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RÜDIGER VALK

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ON SOME ELEMENTARY PROPERTIES OF UNIFORM AUTOMATA

par Rüdiger VALK (1)

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Abstract. — *Uniform automata are special topological automata, where all maps are uniformly continuous. It is shown, that for these automata the existence problem of topological minimal automata and the « topological black box problem » have natural solutions. New uniformities for the state spaces are introduced and their appropriateness for finite approximations is proved.*

1. INTRODUCTION

Many results, known for finite automata can be extended to automata with topological structures and continuous behavior [2]. On the other hand, problems as the questions concerning minimal topological automata involve some difficulties. A deterministic and complete topological automaton $\mathcal{A}$ is called topological minimal [5], if any topological automaton, which is equivalent to $\mathcal{A}$ (in the usual sense), can be mapped by a continuous automaton homomorphism onto $\mathcal{A}$. In [5] it is shown, that for the subclass of topological automata with locally compact input space topological minimal automata always exist. The same is shown in [3] for the subclass of 'compactly generated' automata. Furthermore in [3] automata are introduced, the next state function of which are only independently continuous in every component. Therefore these automata are not topological automata in the usual sense, but the existence of topological minimal automata is proved for this class without additional assumptions [3].

Another question concerning topological automata is a problem, which we call 'the topological black box problem' : Suppose an arbitrary topolo-

(1) Institut für Informatik, Universität Hamburg.

gical automaton $\mathcal{A}$ is given, but the topology of the state space is unknown. Hence, all we know, is the algebraic structure of $\mathcal{A}$ and the 'external' topological structures of the input and output spaces. The question is, whether an 'internal' topology for the state space can be constructed such, that $\mathcal{A}$ is again a topological automaton, i.e. such, that the 'internal' behavior of $\mathcal{A}$ is continuous. If, in particular, this new topology has better compactness properties, it becomes easier to find finite approximations of the automaton. Clearly, this has some importance for practical applications.

In this paper topological automata are studied in the category of uniform spaces and uniformly continuous maps. Such automata are called uniform automata.

It is shown, that the problem of topological minimal automata and the topological black box problem have quite natural solutions for this class of uniform automata. The solutions, which are new uniformities for the state spaces, are characterized as minimal solutions. These uniformities have been introduced in 1973 by the author [6], and it is now proved, that they are precompact under mild assumptions. This shows the appropriateness of these uniformities for finite approximations. Furthermore it turns out, that these uniformities and metrics are strongly related to well-known properties of classical discrete automata.

We first give some general definitions, which are needed in the following expositions.

2. GENERAL DEFINITIONS

Definition 1

a) A partial map f from M into N is denoted by $f : (M) \to N$. $D(f) \subset M$ is the domain of f. If $D(f) = \emptyset$, we write $f = \emptyset$.

b) If $f : (L) \to M$ and $g : (M) \to N$ are partial maps, then $g \circ f : (L) \to N$ is the composition of f and g and is defined by

$$D(g \circ f) := \{ 1 \in D(f) \mid f(1) \in D(g) \}$$

and $(g \circ f)(1) := g(f(1))$ for all $1 \in D(g \circ f)$.

c) If $f : (M) \to N$ and $f' : (M') \to N'$ are partial maps, then we define the product $f \times f' : (M \times M') \to N \times N'$ by $D(f \times f') := D(f) \times D(f')$ and $(f \times f')(m, m') := (f(m), f'(m'))$ for all $(m, m') \in D(f \times f')$.

d) If $f : (M \times M') \to N$ is a partial map, then for all $m \in M$ a partial map $f_m : (M') \to N$ is defined by $D(f_m) := \{ m' \mid (m, m') \in D(f) \}$ and $f_m(m') := f(m, m')$ for all $m' \in D(f_m)$.

e) If M is a set, the identity map on M is denoted by id_M and $\Delta_M : M \to M \times M$ is the map $m \mapsto (m, m)$.

f) If M is a countable cartesian product, then by pr_i we denote the projection to the *i-th* component.

Maps are special cases of partial maps. Therefore definitions 1 *a*)-1 *d*) can be used for maps also.

Definition 2

A family $G = \{ g^{(i)} : (M) \to N \mid i \in I \}$ of partial maps from a uniform space $(M, \mathfrak{U}_M)$ to a uniform space $(N, \mathfrak{U}_N)$ is called uniformly equicontinuous, if

$$\forall W \in \mathfrak{U}_N \, \exists V \in \mathfrak{U}_M \, \forall i \in I \, \forall m, m' \in M : \{ m, m' \} \subset D(g^{(i)}) \wedge (m, m') \in V \Rightarrow (g^{(i)}(m), g^{(i)}(m')) \in W$$

A partial map g is uniformly continuous, if the family $\{ g \}$ is uniformly equicontinuous.

Lemma 1. Composition and product of uniformly continuous partial maps are uniformly continuous. id_M and Δ_M are uniformly continuous for any uniform space $(M, \mathfrak{U}_M)$.

The proofs are obvious, as in the case of total maps (cf. [1]).

Definition 3

The set of all non-empty subsets of a set M is denoted by $P(M) := \{ N \mid N \subset M \wedge N \neq \emptyset \}$. If $(M, \mathfrak{U}_M)$ is a uniform space, then for any entourage $V \in \mathfrak{U}_M$ and any subset $N \in P(M)$ we define $V[N] := \{ m \mid \exists n \in N : (n, m) \in V \}$, and

$$\tilde{V} := \{ (N, N') \in P(M) \times P(M) \mid N \subset V[N'] \wedge N' \subset V[N] \}.$$

$\{ \tilde{V} \mid V \in \mathfrak{U}_M \}$ is a base of a uniformity on $P(M)$, which we call the power uniformity of $(M, \mathfrak{U}_M)$ ([1], II, 1, Ex. 5).

On the other side, if $P(M)$ is a uniform space, M is considered as uniform subspace of $P(M)$ by identifying M with the set $\{ \{ m \} \mid m \in M \} \subset P(M)$. If the uniformity of $P(M)$ is the power uniformity, then the original uniformity and the subspace uniformity of M coincide ([1], II, 2, Ex. 6).

