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On some convexity properties of Orlicz spaces of vector valued functions

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ABSTRACT. A "stability" theorem that is a generalization of Th. 6 in [2] for the modulus of convexity of Banach spaces is given. Necessary and sufficient conditions for $\delta_L\Phi(a)>0$, where $a\in(0,2]$, in Orlicz spaces $L^\Phi(\mu,X)$ of vector valued functions are given. The convexity coefficient $\varepsilon_o(L^\Phi(\mu,X))$ is computed for these spaces. The equality $\varepsilon_o(L^\Phi(\mu,X))=\varepsilon_o(X)$ for Orlicz-Bochner spaces generated by uniformly convex Orlicz functions satisfying the Δ_2-condition is showed.

INTRODUCTION

Throughout this paper $(T,\sum,\mu)$ denotes a non-atomic, infinite and complete measure space and X denotes a Banach space. A function $\Phi:X\to[0,+\infty]$ is said to be an *Orlicz function* if it is convex, even, lower semicontinuous, vanishing and continuous at zero, and $\Phi\not\equiv 0$. $F(T,X)$ stands for the space of all equivalence classes of strongly $\sum$-measurable functions from T into X.

Given an Orlicz function Φ, we define the *Orlicz space* $L^\Phi(\mu,X)$ as the set of all functions $f\in F(T,X)$ such that

$$I_\Phi(\lambda f)=\int_T \Phi(\lambda f(t))d\mu<+\infty$$

for some $\lambda>0$ depending on f. This space equipped with the Luxemburg norm

$$\|f\|_\Phi=\inf\{\lambda>0:I_\Phi(\lambda^{-1}f)\le 1\}$$

is a normed space (see [11-13]), and it is a Banach space if and only if Φ is uniformly large at infinity, i.e.

$\lim_{k\to+\infty}\inf\{\Phi(x):\|x\|=k\}=+\infty$ (see [15]).

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We say an Orlicz function Φ satisfies the Δ_2-condition if there is a constant $K>0$ such that $\Phi(2x)\leq K\Phi(x)$ for all $x\in X$.

The *modulus of convexity* of a normed space $(X,\|\ \|)$ is the function $\delta_X(\cdot)$: $(0,2]\to[0,1]$ defined by

$$\delta_X(\varepsilon)=\inf\{1-\|1/2(x+y)\|:\|x\|\leq 1,\|y\|\leq 1,\|x-y\|\geq \varepsilon\}.$$

The *convexity coeffcient* of a normed space $(X,\|\ \|)$ is defined by

$$\varepsilon_o(X)=\sup\{\varepsilon\in[0,2]:\delta_X(\varepsilon)=0\}(\sup \emptyset \stackrel{\text{def}}{=} o).$$

(see [2]).

RESULTS

To prove the first theorem we shall need the following

Lemma 1. *If* $\delta_X(a)>0$ *for a number* $a\in(0,2)$, *then there is a number* $\gamma>1/a$ *such that* $a\gamma(1-\delta_X(1/\gamma))= 1$.

Proof. By the assumptions and by the continuity of δ_X it follows that there is a number $\alpha>1/a$ such that $\delta_X(1/\alpha)>0$. So, $a\beta(1-\delta_X(1/\alpha))<1$ for a certain $\beta>1/a$. Taking $\alpha_o=\min(\alpha,\beta)$, we have $a\alpha_o(1-\delta_X(1/\alpha_o))<1$.

A function h: $(1/2,+\infty)\to R_+$ defined by $h(\lambda)=a\lambda(1-\delta_X\ (1/\lambda))$ is continuous and $h(\lambda)\to+\infty$ as $\lambda\to+\infty$. Since $h(\alpha_o)<1$, the Darboux property of h yields $h(\gamma)=a\gamma(1\text{-}\delta_X(1/\gamma))=1$ for a certain $\gamma>1/a$ which finishes the proof.

