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ON SOME EPIDEMIC MODELS*

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Abstract. The qualitative behavior of the solution x of the equation

$$x(t) = k\left(p(t) - \int_0^t A(t-s)x(s)\,ds\right)\left(f(t) + \int_0^t a(t-s)x(s)\,ds\right), \qquad t \geq 0$$

is studied. This equation arises in the study of the spread of an infectious disease that does not induce permanent immunity.

1. Introduction. The purpose of this paper is to investigate the qualitative behavior (in particular as time goes to infinity) of solutions of an equation that arises in the study of the spread of an infectious disease that does not induce permanent immunity.

Suppose that we have a population of constant size P (which must be quite large if the deterministic model we describe below is not to be completely unrealistic). It is assumed that the average infectivity of an individual infected at time s is proportional to $a(t-s)$ at time t. If the rate at which individuals susceptible to the disease have become infected up to time t is $x(s)$, $s < t$, then the integral $\int_{-\infty}^t a(t-s)x(s)\,ds$ will be approximately proportional to the "total infectivity". If the cumulative probability function for the loss of immunity of an individual infected at time s is $1 - A(t-s)$, $t \geq s$, then $P - \int_{-\infty}^t A(t-s)x(s)\,ds$ will approximate the number of susceptibles. Our main assumption is that the rate at which susceptibles become infected is proportional to the number of susceptibles and the "total infectivity". This leads us to consider the equation

$$x(t) = k\left(p(t) - \int_0^t A(t-s)x(s)\,ds\right)\left(f(t) + \int_0^t a(t-s)x(s)\,ds\right), \qquad t \in R^+ = [0, \infty) \tag{1.1}$$

where $k > 0$ is a constant and where the functions p and f take into account the effects of the infection before $t = 0$ (for example: $p(t) = P - \int_{-\infty}^0 A(t-s)x(s)\,ds$, $f(t) = \int_{-\infty}^0 a(t-s)x(s)\,ds$ and x is assumed to be known on $(-\infty, 0)$).

When one compares the model used here with the ones used by other authors (see e.g. [1, 2, 3, 7, 11, 12, 13] and the references mentioned there), one can make the following observations. This model is quite closely related to the one used in [3] and the main difference is that it is not here assumed that permanent immunity is induced. Another important feature of this model, not shared by those in e.g. [2] and [13], is that the "infectivity" depends on how long the infectious individuals have been infected, not just the total number of infectives. Thus it is not necessary explicitly to introduce classes of individuals "exposed but not yet infectious" or "recovered but still immune" as in e.g.

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[11] and [12]. The equation (1.1) is essentially time-invariant, i.e., there is not, for example, a periodic contact rate (the constant k), as in [11]. We will show that Eq. (1.1) behaves nicely (as one would expect) in everything that concerns the existence and boundedness of a solution and that under certain restrictive conditions the infection rate and the number of susceptibles converge to limits when $t \to \infty$. The case when Eq. (1.1) has periodic solutions is considered in [7]. Finally, we remark that this model is completely deterministic (for stochastic models see e.g. [1]) and that no space variables are involved (cf. [4]).

2. Statement of results.

THEOREM. Assume that $k > 0$ is a constant and that

$$p \text{ is a positive, continuous and nondecreasing function on } R^+,\quad P \overset{\text{def}}{=} \lim_{t\to\infty} p(t) < \infty, \tag{2.1}$$

$$A \text{ is a nonnegative, right-continuous, nonincreasing function on } R^+,\ A(0) = 1 \text{ and } \lim_{t\to\infty} A(t) = 0, \tag{2.2}$$

$$f \text{ is a nonnegative, continuous function on } R^+, f \not\equiv 0,\ \lim_{t\to\infty} f(t) = 0 \text{ and if } A \notin L^1(R^+), \text{ then } f \in L^1(R^+), \tag{2.3}$$

$$a \text{ is a nonnegative, measurable function on } R^+,\ \int_0^\infty a(s)\,ds = 1, \tag{2.4}$$

$$\text{if } kP > 1, \text{ then } \operatorname{var}(h;\, R^+) < 1 \text{ and if } kP \le 1 \text{ and } A \notin L^1(R^+), \text{ then } \operatorname{var}(h,\, R^+) < \infty \tag{2.5}$$