3. COMPLETE AND DETERMINISTIC UNIFORM AUTOMATA

Definition 4

An automaton $\mathcal{A} = (Z, A, B, f, g)$ is given by a state set Z, an input set A, an output set B a next state map $f : Z \times A \to Z$ and an output map $g : Z \times A \to B$. $\mathcal{A}$ is called a uniform automaton, if Z, A and B are uniform spaces and f and g are uniformly continuous with respect to the product uniformity on $Z \times A$. The uniformities of Z, A and B are denoted by $\mathcal{U}_Z$, $\mathcal{U}_A$ and $\mathcal{U}_B$, respectively.

Definition 5

As usual, the functions f and g are extended to functions $f^+ : Z \times A^+ \to Z$ and $g^+ : Z \times A^+ \to B$, defined for non-empty finite strings over A. Let $N = \{ 0, 1, 2, \dots \}$ denote the set of nonnegative natural numbers. Then $X := A^N$ and $Y := B^N$ are the sets of infinite sequences over A and B, respectively. In the following, X and Y are always used in this sense. If A and B are uniform spaces, we stipulate, that X and Y are endowed with the corresponding product uniformity. The behavior of the automaton is represented by a function $g^N : Z \times X \to Y$, defined by $g^N(z, x)(n) := g^+(z, x(0) \dots x(n))$ for all $z \in Z$, $x \in X$ and $n \in N$. This representation is chosen, because it can be extended easily to continuous-time automata by substituting the non-negative real numbers for N [7]. Using the notation of definition 1d), we define $G := \{ g_z^N : X \to Y \mid z \in Z \}$. Automata $\mathcal{A} = (Z, A, B, f, g)$ and $\mathcal{A}' = (Z', A, B, f', g')$ are called equivalent, if the corresponding families G and G' are identical as sets. Two states $z, z' \in Z$ are equivalent, if $g_z^N = g_{z'}^N$, and $\mathcal{A}$ is reduced, if no two different states are equivalent.

The notions of equivalent and reduced automata coincide with the usual definitions.

Lemma 2 : Let be A a uniform space and $X = A^N$ the corresponding product space. Then the shift operation $\delta : X \to X$, which is defined by $\delta(x)(n) := x(n + 1)$, is uniformly continuous. By δ^n $(n \geqslant 1)$ we denote the n-fold composition of δ.

Proof : For all $n \in N$ we have : $pr_n \circ \delta = pr_{n+1}$.

QED

Proposition 1. For any uniform automaton $\mathcal{A}$ the function $g^N : Z \times X \to Y$ is uniformly continuous.

Proof : For any $n \in N$ we define $f^{(n)} : Z \times X \to Z$ by

$$f^{(n)}(z, x) := f^+(z, x(0) \dots x(n)).$$

The uniform continuity of $f^{(n)}$ is proved for all $n \in N$ by induction : $f^{(0)}$ is uniformly continuous, because it is the composition of $id_Z \times pr_0 : Z \times X \to Z \times A$ and $f : Z \times A \to Z$. $f^{(n+1)}$ is the composition of the following three functions, which are uniformly continuous [1] :

$$\begin{aligned} & id_Z \times \Delta_X && : Z \times X \to Z \times X \times X \\ & f^{(n)} \times \delta^{n+1} && : Z \times X \times X \to Z \times X \\ & f^{(0)} && : Z \times X \to Z \end{aligned}$$

From this follows, that for any $n \in N$ the maps

$$pr_{n+1} \circ g^N : Z \times X \to B \quad , \quad pr_{n+1} \circ g^N = g \circ (f^{(n)} \times pr_{n+1}) \circ (id_Z \times \Delta_X)$$

and

$$pr_0 \circ g^N : Z \times X \to B \quad , \quad pr_0 \circ g^N = g \circ (id_Z \times pr_0)$$

are uniformly continuous. Hence, $g^N : Z \times X \to Y$ is a uniformly continuous map into the product space Y.

QED

Now, for any automaton $\mathcal{A} = (Z, A, B, f, g)$, where B is a uniform space, we define a uniformity for the state space, with respect to which $\mathcal{A}$ becomes – under a natural condition – a uniform automaton.

Definition 6

Let be $(B, \mathcal{U}_B)$ a uniform space and $\mathcal{A} = (Z, A, B, f, g)$ an automaton. The family G of maps (definition 5) is endowed with the uniformity of uniform convergence ([1], X, 1.1) with respect to $Y = B^N$. Now, the initial uniformity on Z with respect to the map $v : Z \to G, z \mapsto g_z^N$ is called uniformity of uniform convergence on Z and is denoted by $\mathcal{U}_{uc}$. The set of all

$$(V) := \{ (z, z') \in Z \times Z \mid \forall x \in X : (g^N(z, x), g^N(z', x)) \in V \},$$

where V runs through all entourages of $\mathcal{U}_Y$, forms a base of $\mathcal{U}_{uc}$.

Theorem 1 : Let $\mathcal{A} = (Z, A, B, f, g)$ be an automaton, where A and B are uniform spaces and Z is endowed with the uniformity of uniform convergence $\mathcal{U}_{uc}$. Then $\mathcal{A}$ is a uniform automaton, if and only if the family G is uniformly equicontinuous.

Proof : If $\mathcal{A}$ is a uniform automaton, then by proposition 1 $g^N : Z \times X \to Y$ is uniformly continuous. This implies, that G is uniformly equicontinuous ([1], X, 2.1).

