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ON SOME CORRESPONDENCES BETWEEN RELATIONAL STRUCTURES AND ALGEBRAS

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1. INTRODUCTION

In this paper , we prove that two categories are isomorphic for any positive integer n: Objects of the first category are $n+1$-ary relational structures where an $n+1$-ary relational structure is a set with one $n+1$-ary relation; morphisms of this category are strong homomorphisms of those structures. Objects of the second category are n-ary algebras where an n-ary algebra is a power set with a totally additive n-ary operation; morphisms of this category are totally additive atom-preserving homomorphisms of those algebras. This result generalizes Main Theorem of [6] and of [5] which represent particular cases of our present result for $n = 1$ and $n = 2$. Definitions presented in [5] and [6] for particular cases are now generalized, which may be of some interest. The proof of our Theorem is omitted because it is almost the same as in [5]. Hence, this article represents a unified methodology for investigations included in [5] and [6]; naturally, this methodology offers further possibilities.

2. TOTALLY ADDITIVE AND ATOM-PRESERVING MAPPINGS

For any set A we denote by $\boldsymbol{P}(A)$ its power set, i.e. $\boldsymbol{P}(A) = \{X\,;\, X \subseteq A\}$.

Let A, A' be sets, H a mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$. The mapping H is said to be *totally additive* if $H(X) = \bigcup \{H(\{x\})\,;\, x \in X\}$ holds for any set $X \in \boldsymbol{P}(A)$. The mapping H is referred to as *atom-preserving* if for any $x \in A$ there exists $x' \in A'$ such that $H(\{x\}) = \{x'\}$.

Let r be a binary relation from A to A'. Then for any $X \in \boldsymbol{P}(A)$ we put $\mathbf{P}[r](X) = \{x' \in A'\,;\,$ there exists $x \in X$ with $(x, x') \in r\}$. Clearly, $\mathbf{P}[r]$ is a mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$.

Let H be a mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$. Then we set $\mathbf{Q}[H] = \{(x, x') \in A \times A';\ x' \in H(\{x\})\}$. Then $\mathbf{Q}[H]$ is a relation from A to A'.

Hence, we have defined two operators $\mathbf{P}$, $\mathbf{Q}$. The operator $\mathbf{P}$ assigns a mapping $\mathbf{P}[r]$ of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$ to any relation r from A to A'; the operator $\mathbf{Q}$ assigns a relation $\mathbf{Q}[H]$ from A to A' to any mapping H of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$.

Example 1. If r is a mapping of A into A', then $\mathbf{P}[r](X) = \{r(x);\ x \in X\}$.

Example 2. If $\leqslant$ is an ordering on A, put $H(X) = \{y \in A;$ there exists $x \in X$ with $x \leqslant y\}$ for any $X \in \boldsymbol{P}(A)$. Clearly, $H(X)$ is the final section generated by X in $(A, \leqslant)$. Then $\mathbf{Q}[H] = \leqslant$.

3. Strong homomorphisms of relational structures

In what follows n is a positive integer. If A is a set, we put $A^n = \underbrace{A \times \ldots \times A}_{n \text{ times}}$. A set $t \subseteq A^n$ is said to be an *n-ary relation* on A and the ordered pair (A, t) is called an *n-ary structure*.

If (A, t), (A', t') are $n+1$-ary structures and h is a mapping of A into A', then h is said to be a *strong homomorphism* of the $n+1$-ary structure (A, t) into (A', t') whenever the following holds: For any $x_1, \ldots, x_n$ in A and x'_{n+1} in A' the condition $(h(x_1), \ldots, h(x_n), x'_{n+1}) \in t'$ is satisfied if and only if there exists $x_{n+1} \in A$ such that $h(x_{n+1}) = x'_{n+1}$ and $(x_1, \ldots, x_n, x_{n+1}) \in t$.

Example 3. For $n = 1$, we obtain $t \subseteq A^2$, $t' \subseteq A'^2$, i.e. (A, t), (A', t') are binary structures. Furthermore, h is a strong homomorphism of (A, t) into (A', t') if and only if the following holds: For any $x \in A$ and any $y' \in A'$ the condition $(h(x), y') \in t'$ is satisfied if and only if there exists $y \in A$ such that $h(y) = y'$ and $(x, y) \in t$. By Lemma 1 of [6], our strong homomorphisms coincide with strong homomorphisms in the sense of [6] for the particular case $n = 1$.