where

$$h \text{ is the solution of the equation}\quad h(t) = A(t) - \int_t^\infty a(s)\,ds + \int_0^t a(t-s)h(s)\,ds, \qquad t \in R^+ \tag{2.6}$$

and

$$\text{if } A \notin L^1(R^+) \text{ and } kP > 1, \text{ then } r \in L^1(R^+) \text{ where } r \text{ is the solution of the equation } r(t) = A(t) - \int_0^t A(t-s)r(s)\,ds,\ t \in R^+. \tag{2.7}$$

Then there exists a unique, nonnegative, continuous and bounded solution x of (1.1) such that $x(t) > 0$ when $t > T$ for some $T \geq 0$ and

$$\lim_{t\to\infty} x(t) = \max\{0, kP - 1\}\left(k\int_0^\infty A(s)\,ds\right)^{-1} \tag{2.8}$$

and

$$\lim_{t\to\infty}\left(p(t) - \int_0^t A(t-s)x(s)\,ds\right) = \min\{1, kP\}/k. \tag{2.9}$$

The assumptions (2.5)–(2.7) are needed only in the proofs of (2.8) and (2.9).

For sufficient conditions for (2.7) to hold, see [8] and [10]. That (2.7) does not follow from (2.2) is shown in [5]. Another way to formulate the crucial condition that $\operatorname{var}(h; R^+) < 1$ is to say that the "quotient" of $A(t)$ and $\int_t^\infty a(s)\, ds$ (when the multiplication is the convolution product) is sufficiently close to unity. In general it can be very difficult to check if this condition is satisfied (and it does not appear to have any obvious biological interpretation), but the following proposition gives some cases when this can be seen to be the case.

PROPOSITION. If h is defined by (2.6), then $\operatorname{var}(h; R^+) < 1$ in the following cases:

$$a(t) = 0, \qquad t \in [0, t_0), \qquad a(t) = ce^{-c(t-t_0)}, \qquad t \geq t_0, \qquad c > 0, \qquad t_0 \geq 0,$$

$$A(t) = \int_t^\infty \int_{[0,s]} a(s-v)\, d\alpha(v)\, ds \tag{2.10}$$

where α is a nonnegative (Borel) measure supported on R^+, $\alpha(R^+) = 1$ and

$$\int_{R^+} t\, d\alpha(t) < (c(1 + c(1 + ct_0/2)(\pi/2)^{1/2}(t_0 + t_0^3(2 + ct_0)^2/12)^{1/2}))^{-1},$$

$$a(t) = 2^{-1}(c^2te^{-ct} + ce^{-ct}), \qquad t \in R^+, \qquad c > 0,$$

$$A(t) = \int_t^\infty \int_{[0,s]} c^2(s-v)e^{-c(s-v)\, d\alpha(v)\, ds}, \qquad t \in R^+ \tag{2.11}$$

where α is a nonnegative (Borel) measure supported on R^+, $\alpha(R^+) = 1$ and $\int_{R^+} t\, d\alpha(t) < c^{-1}$,

$$a(t) = c\int_{[0,t]} e^{-c(t-s)}\, d\beta(s), \qquad c > 0, \tag{2.12}$$

where β is a nonnegative (Borel) measure supported on R^+, $\beta(R^+) = 1$, $\int_{R^+} t\, d\beta(t) < \infty$ and A is defined as in (2.10) and

$$\int_{R^+} t\, d\alpha(t) < c^{-1}\left[1 + \varphi\left(c\int_{R^+} t\, d\beta(t)\right)\right]^{-1}$$

where $\varphi(t) = t/(1-t)$, $0 \leq t < 1 - 10^{-2}$, $\varphi(t) = 50(2t)^{5-4/t}$, $t \geq 1 - 10^{-2}$.