Now let G be uniformly equicontinuous. Then the 'evaluation map' $\text{val} : G \times X \to Y, (g_z^N, x) \mapsto g_z^N(x)$ is uniformly continuous, G being endowed with the uniformity of uniform convergence ([1], X, 2.1). The map

$$w : A \times X \to X, \; w(a, x)(n) := \begin{cases} a & \text{if } n = 0 \\ x(n-1) & \text{if } n > 0 \end{cases}$$ is also uniformly continuous ([1], II, 2.6), which implies the uniform continuity of the following composition of maps : $F := \delta \circ \text{val} \circ (v \times w)$

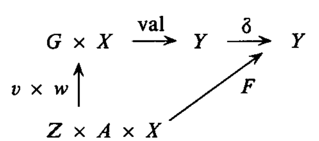

diagram 1

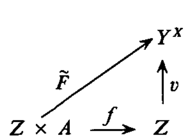

diagram 2

Again by ([1], X, 2.1, Prop. 2) $\{ F_x : Z \times A \to Y \mid x \in X \}$ is uniformly equicontinuous and $\tilde{F} : Z \times A \to Y^X$, defined by $\tilde{F}(z, a)(x) := F(z, a, x)$, is uniformly continuous with respect to the uniformity of uniform convergence on the set Y^X of all maps from X into Y ([1], X, 2.1, Prop. 1).

We now prove, that diagram 2 is commutative. For all $z \in Z$, $a \in A$ and $x \in X$ we have :

$$\begin{aligned} v(f(z, a))(x) &= g^N(f(z, a), x) = \delta(g^N(z, w(a, x)) = (\delta \circ \mathrm{val})(g_z^N, w(a, x)) \\ &= (\delta \circ \mathrm{val} \circ (v \times w))(z, a, x) = F(z, a, x) = \tilde{F}(z, a)(x). \end{aligned}$$

Since the uniformity $\mathcal{U}_{uc}$ of Z is the initial uniformity with respect to the map v, f is uniformly continuous.

To finish the proof, we have to show, that the output map g is uniformly continuous.

$g^N : Z \times X \to Y$ is uniformly continuous, since it is composition of the maps $v \times id_X : Z \times X \to G \times X$ and val : $G \times X \to Y$.

Let $x \in X$ be an arbitrary but fixed element and define

$$p : Z \times A \to Z \times A \times X$$

by $p(z, a) := (z, a, x)$. p is uniformly continuous ([1], II, 2.6).

$$\begin{array}{ccccc} Z \times A & & \xrightarrow{\quad p \quad} & & Z \times A \times X \\ \downarrow g & & & & \downarrow id_Z \times w \\ B & \xleftarrow{pr_0} & Y & \xleftarrow{g^N} & Z \times X \end{array}$$

diagram 3

Diagram 3 is commutative :

$$(pr_0 \circ g^N \circ (id_z \times w) \circ p)(z, a) = pr_0(g^N(z, w(a, x)) = g(z, a)$$

Therefore g is uniformly continuous.

QED

The last proposition and theorem give a solution to the following black box situation.

Given a uniform automaton $\mathcal{A}$, where the (internal) uniformity of the state space Z is unknown. Construct a uniformity for Z such, that $\mathcal{A}$ becomes a uniform automaton.

Clearly, uniformity $\mathcal{U}_{uc}$ for Z is a solution. We give now a characterisation of this solution.

Proposition 2 : For any uniform automaton $\mathcal{A}$ the uniformity $\mathcal{U}_{uc}$ is the coarsest uniformity for the state space, with respect to which $\mathcal{A}$ is again a uniform automaton.

Proof : Let $\mathcal{A}$ be a uniform automaton and Z the state space of $\mathcal{A}$ with uniformity $\mathcal{U}_Z$. By proposition 1 g^N is uniformly continuous and G is uniformly equicontinuous ([1], X, 2.1). Hence, $\mathcal{A}$ is again a uniform automaton if $\mathcal{U}_Z$ is substituted by the uniformity $\mathcal{U}_{uc}$.(theorem 1). If Z_{uc} denotes the set of states of $\mathcal{A}$, supplied with uniformity $\mathcal{U}_{uc}$, we have to prove, that the identity map on Z $id_Z : Z \to Z_{uc}$ is uniformly continuous. Since $g^N : Z \times X \to Y$ is uniformly continuous, $\{ g_x^N : Z \to Y \mid x \in X \}$ is uniformly equicontinuous ([1], X, 2.1, Prop. 2) and from ([1], X, 2.1, Prop. 1) we deduce, that the map $\tilde{g}^N : Z \to Y^X$, $\tilde{g}^N(z)(x) := g^N(z, x)$ is uniformly continuous, if the set Y^X of all maps from X to Y is endowed with the uniformity of uniform convergence. Since Z_{uc} is the initial space with respect to $v : Z_{uc} \to Y^X$, from $\tilde{g}^N = v \circ id_Z$ follows, that $id_Z : Z \to Z_{uc}$ is uniformly continuous.

QED

Definition 7

A uniform automaton $\mathcal{A}$ is called uniformly minimal (or shortly u-minimal), if for any uniform automaton $\mathcal{A}'$, which is equivalent to $\mathcal{A}$, there is an automaton homomorphism from $\mathcal{A}'$ onto $\mathcal{A}$, which is uniformly continuous. $\mathcal{A}$ is called a u-minimal reduction of $\mathcal{A}'$.

This definition is an analogon to the definition of a t-minimal topological automaton in [5].

Theorem 2 : Any uniform automaton $\mathcal{A}$ has a u-minimal reduction and all u-minimal reductions of $\mathcal{A}$ are reduced and uniformly isomorphic.

Proof : For $\mathcal{A} = (Z, A, B, f, g)$ there is an equivalent reduced automaton $\mathcal{A}' = (Z', A, B, f', g')$, which is unique up to automaton isomorphisms and a corresponding automaton homomorphism $\varphi : Z \to Z'$. The families G and G' of $\mathcal{A}$ and $\mathcal{A}'$, respectively, are identical considered as sets. Therefore both G and G' are uniformly equicontinuous and $\mathcal{A}$ and $\mathcal{A}'$ are uniform automata with respect to the uniformity $\mathcal{U}_{uc}$ on their state spaces (theorem 1). The spaces $(Z', \mathcal{U}_{uc})$ and $(G, \mathcal{U}_G)$, where $\mathcal{U}_G$ is the uniformity of uniform convergence on G, are uniformly isomorphic. Hence, $\varphi : (Z, \mathcal{U}_{uc}) \to (Z', \mathcal{U}_{uc})$ is uniformly continuous. But by proposition 2 $\mathcal{U}_{uc}$ is coarser than the original uniformity $\mathcal{U}_Z$ of Z and $\varphi : (Z, \mathcal{U}_Z) \to (Z', \mathcal{U}_{uc})$ is uniformly continuous.

