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ON SOME DOUBLE ALMOST LACUNARY SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS

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Abstract. In this paper we introduce a new concept for almost lacunary strong P-convergent with respect to an Orlicz function and examine some properties of the resulting sequence space. We also introduce and study almost lacunary statistical convergence for double sequences and we shall also present some inclusion theorems.

1. Introduction and Background

Before we enter the motivation for this paper and the presentation of the main results we give some preliminaries.

By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{k,l})$ has **Pringsheim limit** L (denoted by P-lim $x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$ [6]. We shall write more briefly as "**P-convergent**".

Recently Moricz and Rhoades [3] defined almost P- convergent sequences as follows:

Definition 1.1. *A double sequence $x = (x_{k,l})$ of real numbers is called almost P-convergent to a limit L if*

$$P - \lim_{p,q\to\infty} \sup_{m,n\geq 0} \frac{1}{pq} \sum_{k=m}^{m+p-1} \sum_{l=n}^{n+q-1} |x_{k,l} - L| = 0$$

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that is , the average value of $(x_{k,l})$ taken over any rectangle $\{(k,l) : m \leq k \leq m+p-1, n \leq l \leq n+q-1\}$ tends to L as both p and q tend to ∞, and this P-convergence is uniform in m and n. Let denote the set of sequences with this property as $[\hat{c}^2]$.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, ...$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1}$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

Using these notations we now present the following definition:

Definition 1.2. *The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called* **double lacunary** *if there exist two increasing sequences of integers such that*

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Let $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$, and $\theta_{r,s}$ is determine by $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \ \& \ l_{s-1} < l \leq l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\bar{q}_s = \frac{l_s}{l_{s-1}}$, and $q_{r,s} = q_r \bar{q}_s$.

Recall in [2] that an Orlicz function M is continuous, convex, nondecreasing function define for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$. If convexity of Orlicz function is replaced by $M(x+y) \leq M(x)+M(y)$ then this function is called the modulus function which is defined and characterized by Ruckle [7].

Definition 1.3. *Let M be an Orlicz function and $P = (p_{k,l})$ be any factorable double sequence of strictly positive real numbers, we define the following sequence spaces:*

(1) $[AC_{\theta_{r,s}}, M, P] = \{x = (x_{k,l}) :$
$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M \left(\frac{|x_{k+m,l+n} - L|}{\rho} \right) \right]^{p_{k,l}} = 0,$
uniformly in m and n for some L and $\rho > 0\}$

(2) $[AC_{\theta_{r,s}}, M, P]_0 = \{x = (x_{k,l}) :$
$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M \left(\frac{|x_{k+m,l+n}|}{\rho} \right) \right]^{p_{k,l}} = 0,$
uniformly in m and n, for some $\rho > 0\}$

We shall denote $[AC_{\theta_{r,s}}, M, P]$ and $[AC_{\theta_{r,s}}, M, P]_0$, as $[AC_{\theta_{r,s}}, M]$,and $[AC_{\theta_{r,s}}, M]_0$, respectively when $p_{k,l} = 1$ for all k and l. If x is in $[AC_{\theta_{r,s}}, M]$, we shall say that x is almost lacunary strongly P-convergent with respect to the Orlicz function M. Also note if $M(x) = x$, $p_{k,l} = 1$ for all k and l, then $[AC_{\theta_{r,s}}, M, P] = [AC_{\theta_{r,s}}]$ and $[AC_{\theta_{r,s}}, M, P]_0 = [AC^0_{\theta_{r,s}}]$ which are defined as follows:

$$[AC_{\theta_{r,s}}] = \{x = (x_{k,l}) : \text{ for some } L, P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} |x_{k+m,l+n} - L| = 0, \text{ uniformly in } m \text{ and } n\}$$

and

$$[AC^0_{\theta_{r,s}}] = \left\{ x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} |x_{k+m,l+n}| = 0, \text{uniformly in } m \text{ and } n \right\}.$$