Observe that the following assumptions concerning the spread of the disease lead to Eq. (1.1) with a and A as in (2.10): an individual infected at time t becomes infective (i.e., will be able to communicate infectious organisms to other individuals) at $t + t_0$, the infectivity remains constant up to time $t + t_0 + \tau_1$ when it drops to zero and the immunity is lost at time $t + t_0 + \tau_1 + \tau_2$. Here τ_1 and τ_2 are independent random variables, τ_1 has exponential distribution with mean value c^{-1} and τ_2 has probability measure α. Note that we only put a restriction on the mean value of τ_2 in (2.10).

The case (2.11) arises from a similar situation, but here $t_0 = 0$ and the distribution function for τ_1 is c^2te^{-ct} (so that the mean value is $2c^{-1}$).

The condition (2.12) is a generalization of (2.10) and the only difference is that now t_0 is also a random variable with probability measure β.

3. Proof of the Theorem. We consider the equation

$$x(t) = k\left(p(t) - \int_0^t A(t-s)x(s)_+\, ds\right)\left(f(t) + \int_0^t a(t-s)x(s)_+\, ds\right), \qquad t \in R^+ \tag{3.1}$$

where $y_+ = \max\{0, y\}$. If we can show that this equation has a nonnegative continuous solution, then we have also found a nonnegative continuous solution of (1.1). Using the Banach fixed-point theorem we find a continuous solution of (3.1) on some interval $[0, t_0]$. Combining a standard translation argument with the Banach fixed-point theorem, we are able to continue this solution to R^+ if we can show that x remains bounded and that

$$\left(p(t) - \int_0^t A(t-s)x(s)_+ \, ds\right) > 0 \tag{3.2}$$

for as long as the solution exists. Assume that the last statement does not hold and let $t_1 = \min\{t \geq 0 \,|\, p(t) - \int_0^t A(t-s)x(s)_+ \, ds + 0\}$. Since we assume that we have a continuous solution of (3.1) on $[0, t_1]$, it follows from (2.1)–(2.4) and (3.1) that there exists a constant c_1 such that

$$\begin{aligned} 0 \leq x(t) &\leq c_1(p(t) - p(t_1)) + \int_t^{t_1} \left(x(s)_+ + \int_{[0,s]} x(s-v)_+ \, dA(v)\right) ds \\ &\leq c_1 \int_t^{t_1} x(s)_+ \, ds, \qquad t \in [0, t_1]. \end{aligned}$$

But this implies that $x(t) \equiv 0$ on $[0, t_1]$, and we get a contradiction.

Now we conclude from (2.3), (2.4), (3.1) and (3.2) that x is nonnegative for as long as the solution exists. We proceed to establish an a priori bound for the solution.

If $kP < 1$, then we immediately obtain from (2.1)–(2.4), (3.1) and (3.2)

$$x(t) \leq \sup_{t \in R^+} f(t)/(1 - kP).$$

Assume next that $kP \geq 1$. It is clearly possible to choose functions a_1 and a_2 such that

$$a(t) = a_1(t) + a_2(t), \qquad t \in R^+, \qquad a_1 \in BV(R^+), \qquad \int_0^\infty |a_2(t)| \, dt \leq 2^{-6}(kP)^{-2}e^{-4kP}. \tag{3.3}$$

Let

$$\begin{aligned} \omega(\delta, t) &= \sup\{|x(t_1) - x(t_2)| \,|\, 0 \leq t_1 - t_2 \leq \delta,\ t_1 \leq t\}, \\ \beta(t) &= \sup_{s \in [0,t]} x(t). \end{aligned}$$

Using (3.1) and (3.3), we have

$$\begin{aligned} x(t_1) - x(t_2) = k\Bigg(&p(t_1) - p(t_2) - \int_0^{t_1} A(t_1 - s)x(s) \, ds \\ &+ \int_0^{t_2} A(t_2 - s)x(s) \, ds\Bigg)\left(f(t_2) + \int_0^{t_2} a(t_2 - s)x(s) \, ds\right) \\ &+ k\left(p(t_1) - \int_0^{t_1} A(t_1 - s)x(s) \, ds\right)\Bigg(f(t_1) - f(t_2) + \int_0^{t_1} a_1(t_1 - s)x(s) \, ds \end{aligned}$$

$$- \int_0^{t_2} a_1(t_2 - s)x(s)\,ds + \int_{t_2}^{t_1} x(t_1 - s)a_2(s)\,ds$$

$$+ \int_0^{t_2} (x(t_1 - s) - x(t_2 - s))a_2(s)\,ds\Big), \qquad 0 \le t_2 \le t_1. \tag{3.4}$$