Now, we are able to generalize Th. 6 in [2]. The proof is almost the same but we shall give it for the sake of completness.

Theorem 2. *Let X be a Banach space with* $\varepsilon_o(X)$ *in the interval* $[0,a)$, *where* $0<a\leq 2$. *Let* $\gamma>1/a$ *be such that* $a\gamma(1-\delta_X(1/\gamma))= 1$. *If Y is a Banach space with Banach-Mazur distance* $d(X,Y)<a\gamma$, *then* $\varepsilon_o(Y)<a$.

Proof. Without loss of generality we may assume that U is an isomorphism between X and Y such that $\|U^{-1}\|=1$ and $d(X,Y)\leq\|U\|\leq ab\gamma$, where $0<b<1$. Let $y_1, y_2\in S_Y$ (=the unit sphere of Y), $\|y_1-y_2\|\geq\|U\|/\gamma$ and $x_1=U^{-1}y_1$, $x_2=U^{-1}y_2$. Then $\|x_1\|\leq 1,\|x_2\|\leq 1$ and $\|U\|/\gamma\leq\|y_1-y_2\|=\|U(x_1-x_2)\|\leq\|U\|\|x_1-x_2\|$, whence $\|x_1-x_2\|\geq 1/\gamma$. Since $a\gamma>1$, by the equality $a\gamma(1-\delta_X(1/\gamma))=1$, it follows that $\delta_X(1/\gamma)>0$. Therefore

$$\|(y_1+y_2)/2\|=\|U(1/2(x_1+x_2))\|\leq\|U\|\|1/2(x_1+x_2)\|\leq ab\gamma(1-\delta_X\ (1/\gamma))=b.$$

This means that $\delta_Y(\|U\|\ /\gamma)\geq 1-b>0$. Thus, $\varepsilon_o(Y)\leq\|U\|/\ \gamma<a$.

In the fixed point theory the notion of the convexity coefficient is useful (see [2]). We shall give now a basic theorem to compute $\varepsilon_o(L^\Phi(\mu,X))$.

Theorem 3. *Let a be a number in* (0,2]. *The following conditions are equivalent:*

1° $\delta_{L^\Phi}(a)>0$.

2° (*a*) *there is* $\delta\in(0,1)$ *such that for every* $x,y\in X$ *satisfying the equality*

$\Phi((x-y)/a)\geq(1-\delta)\Phi((x+y)/2)$, we have $\Phi((x+y)/2)\leq\frac{1-\delta}{2}\{\Phi(x)+\Phi(y)\}$,

(*b*) Φ *satisfies the* Δ_2*-condition.*

Proof. $2°\Rightarrow1°$. Assume that $\|f\|_\Phi\leq1$, $\|g\|_\Phi\leq1$ and $\|f-g\|_\Phi\geq a$. Then $I_\Phi(f)\leq1$, $I_\Phi(g)\leq1$ and $I_\Phi((f-g)/a)\geq1$. Define

$$A=\{t\in T:\Phi((f(t)-g(t))/a)\geq(1-\delta)\Phi((f(t)+g(t))/2)\}.$$

Then

$$I_\Phi(\frac{f-g}{a}\chi_{T\setminus A})\leq\frac{1-\delta}{2}\{I_\Phi(f\chi_{T\setminus A})+(I_\Phi(g\chi_{T\setminus A})\}\leq1-\delta.$$