Again note if $p_{k,l} = 1$ for all k and l, then $[AC_{\theta_{r,s}}, M, P] = [AC_{\theta_{r,s}}, M]$ and $[AC_{\theta_{r,s}}, M, P]_0 = [AC_{\theta_{r,s}}, M]_0$. We define

$$[AC_{\theta_{r,s}}, M] = \{x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M \left(\frac{|x_{k+m,l+n} - L|}{\rho} \right) \right] = 0, \text{ uniformly in } m \text{ and } n \text{ for some } L \text{ and } \rho > 0\}$$

and

$$[AC_{\theta_{r,s}}, M]_0 = \{x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M \left(\frac{|x_{k+m,l+n}|}{\rho} \right) \right] = 0, \text{ uniformly in } m \text{ and } n, \text{ for some } \rho > 0\}$$

Let us extend almost P-convergent double sequences to Orlicz function as follows:

Definition 1.4. *Let M be an Orlicz function and $P = (p_{k,l})$ be any factorable double sequence of strictly positive real numbers, we define the following sequence space:*

$$[\hat{c}^2, M, P] = \{x = (x_{k,l}) : P - \lim_{pq} \frac{1}{p,q} \sum_{k,l=1,1}^{p,q} \left[M \left(\frac{|x_{k+m,l+n} - L|}{\rho} \right) \right]^{p_{k,l}} = 0, \textit{ uniformly in } m \textit{ and } n, \textit{ for some } \rho > 0\}$$

If we take $M(x) = x$, $p_{k,l} = 1$ for all k and l, then $[\hat{c}^2, M, P] = [\hat{c}^2]$ which was defined above.

With these new concepts we can now consider the following theorem. The proof of the first theorem is standard thus omitted.

Theorem 1.1. *For any Orlicz function M and a bounded factorable positive double number sequence $p_{k,l}$, $[AC_{\theta_{r,s}}, M, P]$ and $[AC_{\theta_{r,s}}, M, P]_0$ are linear spaces.*

Definition 1.5. [2]. *An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$.*

Before the proof of the theorem we need the following lemma.

Lemma 1.1. *Let M be an Orlicz function which satisfies Δ_2- condition and let $0 < \delta < 1$. Then for each $x \geq \delta$ we have $M(x) < K\delta^{-1}M(2)$ for some constant $K > 0$.*

Theorem 1.2. *For any Orlicz function M which satisfies Δ_2- condition we have $[AC_{\theta_{r,s}}] \subseteq [AC_{\theta_{r,s}}, M]$.*

Proof. Let $x \in [AC_{\theta_{r,s}}]$ so that for each m and n

$$A_{r,s} = \left\{ x = (x_{k,l}) : \text{ for some } L, P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} |x_{k+m,l+n} - L| = 0 \right\}.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \epsilon$ for every t with $0 \leq t \leq \delta$. We obtain the following:

$$\begin{aligned}
& \frac{1}{h_{r,s}} \sum_{k \in I_{r,s}} M(|x_{k+m,l+n} - L|) \\
= & \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s} \& |x_{k+m,l+n} - L| \leq \delta} M(|x_{k+m,l+n} - L|) \\
+ & \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s} \& |x_{k+m,l+n} - L| > \delta} M(|x_{k+m,l+n} - L|) \\
\leq & \frac{1}{h_{r,s}} h_{r,s}\epsilon + \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s} \& |x_{k+m,l+n} - L| > \delta} M(|x_{k+m,l+n} - L|) \\
< & \frac{1}{h_{r,s}} (h_{r,s}\epsilon) + \frac{1}{h_{r,s}} K\delta^{-1} M(2) h_{r,s} A_{r,s}.
\end{aligned}$$

Therefore by lemma as r and s goes to infinity in the Pringsheim sense, for each m and n we are granted $x \in [AC_{\theta_{r,s}}, M]$. □

Theorem 1.3. *Let $\theta_{r,s} = \{k_r, l_s\}$ be a double lacunary sequence with $\liminf_r q_r > 1$, and $\liminf_s \bar{q}_s > 1$ then for any Orlicz function M, $[\hat{c}^2, M, P] \subset [AC_{\theta_{r,s}}, M, P]$.*

Proof. It is sufficient to show that $[\hat{c}^2, M, P]_0 \subset [AC_{\theta_{r,s}}, M, P]_0$. The general inclusion follows by linearity. Suppose $\liminf_r q_r > 1$ and $\liminf_s \bar{q}_s > 1$; then there exists $\delta > 0$ such that $q_r > 1 + \delta$ and $\bar{q}_s > 1 + \delta$. This implies