If we use (2.1)–(2.4), (3.3), (3.4) and the fact that x is bounded on $[0, t_0]$ if t_0 is small enough and always nonnegative, then we conclude that there exists a nondecreasing function $q_1(q_1(\delta) \to 0$ as $\delta \to 0)$ such that

$$\omega(\delta, t) \le q_1(\delta)(\beta(t) + 1) + 4k\beta(t)^2\delta, \qquad \delta \in (0, \delta_0)$$

where $\delta_0 \le t_0$ is such that $A(\delta_0) \ge 2^{-1}$ (this fact we will use below). Applying this inequality together with (2.1)–(2.4) and (3.3) once more to (3.4) we get (take $t_2 = t_1 - \delta$ and recall the definition of β)

$$x(t_1) - x(t_1 - \delta) \le q_2(\delta)(\beta(t_1) + 1) + 2^{-4}P^{-1}e^{-4kP}\beta(t_1)^2\delta$$

$$+ k\beta(t_1) \int_{t_1-\delta}^{t_1} (\beta(t_1) - x(v))\,dv, \qquad \delta \in (0, \delta_0) \tag{3.5}$$

where q_2 is some nonnegative, nondecreasing function such that $q_2(\delta) \to 0$ as $\delta \to 0$. Assume next that t_1 is such that $x(t_1) = \beta(t_1)$. Then it follows from Gronwall's inequality applied to (3.5) that

$$x(t_1) - x(t_1 - \delta) \le (q_2(\delta)(\beta(t_1) + 1) + 2^{-4}P^{-1}e^{-4kP}\beta(t_1)^2\delta)e^{k\beta(t_1)\delta}, \qquad \delta \in (0, \delta_0).$$

But this implies that

$$\int_0^{t_1} A(t_1 - s)x(s)\,ds \ge 2^{-1}(\beta(t_1) - (q_2(\delta)(\beta(t_1) + 1)$$

$$+ 2^{-4}P^{-1}e^{-4kP}\beta(t_1)^2\,\delta)e^{k\beta(t_1)\delta})\,\delta, \qquad \delta \in (0, \delta_0)$$

since $A(\delta_0) \ge 2^{-1}$. But if $\beta(t_1)$ is so large that $q_2(\delta)(\beta(t_1) + 1)e^{4kP} < 2^{-2}\beta(t_1)$ if we choose $\delta = 4P\beta(t_1)^{-1}$, then we get a contradiction since $\int_0^{t_1} A(t_1 - s)x(s)\,ds < P$ by (3.2). This shows that the solution x is bounded by an a priori bound and hence it can be continued to R^+.

So far we have shown that there exists a continuous, nonnegative and bounded solution of (1.1) on R^+ and the uniqueness of this solution follows immediately from a contraction mapping argument.

We see from (3.2) and (1.1) that $x(t) > 0$ if $(f(t) + \int_0^t a(t - s)x(s)\,ds) > 0$. Since $f \not\equiv 0$ we conclude from this fact and (2.4) that $x(t) > 0$ if $t \ge T$ for some $T \ge 0$ (we use the result that if $V \subseteq R^+$ is a measurable set with $m(V) > 0$ then $\bigcup_{n=1}^{\infty} nV$ contains an interval $[T, \infty)$ where $nV = \{\sum_{i=1}^n x_i \mid x_i \in V\}$).

Since x is bounded it follows from (2.2) and (2.4) that $\int_0^t A(t - s)x(s)\,ds$ and $\int_0^t a(t - s)x(s)\,ds$ are uniformly continuous functions of t. Therefore the same is by (1.1), (2.1) and (2.3) true for the function x.

