941) p. 529, Lemma IV.

[9] L. Fejér, Mathematische Annalen, Vol. 106, (1932) p. 5.

Theorem 1 does not give a necessary and sufficient condition for the existence of a sequence of polynomials $\varphi_n(x)$ of degree $\leqq n(1 + c)$ with $\varphi_n(x_i) = f(x_i)$ and $\varphi_n(x) \to f(x)$ uniformly in $(-1, 1)$. To obtain such a condition let $x_i^{(n)}$ be a point group, put $\cos \vartheta_i^{(n)} = x_i^{(n)}$ and denote by $N_n(a, b)$ the number of the ϑ_i in (a, b). We have the following:

THEOREM 4. *A necessary and sufficient condition that to every continuous function $f(x)$ and to every $c > 0$ there exists a sequence of polynomials $\varphi_n(x)$ of degree $\leqq n(1 + c)$ such that $\varphi_n(x_i) = f(x_i)$ and $\varphi_n(x) \to f(x)$ uniformly in $(-1, 1)$ is that if $n(b_n - a_n) \to \infty$, $0 \leqq a_n < b_n \leqq \pi$*

$$\text{(1)}\quad \limsup \frac{N_n(a_n, b_n)}{n(b_n - a_n)} \leqslant \frac{1}{\pi} \text{ and } \liminf (\vartheta_i - \vartheta_{i+1})n > 0, (n \to \infty\ i \text{ arbitrary})$$

Condition (1) *states that the number of ϑ_i in (a_n, b_n) can not be much greater than the number of roots of $T_n(x)$ in (a_n, b_n). If the fundamental functions $l_k(x)$ are uniformly bounded* (1) *is satisfied, for then we have*

$$\lim \frac{N_n(a_n, b_n)}{n(b_n - a_n)} = \frac{1}{\pi}, \qquad n(b_n - a_n) \to \infty$$ [10]

We do not give the proof of Theorem 4, but the following proof of Theorem 3 can by a simple modification be applied to it.

PROOF of Theorem 3. Here the fundamental points are

$$x_i^{(n)} = -1 + \frac{2i - 1}{n}$$

First we prove the existence for every n and $c > 0$ of $m = \frac{\pi}{2} n(1 + c)$ points. $y_i^{(m)}$, $i = 1, 2, \cdots n$ such that (I) the $x_i^{(n)}$ occur among the $y_i^{(m)}$ (II) the fundamental functions $L_k(x)$, $k = 1, 2, \cdots m$ are uniformly bounded in $(-1, 1)$ (The $L_k(x)$ are the fundamental functions belonging to the $y_i^{(m)}$). Having constructed the $y_i^{(m)}$ satisfying (I) and (II) we immediately obtain Theorem 3 by applying Theorem 1.

To construct the $y_i^{(m)}$ we first remark that by putting

$$\cos \vartheta_i = -1 + \frac{2i - 1}{n}, \qquad i = 1, 2, \cdots n$$

we obtain by a simple calculation

$$\vartheta_i - \vartheta_{i+1} > \frac{\pi}{m}$$

Now we construct a sequence $y_i^{(m)}$, $i = 1, 2, \cdots m$ such that (1) the $x_i^{(n)}$ occur among the $y_i^{(m)}$ (2) put $\cos \theta_i = y_i$ then $\theta_i = \frac{2i - 1}{m} \frac{\pi}{2} + \frac{d_i}{m}$ where $\sum_{i=1}^{i} d_i$ is uniformly bounded (3) $\theta_i - \theta_{i+1} \geqq \frac{\pi}{4m}$. (2) and (3) insure that the y_i are "very nearly" the roots of $T_m(x)$.

[10] Erdös-Turán, ibid., p. 519.