Consequently, $I_\Phi(\frac{f-g}{a}\chi_A)\geq\delta$. By the Δ_2-condition, we get

$$I_\Phi(\frac{f-g}{a}\chi_A)\leq K\{I_\Phi(f\chi_A)+I_\Phi(g\chi_A)\},$$

where K is a constant depending only on a and Φ. Hence,

$$1-I_\Phi(\frac{f+g}{2})\geq(1/2)\{I_\Phi(f)+I_\Phi(g)\}-I_\Phi(\frac{f+g}{2})$$

$$\geq(1/2)\{I_\Phi(f\chi_A)+I_\Phi(g\chi_A)\}-I_\Phi(\frac{f+g}{2}\chi_A)$$

$$\geq(1/2)\{I_\Phi(f\chi_A)+I_\Phi(g\chi_A)\}-\frac{1-\delta}{2}\{I_\Phi(f\chi_A)+I_\Phi(g\chi_A)\}$$

$$=(\delta/2)\{I_\Phi(f\chi_A)+I_\Phi(g\chi_A)\}\geq\frac{\delta^2}{2K}$$

Equivalently,

$$I_\Phi\left(\frac{f+g}{2}\right) \leq 1 - \frac{\delta^2}{2K}.$$

Applying the Δ_2-condition, we get

$$\left\| \frac{f+g}{2} \right\|_\Phi \leq 1 - p\left(\frac{\delta^2}{2K}\right),$$

where p is a function from $(0,1)$ into itself (in the real case see [4] and [6]).

This yields $\delta_{L^\Phi}(a) \geq p\left(\frac{\delta^2}{2K}\right) > 0.$

1° ⟹ 2°. If Φ does not satisfy the Δ_2-condition, then $L^\Phi(\mu,X)$ contains an isometric copy of l^∞ (see [3] and [4]). Therefore, $\delta_{L^\Phi}(a) \leq \delta_{l^\infty}(a) = 0$ for every $a \in (0,2]$. Assume now that condition 2°(a) is not satisfied, i.e. for every $\delta \in (0,1)$ there exist $x,y \in X$ such that

$$\Phi(1/a(x-y)) \geq (1-\delta)\Phi(x+y)/2) \text{ and } \Phi((x+y)/2) > 1/2(1-\delta)\{\Phi(x)+\Phi(y)\}.$$

Let $B,C \in \sum$, $B \subset C$, be such that $\mu(C) = \mu(B \backslash C)$ and $(\Phi(x)+\Phi(y))\mu(B) = 2$.

Define

$$f = x\chi_C + y\chi_{B\backslash C},\ g = y\chi_C + x\chi_{B\backslash C}$$

We have $I_\Phi(f) = I_\Phi(g) = 1$, whence $\|f\|_\Phi = \|g\|_\Phi = 1$. Moreover,

$$I_\Phi((f-g)/(1-\delta)^2 a) \geq (1/(1-\delta)^2)\int_B \Phi((x-y)/a)d\mu$$

$$\geq (1/(1\text{-}\delta)^2)\Phi((x-y)/a)\mu(B) \geq \left(1/(1-\delta)\right)\Phi((x+y)/2)\mu(B)$$

$$\geq (1/2)\{\Phi(x)+\Phi(y)\}\mu(B) = 1.$$

Therefore $\|(f-g)/a\|_\Phi \geq (1-\delta)^2$. In an analogous way the inequality $\|(f+g)/2\|_\Phi \geq 1-\delta$ can be proved. Since $\delta \in (0,1)$ was arbitrary, this means that $\delta_{L^\Phi}(a) = 0$. The theorem is proved.

To prove the next theorem, we will need the following

Proposition 4. *Let Φ be an Orlicz function satisfying the Δ_2-condition. Then the following assertions are equivalent:*

(+) *there is* $\delta \in (0,1)$ *such that* $\Phi((x+y)/2 \le ((1-\delta)/2)\{\Phi(x)+\Phi(y)\}$ *whenever* $x,y \in X$ *satisfy* $\Phi((x-y)/a) \ge (1-\delta)\Phi((x+y)/2)$.

(++) *there is* $\sigma \in (0,1)$ *such that* $\Phi((x+y)/2) \le ((1-\sigma)/2)\{\Phi(x)+\Phi(y)\}$ *whenever* $x,y \in X$ *satisfy* $\Phi((x-y)/a(1-\sigma)) \ge \Phi((x+y)/2)$.