If $\mathcal{A}'' = (Z'', A, B, f'', g'')$ is a second u-minimal reduction, then the canonical homomorphism $\varphi' : (Z', \mathcal{U}_{uc}) \to (Z'', \mathcal{U}_{uc})$ is a uniform isomorphism.

QED

In order to use the uniformity $\mathcal{U}_{uc}$ in practical applications it is useful to consider a corresponding distance function. If (Y, d_Y) is a metric space, then from the definition of $\mathcal{U}_{uc}$ and ([1], X, 3.1) it follows, that $d_Z(z, z') := \sup \{ d_Y(g^N(z, x), g^N(z', x)) \mid x \in X \}$ is a pseudometric for the state space, which generates the uniformity $\mathcal{U}_{uc}$ corresponding to the uniformity of (Y, d_Y). If d_B is a bounded distance function for $\mathcal{U}_B$, then the product uniformity on Y can be generated by the metric $d_Y(y, y') := \sum_{i=0}^{\infty} 3^{-i} d_B(y(i), y'(i))$. For any $n \in N$, two states z and z' are called n-equivalent, if $g^+(z, w) = g^+(z', w)$ for all non-empty words w of length at most $n + 1$.

Proposition 3 : Let $\mathcal{A} = (Z, A, B, f, g)$ be an automaton and d_B be a bounded metric on B. Then the pseudometric d_Z for the state space, just defined, has the following properties :

a) Any two states z and z' are equivalent, if and only if $d_Z(z, z') = 0$. Hence d_Z is a metric, if an only if $\mathcal{A}$ is reduced.

b) If d_B is the discrete metric on B, then z and z' are n-equivalent, if and only if $d_Z(z, z') \leqslant \frac{1}{2} \cdot 3^{-n}$.

Proof : Statement *a*) follows immediately from the definition of d_Z and the fact, that d_Y is a metric.

To prove statement *b*), let z and z' be n-equivalent. Then we have $d_B(g^N(z, x)(m), g^N(z', x)(m)) = 0$ for all $x \in X$ and $0 \leqslant m \leqslant n$, and $d_Z(z, z') \leqslant \sum_{i=n+1}^{\infty} 3^{-i} = \frac{1}{2} \cdot 3^{-n}$.

Now suppose, that z and z' are not n-equivalent, i.e. there is an input $x \in X$ and a number $0 \leqslant m \leqslant n$ such, that $g^N(z, x)(m) \neq g^N(z', x)(m)$. This gives the result $d_B(g^N(z, x)(m), g^N(z', x)(m)) = 1$ and $d_Z(z, z') \geqslant 3^{-m} \geqslant 3^{-n} > \frac{1}{2} \cdot 3^{-n}$.

QED

4. NONDETERMINISTIC AND INCOMPLETE UNIFORM AUTOMATA

Next state and output relation of a nondeterministic and incomplete uniform automaton are defined by a common transition relation t, whereas continuity of these two relations is required independently. We consider only states, for which at least one transition is defined.

Definition 8

A nondeterministic and incomplete automaton (shortly ND-automaton) $\mathcal{A} = (Z, A, B, t)$ is defined by a non-empty state set Z, an input set A, and an output set B and a transition relation $t \subset Z \times A \times B \times Z$ with the property $pr_1(t) = Z$.

$\mathcal{A}$ is called a uniform ND-automaton, if $P(Z)$, A and B are uniform spaces and the following partial maps f and g are uniformly continuous.

$f : (Z \times A) \to P(Z)$, $D(f) := \{ (z, a) \mid \exists b \, \exists z' : (z, a, b, z') \in t \}$ $f(z,a) := \{ z' \mid \exists b : (z, a, b, z') \in t \}$ for all $(z, a) \in D(f)$, $g : (Z \times A) \to P(B)$, $D(g) := D(f)$, $g(z, a) := \{ b \mid \exists z' : (z, a, b, z') \in t \}$ for all $(z, a) \in D(g)$. Z is considered as uniform subspace of $P(Z)$ (cf. definition 3), $Z \times A$ is the product space of Z and A and $P(B)$ is endowed with the power uniformity.

Definition 9

The global output g^N of a ND-automaton is a partial function : $g^N : (Z \times X) \to P(Y)$ and is defined by $D(g^N) := \{ (z, x) \mid \exists tr \in Z^N \, \exists y \in Y : tr(0) = z \wedge \forall n \in N : (tr(n), x(n), y(n), tr(n+1)) \in t \}$, $g^N(z, x) := \{ y \mid \exists tr \in Z^N : tr(0) = z \wedge \forall n \in N : (tr(n), x(n), y(n), tr(n+1)) \in t \}$ for all $(z, x) \in D(g^N)$. (tr stands for ' state trajectory '.)

Again using the notation of definition 1d), we define

$$G := \{ g_z^N : (X) \to P(Y) \mid z \in Z \}.$$

Given a set $Z_1 \subset Z$, the set of input output pairs $b(Z_1) := \{ (x, y) \mid \exists z \in Z_1 : y \in g^N(z, x) \}$ is called the behavior of Z_1. The behavior of a single state $z \in Z$ is defined by $b(z) := b(\{ z \})$.

Lemma 3 : If $(B, \mathfrak{U}_B)$ is a uniform space, then the map : $\beta : P(B)^N \to P(B^N)$, defined by $\beta(M) := \{ y \mid \forall n \in N : y(n) \in M(n) \}$ is uniformly continuous, with respect to the power and product uniformities.

Proof : The set of all $V_n := \{ (y, y') \mid (y(n), y'(n)) \in V \}$, where V runs through $\mathfrak{U}_B$ and n through N, forms a subbase of the product uniformity $\mathfrak{U}_{B^N}$.