We construct the y_i as follows: Suppose $y_1 < y_2 < \cdots < y_{i-1}$ are already constructed. We further make the hypothesis that if ϑ_r ($\cos \vartheta_r = x_r$) is the greatest $\vartheta < \theta_{i-1}$ then $\theta_{i-1} - \vartheta_r > \frac{\pi}{4m}$. If $\sum_{j=1}^{i-1} d_j < 0$ we choose for y_i either the least $x_r > y_{i-1}$, or if $\vartheta_r < \theta_{i-1} - \frac{4\pi}{m}$ we put $\theta_i = \theta_{i-1} - \frac{2\pi}{m}$. Thus θ_i does not come nearer than $\frac{\pi}{4m}$ to the greatest $\vartheta < \theta_i$. If $\sum_{j=1}^{i-1} d_j > 0$, $y_i = x_r$ if $\vartheta_r > \theta_{i-1} - \frac{\pi}{2m}$ and $\theta_{i-1} - \frac{\pi}{4m}$ otherwise. Thus in any case if ϑ_j is the greatest $\vartheta < \theta_i$ then $\theta_i - \vartheta_j > \frac{\pi}{4m}$. In this we can construct $y_1, y_2, \cdots y_m$. (1) and (3) are clearly satisfied and it is quite immediate that (2) is also satisfied. Now we have to show that the y_i's satisfying (1), (2), and (3) also satisfy (I) and (II). (I) is clearly satisfied, the proof that (II) is satisfied is slightly more difficult. Denote by $z_1, z_2, \cdots z_m$ the roots of $T_m(x)$ and by $L'_k(x)$ the fundamental functions belonging to the z_i. From (2) and (3) it follows by a simple calculation that

$$c_1 \omega'(y_k) < T'_m(z_k) < c_2 \omega'(y_k), \qquad \omega(x) = \prod_{i=1}^{m} (x - y_i) \tag{2}$$

where c_1 and c_2 are independent of m and k. Denote

$$\max_{-1 \leq x \leq 1} |L_k(x)| = A_k, \qquad \max_{-1 \leq x \leq 1} L'_k(x) = B_k$$

Then again from (2) and (3) by a simple calculation [using (2)]

$$c_3 > \frac{A_k}{B_k} < c_4. \tag{3}$$

We know that[11]

$$B_k < \sqrt{2}. \tag{4}$$

Thus from (3) and (4) we obtain (3), and this completes the proof of Theorem 3.

To obtain the second part of Theorem 3 we first have to prove

LEMMA 3. *Let $m = [(\pi/2)n(1 - \epsilon)]$, $\epsilon > 0$ fixed, independent of m and n, n odd. Let $\varphi_m(x)$ be a polynomial of degree m such that $\varphi_m(0) = 1$ and $\varphi_m(-1 + ((2i - 1)/n)) = 0$, $i = 1, 2, \cdots [(n - 1)/2], [(n + 3)/2] \cdots n$. Then*

$$\max_{-1 \leq x \leq 1} |\varphi_m(x)| > c_1^n, \qquad c_1 = c_1(\epsilon) > 1.$$

PROOF. We use the following lemma due to M. Riesz[12]: Let $\varphi_m(x)$ be a poly-

[11] L. Fejér, see footnote 9.

[12] M. Riesz, Jahresbericht der Deutschen Math. Vereinigung, (1915), p. 354–368.

nomial of degree m, it assumes its absolute maximum in $(-1, 1)$ at the point $x_0 = \cos \vartheta_0$. Let $x_i = \cos \vartheta_i$ be the nearest root of $\varphi_m(x)$ in $(-1, +1)$ then

$$|\vartheta_i - \vartheta_0| \geqq \frac{\pi}{2m}.$$

It immediately follows from this lemma that if x_i and x_{i+1} are the nearest roots including x_0, then we have

$$\vartheta_i - \vartheta_{i+1} \geqq \frac{\pi}{m}.$$

Put now $\cos \vartheta_i = -1 + (2i - 1)/n$. A simple calculation shows that there exists a constant $c_2 = c_2(\epsilon)$ such that if $-c_2 \leqq x_i < x_{i+1} \leqq c_2$ then

$$\vartheta_i - \vartheta_{i+1} < \frac{\left(1 - \frac{\epsilon}{2}\right)\pi}{m}.$$