It remains for us to establish (2.8) and (2.9). We define

$$A_0(t) = \int_t^{\infty} a(s)\,ds, \; B(t) = A(t) - A_0(t), \qquad t \in R^+ \tag{3.6}$$

and we observe that it follows from (2.6) that

$$B(t) = \int_{[0,t]} A_0(t-s)\, dh(s), \qquad t \in R^+. \tag{3.7}$$

First we consider the case $kP \leq 1$. Assume that (2.8) does not hold and choose a sequence $\{t_n\}_{n=1}^{\infty}$ (tending to infinity) such that $\lim_{n\to\infty} x(t_n) = \limsup_{t\to\infty} x(t) > 0$. Since x is uniformly continuous and $\lim_{t\to 0+} A(t) = 1$, it follows that there exists a number $\alpha > 0$ such that

$$\int_0^{t_n} A(t_n - s)x(s)\, ds \geq \alpha, \; n = 1, 2, \ldots. \tag{3.8}$$

It follows from (2.3) and (2.4) that

$$\limsup_{n\to\infty} \left(f(t_n) + \int_0^{t_n} a(t_n - s)x(s)\, ds\right) \leq \limsup_{t\to\infty} x(t).$$

Using this inequality combined with (3.8) in (1.1) we get

$$\limsup_{t\to\infty} x(t) \leq k(P-\alpha)\limsup_{t\to\infty} x(t)$$

and since $\alpha > 0$ and $kP \leq 1$ we have a contradiction and (2.8) holds in this case.

If $A \in L^1(R^+)$, then it is clear that (2.9) follows from (2.8). If $A \notin L^1(R^+)$, then we define the function w by

$$w(t) = \int_0^t A_0(t-s)z(s)\, ds + \int_t^{\infty} f(s)\, ds, \qquad t \in R^+. \tag{3.9}$$

Since $kP \leq 1$ and the function x is nonnegative we have $w'(t) \leq 0$ by (1.1) and (3.6). This implies by (3.9) that $\lim_{t\to\infty} \int_0^t A_0(t-s)x(s)\, ds$ exists and then by (2.5) and (3.7) $\lim_{t\to\infty} \int_0^t B(t-s)x(s)\, ds$ exists too. If $\beta \overset{\text{def}}{=} \lim_{t\to\infty} \int_0^t A(t-s)x(s)\, ds$ is zero, then we are done. Assume that $\beta > 0$. We have by (1.1)

$$x(t) \leq c_2\left(f(t) + \int_0^t a(t-s)x(s)\, ds\right), \qquad t \geq T_0 \geq 0, \qquad c_2 = k(P - \beta/2) < 1$$

for some sufficiently large number T_0. Using a comparison argument (see [9, Chap. II]), (2.3) and (2.4) we see that there exists an integrable function g such that if

$$b(t) = g(t) + c_2 \int_0^t a(t-s)b(s)\, ds, \qquad t \in R^+ \tag{3.10}$$

then

$$0 \leq x(t+T_0) \leq b(t), \qquad t \in R^+. \tag{3.11}$$

Since $c_2 \int_0^\infty a(s)\, ds < 1$ we conclude from (3.10) that $b \in L^1(R^+)$ and hence by (3.11) $x \in L^1(R^+)$; hence $\beta = 0$, a contradiction. This completes the proof in the case when $kP \leq 1$.

We proceed to consider the case $kP > 1$ and first we assume that $A \in L^1(R^+)$. We choose a continuous function u such that

$$\int_0^t A_0(t-s)u(s)\,ds = (kP-1)(k+kh(\infty))^{-1}, \qquad t \geq 1 \tag{3.12}$$

where $h(\infty) = \lim_{t\to\infty} h(t)$. This can be done e.g. as follows. Let u be any nonnegative continuous function on $[0, 1]$ such that $\int_0^1 A_0(1-s)u(s)\,ds = (kP-1)(k+kh(\infty))^{-1}$ and $u(1) = \int_0^1 a(1-s)u(s)\,ds$ and for $t \geq 1$ let u be the solution of the equation $u(t) - \int_0^t a(t-s)u(s)\,ds = 0$. If we combine (3.12) with (2.4), (3.6) and (3.7), then we conclude from the standard renewal theorem, since $\int_0^\infty ta(t)\,dt = \int_0^\infty A(t)\,dt(1+h(\infty))^{-1}$, that

$$\lim_{t\to\infty} u(t) = (kP-1)\left(k\int_0^\infty A(s)\,ds\right)^{-1}. \tag{3.13}$$