To prove the uniform continuity of β, with the notation of definition 3, it suffices to show : $(\beta \times \beta)((\tilde{V})_n) \subset (\widetilde{V_n})$. From $(M, M') \in (\tilde{V})_n$ follows $(M(n), M'(n)) \in \tilde{V}$ and $M(n) \subset V[M'(n)]$ and $M'(n) \subset V[M(n)]$. Hence, for any $y \in \beta(M)$ we have $y(n) \in M(n) \subset V[M'(n)]$, i.e. there is an element $y' \in M'$ such, that $(y(n), y'(n)) \in V$. This implies $\forall y \in \beta(M) \, \exists y' \in \beta(M') : (y, y) \in V_n$ and $\beta(M) \subset V_n[\beta(M')]$. In the same way we show $\beta(M') \subset V_n[\beta(M)]$ and obtain $(\beta(M), \beta(M')) \in (\widetilde{V_n})$.

QED

Proposition 4 : If $\mathcal{A}$ is a uniform ND-automaton such, that

$$P(f) : (P(Z) \times A) \to P(Z), \quad (Z_1, a) \mapsto \bigcup_{z \in Z_1} f(z, a)$$

and

$$P(g) : (P(Z) \times A) \to P(B), \quad (Z_1, a) \mapsto \bigcup_{z \in Z_1} g(z, a)$$

are uniformly continuous, then $g^N : (Z \times X) \to P(Y)$ is uniformly continuous and G is uniformly equicontinuous with respect to the power uniformity on $P(Y)$.

In particular, the assumptions concerning $P(f)$ and $P(g)$ hold for uniform ND-automata, where Z is a uniform space and the uniformity of $P(Z)$ is the power uniformity.

Proof : As in the proof of proposition 1 we define for all $n \in N$ the maps $f^{(n)} : (Z \times X) \to P(Z)$ inductively by $f^{(0)} := f \circ (id_Z \times pr_0)$ and $f^{(n+1)} = P(f) \circ (f^{(n)} \times pr_{n+1}) \circ (id_Z \times \Delta_X)$, which are uniformly continuous also in the case of partial maps. Hence, for all $n \in N$ the maps $\alpha_{n+1} : (Z \times X) \to P(B)$, $\alpha_{n+1} := P(g) \circ (f^{(n)} \times pr_{n+1}) \circ (id_Z \times \Delta_X)$ and $\alpha_0 := g \circ (id_Z \times pr_0)$ and $\alpha : (Z \times X) \to P(B)^N$, defined by $\alpha(z, x)(n) := \alpha_n(z, x)$, are all uniformly continuous. By lemma 3, the partial map $\beta \circ \alpha : (Z \times X) \to P(Y)$ is uniformly continuous. Since any partial map, the graph of which is a subset of the graph of a uniformly continuous partial map, is uniformly continuous itself, it suffices to prove, that the graph of g^N is a subset of the graph of $\beta \circ \alpha$. Therefore let be $(z, x) \in D(g^N)$ and $y \in g^N(z, x)$. By definition of g^N there is a sequence $tr \in Z^N$ such, that $tr(0) = z$ and $(tr(n), x(n), y(n), tr(n + 1)) \in t$ for all $n \in N$. Hence, we have for all $n \in N$: $(z, x) \in D(f^{(n)})$ and $tr(n) \in f^{(n)}(z, x)$ and $y(0) \in g(z, x(0)) = \alpha_0(z, x)$ and $y(n + 1) \in P(g)(f^{(n)}(z, x), x(n + 1)) = \alpha_{n+1}(z, x)$. Therefore we obtain $y(n) \in \alpha(z, x)(n)$ for all $n \in N$ and $y \in (\beta \circ \alpha)(z, x)$.

The uniform equicontinuity of G follows immediately. If $(Z, \mathcal{U}_Z)$ is a uniform space and $f : (Z \times X) \to P(Z)$ is uniformly continuous with respect to the power uniformity on $P(Z)$, then $P(f) : (P(Z) \times X) \to P(Z)$ is uniformly continuous : for $W \in \mathcal{U}_Z$ and $\tilde{W} \in \mathcal{U}_{P(Z)}$ there are entourages $V \in \mathcal{U}_Z$ and $V' \in \mathcal{U}_X$ such, that $(z, z') \in V$ and $(x, x') \in V'$ and $\{ (z, x), (z', x') \} \subset D(f)$ imply $(f(z, x), f(z', x')) \in \tilde{W}$. Hence $(Z_1, Z_2) \in \tilde{V}$, $(x, x') \in V'$ and $\{ (Z_1, x), (Z_2, x') \} \subset D(P(f))$ imply $(P(f)(Z_1, x), P(f)(Z_2, x')) \in \tilde{W}$. The same argument applies to $P(g)$.

QED

Since in the case of ND-automata, the elements of G are partial maps, the uniformity of uniform convegence $\mathcal{U}_{uc}$ cannot be defined for the state space. Therefore we now introduce a second uniformity for Z.

Definition 10

Let $\mathcal{A} = (Z, A, B, t)$ be a ND-automaton and A and B uniform spaces. The initial uniformity on Z with respect to the map $b : Z \to P(X \times Y)$, $z \mapsto b(z)$ and the power uniformity on the product $X \times Y$ is called behavior uniformity on Z and is denoted by $\mathcal{U}_b$. Analogously, the behavior uniformity on $P(Z)$ is the initial uniformity with respect to the map $b : P(Z) \to P(X \times Y)$.

For $V \in \mathcal{U}_X$ and $W \in \mathcal{U}_Y$ we define $((V, W)) := \{ (Z_1, Z_2) \in P(Z) \times P(Z) |$

$$\forall (x, y) \in b(Z_1) \exists (x', y') \in b(Z_2) : (x, x') \in V \wedge (y, y') \in W \wedge$$
$$\forall (x', y') \in b(Z_2) \exists (x, y) \in b(Z_1) : (x', x) \in V \wedge (y', y) \in W \}$$

With definition 3 we can conclude, that $\{ ((V, W)) \mid V \in \mathcal{U}_X \wedge W \in \mathcal{U}_Y \}$ forms a base of $(P(Z), \mathcal{U}_b)$.