$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$ and $\frac{\bar{h}_s}{l_s} \geq \frac{\delta}{1+\delta}$. Then for $x \in [\hat{c}^2, M, P]_0$, we can write for each m and n

$$
\begin{aligned}
B_{r,s} &= \frac{1}{h_{rs}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|x_{k+m,l+n}|}{\rho} \right) \right]^{p_{k,l}} \\
&= \frac{1}{h_{rs}} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} \left[M\left(\frac{|x_{k+m,l+n}|}{\rho} \right) \right]^{p_{k,l}} \\
&- \frac{1}{h_{rs}} \sum_{k=1}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} \left[M\left(\frac{|x_{k+m,l+n}|}{\rho} \right) \right]^{p_{k,l}} \\
&- \frac{1}{h_{rs}} \sum_{k=k_{r-1}+1}^{k_r} \sum_{l=1}^{l_{s-1}} \left[M\left(\frac{|x_{k+m,l+n}|}{\rho} \right) \right]^{p_{k,l}} \\
&- \frac{1}{h_{rs}} \sum_{l=l_{s-1}+1}^{l_s} \sum_{k=1}^{k_{r-1}} \left[M\left(\frac{|x_{k+m,l+n}|}{\rho} \right) \right]^{p_{k,l}} \\
&= \frac{k_r k_s}{h_{rs}} \left(\frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} \left[M\left(\frac{|x_{k+m,l+n}|}{\delta} \right) \right]^{p_{k,l}} \right) \\
&- \frac{k_{r-1} l_{s-1}}{h_{rs}} \left(\frac{1}{k_{r-1} l_{s-1}} \sum_{k=1}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} \left[M\left(\frac{|x_{k+m,l+n}|}{\rho} \right) \right]^{p_{k,l}} \right) \\
&- \frac{1}{h_r} \sum_{k=k_{r-1}+1}^{k_r} \frac{l_{s-1}}{h_s} \frac{1}{l_{s-1}} \sum_{l=1}^{l_{s-1}} \left[M\left(\frac{|x_{k+m,l+n}|}{\rho} \right) \right]^{p_{k,l}} \\
&- \frac{1}{h_s} \sum_{l=l_{s-1}+1}^{l_s} \frac{k_{r-1}}{h_r} \frac{1}{k_{r-1}} \sum_{k=1}^{k_{r-1}} \left[M\left(\frac{|x_{k+m,l+n}|}{\rho} \right) \right]^{p_{k,l}}.
\end{aligned}
$$

Since $x \in [\hat{c}^2, M, P]$ the last two terms tends to zero uniformly in m, n in the Pringsheim sense, thus for each m and n

$$
\begin{aligned}
B_{r,s} &= \frac{k_r k_s}{h_{rs}} \left(\frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} \left[M\left(\frac{|x_{k+m,l+n}|}{\rho} \right) \right]^{p_{k,l}} \right) \\
&- \frac{k_{r-1} l_{s-1}}{h_{rs}} \left(\frac{1}{k_{r-1} l_{s-1}} \sum_{k=1}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} \left[M\left(\frac{|x_{k+m,l+m}|}{\rho} \right) \right]^{p_{k,l}} \right) + o(1).
\end{aligned}
$$

Since $h_{rs} = k_r l_s - k_{r-1} l_{s-1}$ we are granted for each m and n the following:

$$\frac{k_r l_s}{h_{rs}} \leq \frac{1+\delta}{\delta} \text{ and } \frac{k_{r-1} l_{s-1}}{h_{rs}} \leq \frac{1}{\delta}.$$

The terms

$$\frac{1}{k_r l_s}\sum_{k=1}^{k_r}\sum_{l=1}^{l_s}\left[M\left(\frac{|x_{k+m,l+n}|}{\rho}\right)\right]^{p_{k,l}}$$

and

$$\frac{1}{k_{r-1} l_{s-1}}\sum_{k=1}^{k_{r-1}}\sum_{l=1}^{l_{s-1}}\left[M\left(\frac{|x_{k+m,l+n}|}{\rho}\right)\right]^{p_{k,l}}$$

are both Pringsheim null sequences for all m and n. Thus $B_{r,s}$ is a Pringsheim null sequence for each m and n. Therefore x is in $[AC_{\theta_{r,s}}, M, P]_0$. This completes the proof of this theorem. $\square$