Proof. (++)⟹(+). Assume that $\Phi((x-y)/a) \ge (1-\sigma)\Phi((x+y)/2)$. Then $\Phi((x-y)/a(1-\sigma)) \ge (1/(1-\sigma))\Phi((x-y)/a) \ge \Phi((x+y)/2)$. In view of condition (++), we have $\Phi((x+y)/2) \le ((1-\sigma)/2)\{\Phi(x)+\Phi(y)\}$.
Thus, it suffices to put $\delta=\sigma$.

(+)⟹(++). Assume that $\Phi((x-y)/a(1-\sigma_1)) \ge \Phi((x+y)/2)$, where σ_1 is a constant in (0,1) satisfying $\Phi(x/(1-\sigma_1)) \le (1/(1-\delta))\Phi(x)$ for every $x \in X$ (by the Δ_2-condition such a constant exists) and δ is the constant from condition (+). Then

$$\Phi((x-y)/a)(1/(1-\delta)) \ge \Phi((x-y)/a(1-\sigma_1)) \ge \Phi((x+y)/2).$$

Therefore, by condition (+), we get

$$\Phi((x+y)/2) \le ((1-\delta)/2)\{\Phi(x)+\Phi(y)\}.$$

It suffices to put $\sigma=\min(\sigma_1, \delta)$.

Theorem 5. *Let* Φ *be a uniformly convex Orlicz function defined on the real line, i.e. for every* $a \in (0,1)$ *there exists* $\delta(a) \in (0,1)$ *such that for every* $u \in R$ *we have* $\Phi((u+au)/2) \le (1/2)(1-\delta(a))\{\Phi(u)+\Phi(au)\}$, *and let* Φ *satisfy the* Δ_2*-condition and* $(X,\|\ \|)$ *be a Banach space. Then* $\delta_{L^\Phi}(\varepsilon)>0$ *for the Orlicz-Bochner space* $L^\Phi = L^\Phi(\mu,X)$ *if and only if* $\delta_X(\varepsilon)>0$.

Proof. Since X can be isometrically embedded into $L^\Phi(\mu,X)$, the condition $\delta_X(\varepsilon)=0$ implies $\delta_{L^\Phi}(\varepsilon)=0$.

Now, in view of Proposition 4 assume that $\delta_X(\varepsilon)>0$. It suffices to show that there exists a constant $\sigma \in (0,1)$ such that for every $x,y \in X$, we have

(1) $\left\| \frac{x-y}{\varepsilon\sigma} \right\| \ge \left\| \frac{x+y}{2} \right\|$ implies $\Phi\left(\left\| \frac{x+y}{2} \right\|\right) \le \frac{\sigma}{2}\{\Phi(\|x\|)+\Phi(\|y\|)\}$.

Since $\delta_X(\varepsilon)>0$ by the assumption, there exists $\delta \in (0,1)$ such that

(2) $\left\| \frac{x+y}{2} \right\| \le \delta$ whenever $x,y \in B_X$ (= the unit ball of X) and

$$\left\| \frac{x-y}{\varepsilon\delta} \right\| \ge \left\| \frac{x+y}{2} \right\|.$$

Assume that $x,y \in X$ and $\left\| \frac{x-y}{\varepsilon\delta} \right\| \geq \left\| \frac{x+y}{2} \right\|$. We can assume without loss of generality that $\|x\| \leq \|y\|$. Then $\frac{x}{\|y\|}$, $\frac{y}{\|y\|} \in B_X$ and

$\left\| \frac{x-y}{\varepsilon\delta\|y\|} \right\| \geq \left\| \frac{x+y}{2\|y\|} \right\|$. Thus, in virtue of condition (2), we get

$\left\| \frac{x+y}{2} \right\| \leq \delta\|y\|$. Now, we shall consider two cases.