The behavior uniformity on Z may be considered as a subspace of the behavior uniformity on $P(Z)$. The elements of the corresponding restriction of the given base are denoted by $(V, W) := \{ (z, z') \mid (\{ z \}, \{ z' \}) \in ((V, W)) \}$.

Proposition 5 : Let $\mathcal{A} = (Z, A, B, t)$ be a ND-automaton, where A and B are uniform spaces and $G = \{ g_z^N : (X) \to P(Y) \mid z \in Z \}$ is uniformly equicontinuous. Furthermore assume :

$$\forall z, z' \in Z \; \forall x \in D(g_z^N) \; \forall V \in \mathcal{U}_X \; \exists x' \in D(g_{z'}^N) : (x, x') \in V.$$

Then the behavior uniformity $\mathcal{U}_b$ is the coarsest uniformity on Z such, that $g^N : (Z \times X) \to P(Y)$ is uniformly continuous.

Proof : Given $\tilde{W} \in \mathcal{U}_{P(Y)}$ there is an entourage $V \in \mathcal{U}_X$ such, that for all $z \in Z : (x, x') \in V^{-1} \circ V \wedge \{ x, x' \} \subset D(g_z^N) \Rightarrow (g_z^N(x), g_z^N(x')) \in \tilde{W}$. Now define $V' := (V, W) \in \mathcal{U}_b$ (cf. definition 10).

Then $(x, x') \in V$ and $(z, z') \in V'$ and $(z, x), (z', x') \in D(g^N)$ imply : for all $(x, y) \in b(z)$, there is a pair $(\tilde{x}, \tilde{y}) \in b(z')$ such, that $(x, \tilde{x}) \in V$ and $(y, \tilde{y}) \in W$. Hence we have $(x', \tilde{x}) \in V^{-1} \circ V$ and $(\tilde{y}, y') \in W$ and $(y, y') \in W \circ W$, which proves the uniform continuity of g^N wih respect to $\mathcal{U}_b$.

If $\mathcal{U}_Z$ is an arbitrary uniformity for Z, with respect to which g^N is uniformly continuous, we must prove $\mathcal{U}_Z \supset \mathcal{U}_b$.

Let be $(V, W) \in \mathcal{U}_b$.

Since g^N is uniformly continuous with respect to $\mathcal{U}_Z$, there are entourages $V_1 \in \mathcal{U}_Z$, $V_2 \in \mathcal{U}_X$ such, that $(z, z') \in V_1$, $(x, x') \in V_2$ and $(z, x), (z', x') \in D(g^N)$ imply $(g^N(z, x), g^N(z', x')) \in \tilde{W}$. By assumption of the proposition for $(z, z') \in V_1$ and $x \in D(g_z^N)$ there is an element x' such, that $(x, x') \in V \cap V_2$. This implies $(g^N(z, x), g^N(z', x')) \in \tilde{W}$. Hence for any $(x, y) \in b(z)$ there is a pair $(x', y') \in b(z')$ with the property $(x, x') \in V$ and $(y, y') \in W$. In the same way one can show : $\forall (x', y') \in b(z') \exists (x, y) \in b(z) : (x, x') \in V \wedge (y, y') \in W$. This gives the desired result : $(z, z') \in (V, W)$ and $V_1 \subset (V, W)$.

QED

Proposition 6 : Let $\mathcal{A} = (Z, A, B, f, g)$ be a (complete and deterministic) automaton and A and B uniform spaces.

Then the behavior uniformity $\mathcal{U}_b$ is coarser than the uniformity of uniform convergence $\mathcal{U}_{uc}$ on Z. The uniformities are identical, if G is uniformly equicontinuous.

Proof : The first statement follows from the definitions of $\mathcal{U}_{uc}$ and $\mathcal{U}_b$ and the observation, that $(W) \subset (V, W)$ holds for all $V \in \mathcal{U}_X$ and $W \in \mathcal{U}_Y$.

If G is uniformly equicontinuous, then by theorem 1 $\mathcal{A}$ is a uniform automaton with respect to $\mathcal{U}_{uc}$. By proposition 5 g^N is uniformly continuous with respect to $\mathcal{U}_b$. But $\mathcal{U}_{uc}$ is the coarsest uniformity with this property and therefore coarser than $\mathcal{U}_b$.

QED

By the last proposition the uniformities $\mathcal{U}_{uc}$ and $\mathcal{U}_b$ are identical for all uniform automata. This result suggests, that the behavior uniformity $\mathcal{U}_b$ is a suitable extension of the uniformity of uniform convergence $\mathcal{U}_{uc}$ to incomplete uniform automata.

Definition 11

A ND-automaton $\mathcal{A} = (Z, A, B, t)$ is called input concatenation preserving, if for any arbitrary state $z \in Z$ and inputs $\{ x, x' \} \subset D(g_z^N)$ also the following input $x(0)x'$, defined by

$$(x(O)x')(n) := \begin{cases} x(O) & \text{if} \quad n = 0 \\ x'(n) & \text{if} \quad n > 0 \end{cases}, \quad \text{is an input for } z, \text{ i.e. } x(O)x' \in D(g_z^N)$$

Intuitively, this property means, that after a transition from a state z to a next state z' by an input $x(0)$ we can switch over to the input $\delta(x')$, if x' was defined for the state z. This property is trivially satisfied for complete automata.

Theorem 3 : Let $\mathcal{A} = (Z, A, B, t)$ be an input concatenation preserving ND-automaton, where A, B and $P(Z)$ are uniform spaces and the uniformity of $P(Z)$ is the behavior uniformity.

Then, if G is a uniformly equicontinuous family of partial maps, $\mathcal{A}$ is a uniform ND-automaton.