Theorem 1.4. *Let $\theta_{r,s} = \{k, l\}$ be a double lacunary sequence with $\limsup_r q_r < \infty$ and $\limsup_r \bar{q}_r < \infty$ then for any Orlicz function M, $[AC_{\theta_{r,s}}, M, P] \subset [\hat{c}^2, M, P]$.*

Proof. Since $\limsup_r q_r < \infty$ and $\limsup_s \bar{q}_s < \infty$ there exists $H > 0$ such that $q_r < H$ and $\bar{q}_s < H$ for all r and s. Let $x \in [AC_{\theta_{r,s}}, M, P]$ and $\epsilon > 0$. Also there exist $r_0 > 0$ and $s_0 > 0$ such that for every $i \geq r_0$ and $j \geq s_0$ and all m and n,

$$A'_{i,j} = \frac{1}{h_{ij}}\sum_{(k,l)\in I_{i,j}}\left[M\left(\frac{|x_{k+m,l+n}|}{\rho}\right)\right]^{p_{k,l}} < \epsilon.$$

Let $M^{'} = \max\{A^{'}_{i,j} : 1 \le i \le r_0 \text{ and } 1 \le j \le s_0\}$, and p and q be such that $k_{r-1} < p \le k_r$ and $l_{s-1} < q \le l_s$. Thus we obtain the following:

$$
\begin{aligned}
&\frac{1}{pq}\sum_{k,l=1,1}^{p,q}\left[M\left(\frac{|x_{k+m,l+n}|}{\rho}\right)\right]^{p_{k,l}} \\
\le\ & \frac{1}{k_{r-1}l_{s-1}}\sum_{k,l=1,1}^{k_r l_s}\left[M\left(\frac{|x_{k+m,l+n}|}{\rho}\right)\right]^{p_{k,l}} \\
\le\ & \frac{1}{k_{r-1}l_{s-1}}\sum_{t,u=1,1}^{r,s}\left(\sum_{k,l\in I_{t,u}}\left[M\left(\frac{|x_{k+m,l+n}|}{\rho}\right)\right]^{p_{k,l}}\right) \\
=\ & \frac{1}{k_{r-1}l_{s-1}}\sum_{t,u=1,1}^{r_0,s_0} h_{t,u}A^{'}_{t,u} + \frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<t\le r)\cup(s_0<u\le s)} h_{t,u}A^{'}_{t,u} \\
\le\ & \frac{M^{'}}{k_{r-1}l_{s-1}}\sum_{t,u=1,1}^{r_0,s_0} h_{t,u} + \frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<t\le r)\cup(s_0<u\le s)} h_{t,u}A^{'}_{t,u} \\
\le\ & \frac{M^{'}k_{r_0}l_{s_0}r_0 s_0}{k_{r-1}l_{s-1}} + \frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<t\le r)\cup(s_0<u\le s)} A^{'}_{t,u}h_{t,u} \\
\le\ & \frac{M^{'}k_{r_0}l_{s_0}r_0 s_0}{k_{r-1}l_{s-1}} + \left(\sup_{t\ge r_0\cup u\ge s_0} A^{'}_{t,u}\right)\frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<t\le r)\cup(s_0<u\le s)} h_{t,u} \\
\le\ & \frac{M^{'}k_{r_0}l_{s_0}r_0 s_0}{k_{r-1}l_{s-1}} + \frac{1}{k_{r-1}l_{s-1}}\epsilon\sum_{(r_0<t\le r)\cup(s_0<u\le s)} h_{t,u} \\
\le\ & \frac{M^{'}l_{r_0}l_{s_0}r_0 s_0}{k_{r-1}l_{s-1}} + \epsilon H^2.
\end{aligned}
$$

Since k_r and l_s both approaches infinity as both p and q approaches infinity, it follows that

$$\frac{1}{pq}\sum_{k,l=1,1}^{p,q}\left[M\left(\frac{|x_{k+m,l+n}|}{\rho}\right)\right]^{p_{k,l}} \to 0, \text{uniformly in } m \text{ and } n.$$