Hence $\varphi_m(x)$ can not assume its absolute maximum for $-c_2 \leqq x \leqq c_2$ except if

$$x_{\frac{n-1}{2}} < x < x_{\frac{n+3}{2}} \qquad \text{(i. e. in the neighborhood of 0)}$$

Consider now a polynomial $h_m(x)$ with highest coefficient the same as that of $\varphi_m(x)$ whose roots are defined as follows: Let $-c_2 < z_i < c_2$ then $z_i = (1 + \delta)x_i$ where δ is chosen so small that

$$\theta_i - \theta_{i+1} < \frac{1 - \frac{\epsilon}{4}}{m}\pi \qquad (\cos \theta_i = z_i)$$

The other roots of $h_m(x)$ coincide with those of $\varphi_m(x)$. Clearly the degree of $h_m(x)$ is m. Define

$$g(x) = \left(x + \frac{1}{4m}\right)\left(x - \frac{1}{4m}\right)h_m(x).$$

By the lemma of M. Riesz $g(x)$ does not assume its absolute maximum in $(-c_2, c_2)$. It follows from the inequality of the arithmetic and geometric means that

(5) $$|g(x)| < |\varphi_m(x)| \text{ for } c_2(1 + \delta) \leqq |x| \leqq 1$$

Denote by $A(c_2)$ the number of x_i in $(-c_2, +c_2)$. We evidently have $A(c_2) > c_3 n$. Thus

(6) $$|g(0)| > \varphi_m(0)|(1 + \delta)^{c_3 n}\frac{1}{16m^2} > \varphi_m(0)c_1^n = c_1^n (c_1 > 1)$$

But since $g(x)$ assumes its absolute maximum in $(-1, 1)$ for some $|x_0| > c_2(1 + \delta)$ we have by (5) and (6)

$$|\varphi_m(x_0)| > |g(x_0)| > c_1^n \quad \text{q.e.d.}$$

Let now $n_1, n_2, \cdots$ be an infinite sequence of odd integers, which tend to infinity sufficiently quickly. We define a polynomial $\psi_i(x)$ as follows

$$\psi_i\left(-1 + \frac{2j-1}{n_r}\right) = 0, \quad r \leqq i, \quad j \neq \frac{1+n_r}{2}, \quad j \leqq n_r,$$

$$\psi_i(0) = 1, \quad |\psi_i(x)| \leqq 2, \quad -1 \leqq x \leqq 1.$$

From the approximation theorem of Weierstrass it follows that such a $\psi_i(x)$ exists. Consider now the continuous function

$$f(x) = \sum_{k=1}^{\infty} \frac{\psi_k(x)}{2^k}.$$

If the second part of Theorem 3 would not be true, we could find a sequence of polynomials $\varphi_i(x)$ of degree $\leqq n_i(\pi/2)(1-\epsilon)$ such that $\varphi_i(-1 + ((2j-1)/n_i)) = f[1 + ((2j-1)/n_i)]$ and $\varphi_i(x) \to f(x)$ uniformly in $(-1, 1)$. For $k > i$

$$\psi_k\left(-1 + \frac{2j-1}{n_i}\right) = 0, \quad j \neq \frac{1+n_i}{2}.$$

Thus $\varphi_i(x)$ coincides with

$$\sum_{r=1}^{i-1} \frac{\psi_r(x)}{2^r} = g(x)$$

at the points $-1 + ((2j-1)/n_i)$, $j \neq ((1+n_i)/2)$.