Let

$$y(t) = x(t) - u(t),\ v(t) = \int_0^t A_0(t-s)y(s)\,ds,$$

$$z(t) = P - \int_0^t A_0(t-s)x(s)\,ds, \qquad t \in R^+. \tag{3.14}$$

We obviously have by (1.1), (3.6) and (3.14)

$$z'(t) = \left(f(t) + \int_0^t a(t-s)x(s)\,ds\right)\left\{1 - f(t)\left[f(t) + \int_0^t a(t-s)x(s)\,ds\right]^{-1}\right.$$

$$\left. - k\left[p(t) - \int_0^t A(t-s)x(s)\,ds\right]\right\}, \qquad t \geq T_1 \tag{3.15}$$

where $T_1 \geq 1$ is chosen so that

$$q(t) \overset{\text{def}}{=} f(t) + \int_0^t a(t-s)x(s)\,ds > 0, \qquad t \geq T_1. \tag{3.16}$$

Since $\lim_{t\to\infty} \int_0^t B(t-s)u(s)\,ds = (kP-1)h(\infty)(k+kh(\infty))^{-1}$ by (2.5), (3.7) and (3.12), it is easy to conclude from (1.1), (2.1), (3.6), (3.7) and (3.14)–(3.16) (note that $v'(t) = -z'(t)$, $t \geq T_1$), that if we define $e_1(t) = k^{-1} - p(t) + \int_0^t A_0(t-s)u(s)\,ds + \int_0^t B(t-s)u(s)\,ds$, then e_1 is continuous and

$$\lim_{t\to\infty} e_1(t) = 0 \tag{3.17}$$

and

$$v'(t) = -kq(t)(-f(t)(kq(t))^{-1} + e_1(t) + v(t) + \int_{[0,t]} v(t-s)\,dh(s)), \qquad t \geq T_1. \tag{3.18}$$

Assume that $-\liminf_{t\to\infty} v(t) \geq \limsup_{t\to\infty} v(t)$ and that $v(t)$ does not converge as $t\to\infty$. Then there exists a sequence of numbers $\{t_n\}_{n=1}^\infty$ (tending to infinity) such that

$$\lim_{n\to\infty} v(t_n) = \liminf_{t\to\infty} v(t) < 0 \quad \text{and} \quad v'(t_n) = 0, \qquad n = 1, 2, \ldots. \tag{3.19}$$

Since we clearly have $\limsup_{n\to\infty} \int_{[0,t_n]} v(t_n - s)\,dh(s) \leq -\text{var}(h; R^+)\liminf_{t\to\infty} v(t)$ it follows from (2.5) and (3.16)–(3.19) that $v'(t_n) > 0$ for n large enough and we have a contradiction.

If $\lim_{t\to\infty} v(t)$ exists and is negative, then we easily see from (1.1), (2.1), (2.5), (3.6), (3.7), (3.13) and (3.14) that there exists a constant $c_3 > 1$ such that

$$x(t) > c_3\left(f(t) + \int_0^t a(t-s)x(s)\,ds\right)$$

when t is large enough. But then a straightforward argument using the fact that $c_3 \int_0^\infty a(s)\,ds > 1$ shows that x cannot be bounded and we have a contradiction.

It follows from the results above that unless $\lim_{t\to\infty} v(t)$ exists and is nonnegative we have

$$\gamma \overset{\text{def}}{=} \limsup_{t\to\infty} v(t) > 0, \qquad \liminf_{t\to\infty} v(t) > -\gamma. \tag{3.20}$$

Let $\gamma_1 = \gamma(3 + \operatorname{var}(h; R^+))/4$. By (2.5) and (3.20) there exists a number $T_2 \geq T_1$ such that

$$\left|\int_{[0,t]} v(t-s)\,dh(s)\right| \leq (1 + \operatorname{var}(h; R^+))\gamma/2, \qquad t \geq T_2. \tag{3.21}$$