Therefore $x \in [\hat{c}^2, M, P]$. □

The following is an immediate consequence of Theorem 2 and Theorem 3

Theorem 1.5. *Let $\theta_{r,s} = \{k,l\}$ be a double lacunary sequence with $1 < \liminf_{r,s} q_{rs} \le \limsup_{r,s} q_{r,s} < \infty$, then for any Orlicz function M, $[AC_{\theta_{r,s}}, M, P] = [\hat{c}^2, M, P]$.*

Quite recently, Savaş and Nuray [8] defined almost lacunary statistical convergence for single sequence by combining lacunary sequence and almost convergence as follows:

Definition 1.6. *Let $\theta_{r,s}$ be a lacunary sequence; the number sequence x is $\hat{S}_{\theta_{r,s}} - P -$ convergent to L provided that for every $\epsilon > 0$,*

$$P - \lim_{r} \frac{1}{h_r} \max_{m} |\{k \in I_r : |x_{k+m} - L| \geq \epsilon\}| = 0.$$

In this case we write $\hat{S}_\theta - \lim x = L$. Now we extend this definition for double sequences.

Definition 1.7. *Let $\theta_{r,s}$ be a double lacunary sequence; the double number sequence x is $\hat{S}_{\theta_{r,s}} - P -$ convergent to L provided that for every $\epsilon > 0$,*

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \max_{m,n} |\{(k,l) \in I_{r,s} : |x_{k+m,l+n} - L| \geq \epsilon\}| = 0.$$

In this case we write $\hat{S}_{\theta_{r,s}} - \lim x = L$. The following theorem is a multidimensional analog of Savas and Nuray theorem presented in [8].

Theorem 1.6. *Let $\theta_{r,s}$ be a double lacunary sequence then*

- **A:** *$x_{k,l} \xrightarrow{P} L[AC_{\theta_{r,s}}]$ implies $x_{k,l} \xrightarrow{P} L(\hat{S}_{\theta_{r,s}})$.*
- **B:** *$[AC_{\theta_{r,s}}]$ is a proper subset of $\hat{S}_{\theta_{r,s}}$*
- **C:** *If $x \in l''_\infty$ and $x_{k,l} \xrightarrow{P} L(\hat{S}_{\theta_{r,s}})$ then $x_{k,l} \xrightarrow{P} L[AC_{\theta_{r,s}}]$*
- **D:** *$\hat{S}_{\theta_{r,s}} \cap l''_\infty = [AC_{\theta_{r,s}}] \cap l''_\infty$*

where l''_∞ is the space of all bounded double sequence.

Proof. (A) Since for all m and n

$$\begin{aligned} |\{(k,l) \in I_{r,s} : |x_{k+m,l+n} - L| \geq \epsilon\}| &\leq \sum_{(k,l)\in I_{r,s} \& |x_{k+m,l+n} - L| \geq \epsilon} |x_{k+m,l+n} - L| \\ &\leq \sum_{(k,l)\in I_{r,s}} |x_{k+m,l+n} - L| \end{aligned}$$

and for all m and n

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |x_{k+m,l+n} - L| = 0.$$

This implies that for all m and n

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : |x_{k+m,l+n} - L| \geq \epsilon\}| = 0.$$

This complete the proof of (A).

(B) let x be defined as follows:

$$x_{k,l} := \begin{pmatrix} 1 & 2 & 3 & \cdots & [\sqrt[3]{h_{r,s}}] & 0 & \cdots \\ 2 & 2 & 3 & \cdots & [\sqrt[3]{h_{r,s}}] & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & [\sqrt[3]{h_{r,s}}] & \cdots & \cdots & [\sqrt[3]{h_{r,s}}] & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It is clear that x is an unbounded double sequence and for $\epsilon > 0$ and for all m and n

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : |x_{k+m,l+n} - L| \geq \epsilon\}| = P - \lim_{r,s} \frac{[\sqrt[3]{h_{r,s}}]}{h_{r,s}} = 0.$$

Therefore $x_{k,l} \xrightarrow{P} 0(\mathbf{S}_{\theta_{r,s}})$. Also note

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |x_{k,l}| = P - \lim_{r,s} \frac{[\sqrt[3]{h_{r,s}}]([\sqrt[3]{h_{r,s}}]([\sqrt[3]{h_{r,s}}]+1))}{2h_{r,s}} = \frac{1}{2}.$$

Therefore $x_{k,l} \overset{P}{\not\to} 0[AC_{\theta_{r,s}}]$. This completes the proof of (B).