Proof : To prove the uniform continuity of $f : (Z \times X) \to P(Z)$, let be $((V, W))$ an entourage of the base of $\mathcal{U}_{P(Z)}$ (definition 10). By definition of uniform spaces there exists an entourage $\mathring{W} \in \mathcal{U}_Y$ such, that $\mathring{W}^{-1} = \mathring{W}$ (symmetry) and $\mathring{W} \circ \mathring{W} \subset W$ ($\circ$ = relational composition). Since the shift operation δ is uniformly continuous, there are elements $W_1 \in \mathcal{U}_Y$ and $V_1 \in \mathcal{U}_X$ such, that $(y, y') \in W_1$ and $(x, x') \in V_1$ imply $(\delta(y), \delta(y')) \in \mathring{W}$ and $(\delta(x), \delta(x')) \in V$.

Since G is uniformly equicontinuous, an entourage $V_2 \in \mathcal{U}_X$ must exist, which satisfies the following condition :

$$\text{(I)} \quad \forall z \in Z\, \forall x, x' \in X : \{ x, x' \} \subset D(g_z^N) \wedge (x, x') \in V_2 \Rightarrow$$
$$\Rightarrow \forall y \in g_z^N(x)\, \exists y' \in g_z^N(x') : (y, y') \in W_1$$
$$\wedge\ \forall y' \in g_z^N(x')\, \exists y \in g_z^N(x) : (y', y) \in W_1$$

As a subbase of X we consider the set of all V_n, which are defined in the proof of lemma 3. We can choose entourages $\mathring{V}_2 \in \mathcal{U}_X$, $V' \in \mathcal{U}_X$ and $V'' \in \mathcal{U}_A$ with the following properties : $(\mathring{V}_2)^{-1} \circ \mathring{V}_2 \subset V_2$, $V' \subset \mathring{V}_2 \cap V_1$, $V'' \subset (pr_0 \times pr_0)(V')$ and V' belongs to the base of $\mathcal{U}_X$.

To prove the uniform continuity of f, it suffices to verify :

$$(z, z') \in (V', W_1) \wedge (a, a') \in V'' \wedge \{ (z, a), (z', a') \} \subset D(f) \wedge$$
$$\hat{Z} = f(z, a) \wedge \hat{Z}' = f(z', a') \Rightarrow (\hat{Z}, \hat{Z}') \in ((V, W)).$$

Therefore let be $(x, y) \in b(\hat{Z})$. Now we must find a pair $(x', y') \in b(\hat{Z}')$ such, that $(x, x') \in V$ and $(y, y') \in W$ is satisfied.

$(x, y) \in b(\hat{Z})$ implies, that there exist a pair $(\hat{x}, \hat{y}) \in b(z)$ satisfying $(\delta(\hat{x}), \delta(\hat{y})) = (x, y)$. By $(z, z') \in (V', W_1)$ we know, that there is a pair $(\tilde{x}, \tilde{y}) \in b(z')$ with the properties :

$$\text{(II)} \qquad (\hat{x}, \tilde{x}) \in V'$$

and

$$\text{(III)} \qquad (\hat{y}, \tilde{y}) \in W_1.$$

From $(a, a') \in V''$ follows $(a\tilde{x}, a'\tilde{x}) \in V' \subset \mathring{V}_2$ and formula (II) implies $(a\tilde{x}, \tilde{x}) \in V' \subset \mathring{V}_2$. Combining these results, we obtain $(a'\tilde{x}, \tilde{x}) \in (\mathring{V}_2)^{-1} \circ \mathring{V}_2 \subset V_2$. By assumption of the theorem we know $\{ a'\tilde{x}, \tilde{x} \} \subset D(g_{z'}^N)$. Using formula (I), for $\tilde{y} \in g_{z'}^N(\tilde{x})$ we get an element $y'' \in g_{z'}^N\ (a'\tilde{x})$, which satisfies :

$$\text{(IV)} \qquad (\tilde{y}, y'') \in V_1.$$

By $(a'\tilde{x}, y'') \in b(z')$ we obtain $(x', y') := (\delta(a'\tilde{x}), \delta(y'')) \in b(\hat{Z}')$.

From (III), (IV) and the definition of W_1 follows $(\delta(\tilde{y}), \delta(y'')) \in \mathring{W}$, $(\delta(\hat{y}), \delta(\tilde{y})) = (y, \delta(\tilde{y})) \in \mathring{W}$ and $(y, y') \in W$. (II) implies $(\hat{x}, \tilde{x}) \in V_1$ and $(\delta(\hat{x}), \delta(\tilde{x})) = (x, x') \in V$.

In the same way for all $(x', y') \in b(\hat{Z}')$ we can find a pair $(x, y) \in b(\hat{Z})$ satisfying $(x', x) \in V$ and $(y', y) \in W$, which establishes the result $(\hat{Z}, \hat{Z}') \in ((V, W))$.

We now prove, that $g : (Z \times A) \to P(B)$ is uniformly continuous. Let be $W \in \mathcal{U}_B$ and $\tilde{W} \in \mathcal{U}_{P(B)}$. Then there is an entourage $W' \in \mathcal{U}_Y$ such, that

$(pr_0 \times pr_0)(W') \subset W$. Let be $W_1 \in \mathcal{U}_Y$ having the property $W_1 \circ W_1 \subset W'$. For this entourage we can find an entourage $V_2 \in \mathcal{U}_X$, which satisfies formula (I). Furthermore let be $\mathring{V}_2 \in \mathcal{U}_X$ and $V_3 \in \mathcal{U}_A$ entourages with the properties $(\mathring{V}_2)^{-1} \circ \mathring{V}_2 \subset V_2$ and $V_3 \subset (pr_0 \times pr_0)(\mathring{V}_2)$ and $\mathring{V}_2$ belongs to the base of $\mathcal{U}_X$, mentioned above.