Example 4. For $n = 2$, we obtain $t \subseteq A^3$, $t' \subseteq A'^3$, i.e. (A, t), (A', t') are ternary structures. For any ternary relation t on A put $\hat{t} = \{(x, z, y) \in A^3;\ (x, y, z) \in t\}$. Thus, (A, t) and $(A, \hat{t})$ differ only in the way of notation. Let h be a mapping of A into A'. Then h is a strong homomorphism of (A, t) into (A', t') if and only if the following holds: For any $x \in A$, $y \in A$, $z' \in A'$ the condition $(h(x), h(y), z') \in t'$ is satisfied if and only if there exists $z \in A$ such that $h(z) = z'$, $(x, y, z) \in t$. The condition may be reformulated as follows. For any $x \in A$, $y \in A$, $z' \in A'$ the condition $(h(x), z', h(y)) \in \hat{t}'$ is satisfied if and only if there exists $z \in A$ such that $h(z) = z'$, $(x, z, y) \in \hat{t}$. Hence, h is a strong homomorphism of (A, t) into (A', t') if and only if it is a strong homomorphism in the sense of [5] of the structure $(A, \hat{t})$ into $(A', \hat{t}')$.

4. Algebras on power sets

Let n be a positive integer and let (A, t) be an $n+1$-ary structure. For arbitrary sets $X_1, \ldots, X_n$ in $\boldsymbol{P}(A)$ we put

$$\mathbf{R}[t](X_1, \ldots, X_n) = \{x_{n+1} \in A;\ \text{there exist } x_1 \in X_1, \ldots, x_n \in X_n \text{ such that } (x_1, \ldots, x_n, x_{n+1}) \in t\}.$$

Clearly, $\mathbf{R}[t]$ is an n-ary operation on the set $\boldsymbol{P}(A)$. Hence $\mathbf{R}$ is an operator assigning an n-ary operation on $\boldsymbol{P}(A)$ to any $n+1$-ary relation on A.

Example 5. Let $(A, \leqslant)$ be an ordered set. Then $n = 1$ and $\mathbf{R}[\leqslant](X) = \{y \in A;$ there exists $x \in X$ such that $x \leqslant y\}$, i.e. $\mathbf{R}[\leqslant](X)$ is the final segment generated by the set X for any $X \in \boldsymbol{P}(A)$.

Let n be a positive integer, A a set, and N an n-ary operation on $\boldsymbol{P}(A)$. Then the ordered pair $(\boldsymbol{P}(A), N)$ will be referred to as an *n-ary algebra*. The operation N is said to be *totally additive* if $N(X_1, \ldots, X_n) = \bigcup \{N(\{x_1\}, \ldots, \{x_n\});\ (x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n\}$ holds for any $X_1, \ldots, X_n$ in $\boldsymbol{P}(A)$. By Lemma 7 of [5], this definition generalizes the definition included in [5]; clearly, a totally additive unary operation is a totally additive mapping.

If n is a positive integer, A a set, and N an n-ary operation on $\boldsymbol{P}(A)$, we put

$$\mathbf{S}[N] = \Big\{(x_1, \ldots, x_n, x_{n+1}) \in A^{n+1};\ x_{n+1} \in N\big(\{x_1\}, \ldots, \{x_n\}\big)\Big\}.$$

Example 6. Let A be a set and $N(X, Y) = X \cup Y$ for any X, Y in $\boldsymbol{P}(A)$. Then $\mathbf{S}[N] = \{(x, y, z) \in A^3;\ z \in \{x, y\}\} = \{(x, y, x);\ x, y \in A\} \cup \{(x, y, y);\ x, y \in A\}$.

Example 7. Let A be a set and $N(X, Y) = X \cap Y$ for any X, Y in $\boldsymbol{P}(A)$. Then $\mathbf{S}[N] = \{(x, y, z) \in A^3;\ z \in \{x\} \cap \{y\}\} = \{(x, x, x);\ x \in A\}$.

5. Category $\mathsf{REL}n+1$ and category $\mathsf{ALG}n$

We now introduce two categories. The instruments of the theory of categories needed in the sequel may be easily found in [1].

Let n be a positive integer.

Objects of the category $\mathsf{REL}n+1$ are $n+1$-ary structures of the form (A,t). By a morphism of the object (A,t) into (A',t') in $\mathsf{REL}n+1$ we mean a strong homomorphism of the structure (A,t) into (A',t'). Since $1_{(A,t)}$ is a strong homomorphism of (A,t) into itself and since the composite of two strong homomorphisms is a strong homomorphism (this may be proved similarly as in [5], p. 91), $\mathsf{REL}n+1$ is a category.

Objects of the category $\mathsf{ALG}n$ are n-ary algebras of the form $(\boldsymbol{P}(A),N)$ where A is a set and N is a totally additive n-ary operation on $\boldsymbol{P}(A)$. By a morphism of the object $(\boldsymbol{P}(A),N)$ into the object $(\boldsymbol{P}(A'),N')$ in $\mathsf{ALG}n$ we mean a totally additive atom-preserving homomorphism of the algebra $(\boldsymbol{P}(A),N)$ into $(\boldsymbol{P}(A'),N')$. Since $1_{(\boldsymbol{P}(A),N)}$ is a totally additive atom-preserving homomorphism of $(\boldsymbol{P}(A),N)$ into itself and since the composite of two totally additive atom-preserving homomorphisms is a totally additive atom-preserving homomorphism, $\mathsf{ALG}n$ is a category.