(C)If $x \in l''_{\infty}$ and $x_{k,l} \xrightarrow{P} L(\hat{S}_{\theta_{r,s}})$then $x_{k,l} \xrightarrow{P} L[AC_{\theta_{r,s}}]$. Suppose $x \in l''_{\infty}$ then for all m and n, $|x_{k+m,l+n} - L| \leq M$ for all K. Also for given $\epsilon > 0$ and r and s large for all m and n we obtain the following:

$$\begin{aligned} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |x_{k,l} - L| &= \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s} \& |x_{k+m,l+n}-L|\geq\epsilon} |x_{k+m,l+n} - L| \\ &+ \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s} \& |x_{k+m,l+n}-L|\leq\epsilon} |x_{k+m,l+n} - L| \\ &\leq \frac{M}{h_{r,s}} |\{(k,l) \in I_{r,s} : |x_{k+m,l+n} - L| \geq \epsilon\}| + \epsilon. \end{aligned}$$

Therefore $x \in l''_{\infty}$ and $x_{k,l} \xrightarrow{P} L(\hat{S}_{\theta_{r,s}})$ implies $x_{k,l} \xrightarrow{P} L[AC_{\theta_{r,s}}]$

(D) $[AC_{\theta_{r,s}}] \cap l''_{\infty} = \hat{S}_{\theta_{r,s}} \cap l''_{\infty}$ follows directly from (A), (B), and (C). $\square$

We shall now establish an inclusion theorem between $[AC_{\theta_{r,s}}, M]$ and $\hat{S}_{\theta_{r,s}}$.

Theorem 1.7. *For any Orlicz function M, $[AC_{\theta_{r,s}}, M] \subset \hat{S}_{\theta_{r,s}}$.*

Proof. Let $x \in [AC_{\theta_{r,s}}, M]$ and $\epsilon > 0$. Then for all m and n,

$$\frac{1}{h_{rs}} \sum_{(k,l)\in I_{r,s}} M\left(\frac{|x_{k+m,l+n} - L|}{\rho}\right)$$

$$\geq \frac{1}{h_{rs}} \sum_{(k,l)\in I_{r,s}:|x_{k+m,l+n}-L|\geq\epsilon} M\left(\frac{|x_{k+m,l+n} - L|}{\rho}\right)$$

$$> \frac{1}{h_{rs}} M\left(\frac{\epsilon}{\rho}\right) |\{(k,l) \in I_{r,s} : |x_{k+m,l+n} - L| \geq \epsilon\}|.$$

This implies that $x \in \hat{S}_{\theta_{r,s}}$. □

References

[1] A. R. Freedman, J. J. Sember, and M. Raphael, Some Cesàro Type Summability Spaces, Proc. London Math. Soc., 37 (1978) 508-520.

[2] M. A. Krasnoselskii and Y. B. Rutisky, Convex function and Orlicz spaecs, Groningen, Netherlands, 1961.

[3] F. Moricz and B. E. Rhoades, Almost convergence of double sequences and strong regularity of Summability matrices, Math. Proc. Camb. Phil. Soc. 104 (1988) 283-293.

[4] Mursaleen and O. H. Edely, Statistical convergence of double sequences, J. Math. Anal.Appl. 288 (2003), no. 1, 223–231.

[5] R. F. Patterson, Analogues of some Fundamental Theorems of Summability Theory, Internat. J. Math. & Math. Sci. 23(1) 2000, 1-9.

[6] A. Pringsheim, Zur theorie der zweifach unendlichen Zahlenfolgen, Mathematische Annalen, 53 (1900) 289 - 321.

[7] W. H. Ruckle, FK Spaces in which the sequence of coordinate vectors in bounded , Canada J. Math. 25 (1973) 973-978.

[8] E. Savas, F. Nuray, On σ - statistically convergence and Lacunary σ-statistically convergence, Math.Slovaca.43(1993), 309 - 315.

[9] E. Savas, R. F. Patterson, Lacunary statistical convergence of multiple sequence, Appl. Math. Lett.(Preprint).

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