Now for $(z, z') \in (\mathring{V}_2, W_1)$, $(a, a') \in V_3$ and $\{ (z, a), (z', a') \} \in D(g)$ it suffices to prove $(g(z, a), g(z', a')) \in (W)$ or equivalently

$$\forall b \in g(z, a) \, \exists b' \in g(z', a') : (b, b') \in W$$

and

$$\forall b' \in g(z', a') \, \exists b \in g(z, a) : (b', b) \in W.$$

Let be $b \in g(z, a)$. By the definition of a ND-automaton an element $x \in D(f(z, a))$ exists. Furthermore there is a pair $(\hat{x}, \hat{y}) \in b(z)$ such, that $\delta(\hat{x}) = x$, $\hat{x}(0) = a$ and $\hat{y}(0) = b$ and by $(z, z') \in (\mathring{V}_2, W_1)$ another pair $(\tilde{x}, \tilde{y}) \in b(z')$ exists, which satisfies

$$\text{(V)} \qquad (\hat{x}, \tilde{x}) \in \mathring{V}_2$$

and

$$\text{(VI)} \qquad (\hat{y}, \tilde{y}) \in W_1.$$

From (V) follows $(a\tilde{x}, \tilde{x}) \in \mathring{V}_2$ and from $(a, a') \in V_3$ we obtain $(a\tilde{x}, a'\tilde{x}) \in \mathring{V}_2$ and $(a'\tilde{x}, \tilde{x}) \in (\mathring{V}_2)^{-1} \circ \mathring{V}_2 \subset V_2$. Since $\tilde{x} \in D(g^N_{z'})$ and $a'\tilde{x} \in D(g^N_{z'})$, which is true by assumption of the theorem, we can use formula (I). Thus, for $\tilde{y} \in g^N_{z'}(\tilde{x})$ there is an element $y' \in g^N_{z'}(a'\tilde{x})$, which satisfies $(\tilde{y}, y') \in W_1$. Together with (VI) this implies $(\hat{y}, y') \in W_1 \circ W_1 \subset W'$ and $(\hat{y}(0), y'(0)) = (b, y'(0)) \in W$ and $b' := y'(0)$ is the required element $b' \in g(z', a')$.

In the same manner for a given $b' \in g(z', a')$, we can find an element $b \in g(z, a)$ such, that $(b', b) \in W$ is satisfied.

QED

Definition 12

An incomplete automaton $\mathcal{A}$ is a ND-automaton such, that the range of f is Z and the range of g is B (considered as subsets of $P(Z)$ and $P(B)$, respectively).

$\mathcal{A}$ is a uniform incomplete automaton, if A, B and Z are uniform spaces and f and g are uniformly continuous partial maps.

Similar to theorem 1 the condition in theorem 3 is also necessary, if we consider incomplete automata. We recall, that in this case again the uniformity of uniform convergence cannot be used.

Corollary : Let $\mathcal{A} = (Z, A, B, t)$ be an input concatenation preserving incomplete automaton, where A and B are uniform spaces and Z is endowed with the behavior uniformity $\mathcal{U}_b$.

Then $\mathcal{A}$ is a uniform incomplete automaton, if and only if G is a uniformly equicontinuous family of partial maps.

Proof : If $\mathcal{A}$ is a uniform incomplete automaton in the proof of proposition 4 the maps $P(f)$ and $P(g)$ can be substituted by f and g, respectively. Since f and g are uniformly continuous by assumption, G is uniformly equicontinuous.

On the other hand, $(Z, \mathcal{U}_b)$ can be considered as a subspace of $(P(Z), \mathcal{U}_b)$. Therefore, if G is uniformly equicontinuous $f : (Z \times A) \to Z$ is uniformly continuous by theorem 3 and the same holds for $g : (Z \times A) \to B$.

QED

5. PRECOMPACTNESS OF THE STATE SPACES

For practical applications, for modelling and simulation only finite models can be used. Therefore the problem of finite approximation of uniform automata has some importance. Suppose the uniformity of the state space Z of a uniform automaton $\mathcal{A}$ is given by a metric d. Let ε be a positive real number. A finite approximation of Z of degree ε can be defined as a finite subset $Z_1 \subset Z$, which has the property, that every state $z \in Z$ of $\mathcal{A}$ has a distance smaller than ε from at least one element z_1 of the subset Z_1. The property, that approximations of arbitrary small degree ε exist, is well-known as the property of precompactness.

Definition 13

A uniform space $(Z, \mathcal{U}_Z)$ is precompact, if for any arbitrary entourage $V \in \mathcal{U}_Z$, there is a finite subset $Z_1 \subset Z$ having the property $Z = V[Z_1]$ (cf. definition 3) [4].

In the following we assume, that the input space A and the output space B are precompact uniform spaces. This is a natural assumption, since in many practical cases A and B are compact real intervals.

Proposition 7 : The state space Z and the power set $P(Z)$ of a ND-automaton $\mathcal{A} = (Z, A, B, t)$ or an automaton $\mathcal{A} = (Z, A, B, f, g)$ are precompact with respect to the behavior uniformity $\mathcal{U}_b$.

Proof : The product spaces $X = A^N$, $Y = B^N$ and $X \times Y$ are precompact [1]. Furthermore it is easy to prove, that the same holds for the power uniformity on $P(X \times Y)$. Since the behavior uniformity $\mathcal{U}_b$ is the initial uniformity with respect to the maps $b : Z \to P(X \times Y)$ or $b : P(Z) \to P(X \times Y)$, the uniformity $\mathcal{U}_b$ is precompact ([1], II, 4.2).

QED

The preceeding proposition holds under a stronger condition for the uniformity $\mathcal{U}_{uc}$. The following proposition is an application of the theorem of Ascoli to automata theory.

Proposition 8 : The state space Z of an automaton $\mathcal{A} = (Z, A, B, f, g)$ is precompact with respect to the uniformity of uniform convergence $\mathcal{U}_{uc}$, if the family of maps $G = \{ g_z^N : X \to Y \mid z \in Z \}$ is uniformly equicontinuous. The condition is also necessary, if all the maps of G are uniformly continuous.

In particular, the state spaces of all uniform automata are precompact with respect to the uniformity $\mathcal{U}_{uc}$.

Proof : Since $\mathcal{U}_{uc}$ is the initial uniformity with respect to the map $v : Z \to G$, $(Z, \mathcal{U}_{uc})$ is precompact, if and only if G is precompact with respect to the uniformity of uniform convergence ([1], II, 4.3, Prop. 3).

Now the proposition follows from the theorem of Ascoli ([1], X, 2.5, Theorem 2).

The results given in this exposition suggest, that a general approximation theory of sequential systems can be developed.

QED

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