1°. $\sqrt{\delta}\|y\| \leq \|x\|$. Then

$$\left\| \frac{x+y}{2} \right\| \leq \delta\|y\| = \delta \frac{\|y\| + \|y\|}{2} \leq \delta \frac{\|x\|/\sqrt{\delta} + \|y\|}{2} \leq \delta \frac{\|x\| + \|y\|}{2\sqrt{\delta}}$$

$$= \frac{\sqrt{\delta}}{2}(\|x\| + \|y\|).$$

Hence

$$\Phi\left(\left\| \frac{x+y}{2} \right\|\right) \leq \frac{\sqrt{\delta}}{2} \{\Phi(\|x\|) + \Phi(\|y\|)\}.$$

2°. $\|x\| < \sqrt{\delta}\|y\|$. By uniform convexity of Φ, we have

$$\Phi\left(\left\| \frac{x+y}{2} \right\|\right) \leq \Phi\left(\frac{\|x\| + \|y\|}{2}\right) \leq \frac{1-\eta(\delta)}{2} \{\Phi(\|x\|) + \Phi(\|y\|)\}.$$

Therefore, for every $x,y \in X$ such that $\left\| \frac{x-y}{a\delta} \right\| \geq \left\| \frac{x+y}{2} \right\|$, we have

$$\Phi\left(\left\| \frac{x+y}{2} \right\|\right) \leq \frac{\max((1-\eta(\delta)),\sqrt{\delta})}{2} \{\Phi(\|x\|) + \Phi(\|y\|)\}.$$

To prove condition (1) it suffices to put $\sigma = \max((1-\eta(\delta),\sqrt{\delta})$.

Note. The thesis of Theorem 5 means that $\varepsilon_o(L^\Phi(\mu,X)) = \varepsilon_o(x)$. It is a generalization of Theorem 9 of Downing and Turett [2]. However, our method of the proof is quite different than the method used there.

An Orlicz function Φ is said to satisfy condition C_a ($a \in (0,2)$) if there exists a number $\sigma \in (0,1)$ such that

$$\Phi((x+y)/2) \leq (\sigma/2)\{\Phi(x) + \Phi(y)\},$$

whenever $x,y \in X$ and $\Phi((x-y)/a\sigma) \geq \Phi((x+y)/2)$.

For any Orlicz function Φ we define the parameter

$$\alpha(\Phi) = \inf\{a \in (0,2): \Phi \text{ satisfies condition } C_a\}.$$

We shall give now an inmediate consequence of Th. 3 and Prop. 4.

Corollary 6. *Let Φ be an Orlicz function. Then $\varepsilon_o(L^\Phi(\mu,X)) = 2$ whenever Φ does not satisfy the Δ_2-condition and $\varepsilon_o(L^\Phi(\mu,X)) = \alpha(\Phi)$ in the opposite case.*

Note. Theorem 3 and Corollary 6 generalize the results of [6] to Orlicz spaces of vector valued functions. Theorem 3 generalizes also some results of [4], [5] and [7]. These results are also connected with the results of [8], [9] and [10].

Corollary 7. *Let Φ be an Orlicz function defined on the real line R and $(X,\| \ \|)$ be a Banach space. Then the Orlicz-Bochner space $L^\Phi(\mu,X)$ is uniformly rotund if and only if both spaces $L^\Phi(\mu,R)$ and X are uniformly rotund.*

Recall that $L^\Phi(\mu,R)$ is uniformly rotund if and only if Φ is uniformly convex and satisifes the Δ_2-condition (see [7]).

Problem. Is the equality $\varepsilon_o(L^\Phi(\mu,X)) = max(\varepsilon_o(L^\Phi(\mu, R), \varepsilon_o(X))$ true for every Orlicz-Bochner space?

Added in proof. The problem has negative answer. We refer to the paper of the author and T. Landes entitled *"Characteristic of convexity of Köthe function spaces"* (preprint).

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