Let now n_i tend to infinity so quickly that n_i is greater than the degree of $g(x)$. Then $\varphi_i(x)$ can be written as

$$\varphi_i(x) = \varphi_i^{(1)}(x) + \varphi_i^{(2)}(x),$$

where $\varphi_i^{(1)} = g(x)$, and $\varphi_i^{(2)}(x)$ is of degree $\leqq ((\pi/2) - c_1)n_2$ and $\varphi_i''(-1 + ((2j-1)/n_i)) = 0$, $j \neq ((1+n_i)/2)$, $j \leqq n_i$, also $\varphi_i^{(2)}(0) = \sum_{k \geqq i}((\psi_k(0))/2^k) = (1/2^{i-1})$. Thus by lemma 3

$$\max_{-1 \leqq x \leqq 1} |\varphi_i(x)| \geqq \max_{-1 \leqq x \leqq 1} |\varphi_i^{(2)}(x)| - 2 > \frac{c_2^{n_i}}{2^{i-1}} > c_3^{n_i} \; (c_2 \text{ and } c_3 \text{ are} > 1$$

if n_i tends to infinity sufficiently quickly. Hence $\varphi_i(x)$ can not converge uniformly to $f(x)$, and this completes the proof of Theorem 3.

By a more complicated argument we could prove that a point x_0 exists such that $\varphi_n(x_0)$ diverges. We give only the sketch of the proof. Since $\max_{-1 \leqq x \leqq 1} |\varphi_{n_i}(x)| > (1+\delta)^{n_i}$ it follows from a theorem of Remes[13] that there exists in $(-1, 1)$ a set of measure $> c = c(\delta)$ such that on this set $|\varphi_{n_i}(x)| > (1 + (\delta/2))^{n_i}$. Then, it follows easily that there exist a point x_0 with $\limsup |\varphi_{n_i}(x_0)| = \infty$.

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[13] E. Remes, *Sur une propriété extremale des polynomes de Tchebycheff*. Comm. de l'Institut des Sciences etc. Kharkov, (1936) série 4, XIII fasc. 1, p. 93–95.

By the same method we can prove the following:

THEOREM 5. *Let* $\begin{matrix} x_1^{(1)} \\ x_1^{(2)}, x_2^{(2)} \\ \cdots \end{matrix}$ *be a point group and put* $\cos(\vartheta_i^{(n)}) = x_i^{(n)}$. *Suppose that*

$$\liminf n(\vartheta_i^{(n)} - \vartheta_{i+1}^{(n)}) = \frac{\pi}{d}, \ (n \to \infty \ , i \text{ arbitrary})$$

Then to every continuous $f(x)$ *and constant* $c > 0$ *there exists a sequence of polynomials* $\varphi_n(x)$ *of degree* $< d(1 + c)n$ *such that* $\varphi_n(x_i^{(n)}) = f(x_i^{(n)})$ *and* $\varphi_n(x) \to f(x)$ *uniformly in* $(-1, 1)$.

The constant d_1 of Theorem 5 is not best possible. We can obtain the best possible constant d_1 as follows: Let a_n and b_n be two arbitrary sequences of real numbers, such that $0 \leqq a_n < b_n \leqq \pi$, $n(b_n - a_n) \to \infty$. Then if $d < \infty$

$$\limsup \frac{N_n(a_n, b_n)}{n(b_n - a_n)} = \pi \, d_1 .$$

Lemma 3 would not suffice for the proof of Theorem 5. Here we need

LEMMA 4. *Let* $\varphi_n(x)$ *be a polynomial of degree* n, $\varphi_n(0) = 1$. *Let* $\psi(n)$ *be any function of* n *tending to infinity together with* n *and let* c_1 *be a constant independent of* n. *Then if* $\varphi_n(x)$ *is such that for every* $c_1 < A < \psi(n)$ *the number of roots of* $\varphi_n(\cos \vartheta)$ *in* $((\pi/2) - (A/n), (\pi/2) + (A/n))$ *is greater than* $[((1 + c_2)2)/\pi]$ *we have* $\max_{-1 \leq x \leq 1} |\varphi_n(x)| \to \infty$. *Our condition means that the number of roots of* $\varphi_n(x)$ *in the neighborhood of* 0 *is substantially larger than the number of roots of* $T_n(x)$. *The proof of Lemma 4 is similar, but more complicated than the proof of Lemma 3.*

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