Example 8. in [6], REL2 was denoted by STR and ALG1 by PMA. The category REL3 appeared in [5] under the name TER, and ALG2 was denoted by PGR there.

6. Isomorphisms of categories $\mathsf{REL}n+1$ and $\mathsf{ALG}n$

We now introduce two functors. F will be a functor of the category $\mathsf{REL}n+1$ into $\mathsf{ALG}n$ and G will be a functor of the category $\mathsf{ALG}n$ into $\mathsf{REL}n+1$. These functors will be defined by presenting the object mappings Fo, Go and the morphism mappings Fm, Gm.

If (A,t) is an object in the category $\mathsf{REL}n+1$ and h a morphism in this category, we put

$$Fo(A,t) = (\boldsymbol{P}(A), \mathbf{R}[t]), \qquad Fm(h) = \mathbf{P}[h].$$

If $(\boldsymbol{P}(A),N)$ is an object in the category $\mathsf{ALG}n$ and H is a morphism in this category, we set

$$Go(\boldsymbol{P}(A),N) = (A, \mathsf{S}[N]), \qquad Gm(H) = \mathbf{Q}[H].$$

Theorem. *Let n be a positive integer. Then F is a functor of the category $\mathsf{REL}n+1$ into $\mathsf{ALG}n$ and G is a functor of the category $\mathsf{ALG}n$ into $\mathsf{REL}n+1$ such that $F\circ G$ and $G\circ F$ are identity functors.*

P r o o f is the same as the proof of Main Theorem of [5]. All results included in [5] that are needed in the proof of Main Theorem may be easily generalized to an arbitrary positive arity, which makes the proof of our Theorem possible. □

Corollary 1. *Let n be a positive integer. Then the functor F is an isomorphism of the category* $\mathsf{REL}n+1$ *into* $\mathsf{ALG}n$ *and the functor G is an isomorphism of the category* $\mathsf{ALG}n$ *into* $\mathsf{REL}n+1$.

Corollary 2. *Let n be a positive integer and (A,t), (A',t') $n+1$-ary structures.*

(i) *For any strong homomorphism h of the structure (A,t) into (A',t') there exists a totally additive atom-preserving homomorphism H of the algebra $(\boldsymbol{P}(A),\mathbf{R}[t])$ into $(\boldsymbol{P}(A'),\mathbf{R}[t'])$ such that $h=\mathbf{Q}[H]$.*

(ii) *If H is an arbitrary totally additive atom-preserving homomorphism of the algebra $(\boldsymbol{P}(A),\mathbf{R}[t])$ into $(\boldsymbol{P}(A'),\mathbf{R}[t'])$, then $\mathbf{Q}[H]$ is a strong homomorphism of the structure (A,t) into (A',t').*

Example 9. By Corollary 2, all strong homomorphisms of a binary structure (A,t) into another one (A',t') may be constructed. We construct mono-unary algebras $(\boldsymbol{P}(A),\mathbf{R}[t])$ and $(\boldsymbol{P}(A'),\mathbf{R}[t'])$. The construction of all homomorphisms of the first algebra into the latter is known (cf., e.g., [2], [3], [4]). We take only totally additive atom-preserving homomorphisms; for any such homomorphism H the mapping $\mathbf{Q}[H]$ is a strong homomorphism of (A,t) into (A',t'), and any strong homomorphism of (A,t) into (A',t') may be constructed in this way. If A, A' are finite, the construction is effective. Cf. Section 5 of [6].

Example 10. Theorem gives the possibility to describe subclasses of $\mathsf{REL}n+1$ by means of subclasses of $\mathsf{ALG}n$. We give a concrete example. Let (A,t) be a binary structure. Then t is a preordering if and only if $\mathbf{R}[t]$ is a totally additive closure operator. Cf. Section 6 of [6].

References

[1] *S. Mac Lane*: Categories for the working mathematician, Springer, New York-Heidelberg-Berlin, 1971.

[2] *M. Novotný*: O jednom problému z teorie zobrazení (Sur un problème de la théorie des applications), Publ. Fac. Sci. Univ. Masaryk Brno, No *344* (1953), 53–64.

[3] *M. Novotný*: Über Abbildungen von Mengen, Pac. J. Math. *13* (1963), 1359–1369.

[4] *M. Novotný*: Mono-unary algebras in the work of Czechoslovak mathematicians, Archivum Mathematicum (Brno) *26* (1990), 155–164.

[5] *M. Novotný*: Ternary structures and groupoids, Czech. Math. J. *41 (116)* (1991), 90–98.

[6] *M. Novotný*: Construction of all strong homomorphisms of binary structures, Czech. Math. J. *41 (116)* (1991), 300–311.

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