Now we pick two sequences $\{s_n\}_{n=1}^\infty$, $\{t_n\}_{n=1}^\infty$ (tending to infinity), $T_2 \leq s_n \leq t_n$, such that

$$v(t_n) \geq v(t) \geq \gamma_1 \quad \text{on} \quad [s_n, t_n] \quad \text{and} \quad v(t_n) \geq (\gamma + \gamma_1)/2,\ n = 1, 2, \ldots. \tag{3.22}$$

It follows from (3.18), (3.21) and (3.22) that if $t \in [s_n, t_n]$ is such that $v'(t) > 0$, then we must have

$$-f(t)(kq(t))^{-1} + e_1(t) + (1 - \operatorname{var}(h; R^+))\gamma/4 < 0$$

or, if n is large enough (see (3.17))

$$q(t) \leq 8f(t)((1 - \operatorname{var}(h; R^+))\gamma k)^{-1}. \tag{3.23}$$

Since $\int_0^t a(t-s)x(s)\,ds \leq q(t)$ and $x(t) \leq kPq(t)$ it follows from (2.3), (2.5) (3.6), (3.14) and (3.23) that

$$\lim_{n\to\infty} \sup_{t\in[s_n, t_n]} v'(t) = 0.$$

Hence we can choose the sequences $\{s_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ so that

$$\lim_{n\to\infty} (t_n - s_n) = \infty. \tag{3.24}$$

It follows from (1.1), (2.1), (2.5), (3.6), (3.7), (3.13), (3.14), (3.20)–(3.22) that there exists a constant $c_4 < 1$ such that

$$x(t) \leq c_4\left(f(t) + \int_0^t a(t-s)x(s)\,ds\right), \qquad t \in [s_n, t_n], \qquad n \geq N_0 \tag{3.25}$$

for some integer N_0. By (2.3), (2.4) and (3.25) it is easy to construct a nonnegative continuous function g on R^+ such that $g(t) \to 0$ as $t \to \infty$ and such that if

$$b(t) = g(t) + c_4 \int_0^t a(t-s)b(s)\,ds, \qquad t \in R^+$$

then

$$b(t) \geq x(t + s_n), \qquad t \in [0, t_n - s_n], \qquad n \geq N_0 \tag{3.26}$$

(see [7, Chap. II]). Since $c_4 \int_0^\infty a(s)\,ds < 1$ it follows that $\lim_{t\to\infty} b(t) = 0$, and then by (3.24) and (3.26) we have $\int_0^{t_n} A(t_n - s)x(s)\,ds \to 0$ as $n \to \infty$, which gives a contradiction in view of (1.1), (2.1), (3.25) and the fact that $kP > 1$. Hence it follows that $v(\infty) = \lim_{t\to\infty} v(t)$ exists and is nonnegative. It follows from (3.6), (3.7) and the assumptions that $A \in L^1(R^+)$ and $\operatorname{var}(h, R^+) < 1$ that $A_0 \in L^1(R^+)$ and, since A_0 is nonincreasing, we deduce from Wiener's Tauberian theorem that $\lim_{t\to\infty} y(t) = v(\infty)(\int_0^\infty A_0(s)\,ds)^{-1}$. We see from Eq. (1.1) that if (2.8) does not hold but $x(t)$ converges, then the limit must be 0. Thus it follows from (3.13) and (3.14) that $v(\infty) = 0$, i.e. (2.8) holds. As $A \in L^1(R^+)$ the statement (2.9) is a direct consequence of (2.8). This completes the proof in the case when $kP > 1$ and $A \in L^1(R^+)$.

Finally we consider the case when $kP > 1$ and $A \notin L^1(R^+)$. We define the functions u and y as before and we note that (3.13) still holds. But this time we let

$$v(t) = \int_0^t A_0(t-s)y(s)\,ds + \int_t^\infty f(s)\,ds, \qquad t \in R^+. \tag{3.27}$$

In the same way as we deduced (3.18) above we can now conclude that

$$v'(t) = -kq(t)\left(e_2(t) + v(t) + \int_{[0,\,t]} v(t-s)\,dh(s)\right), \qquad t \geq T_1 \tag{3.28}$$

where e_2 is a continuous function satisfying

$$\lim_{t\to\infty} e_2(t) = 0. \tag{3.29}$$

If $\lim_{t\to\infty} v(t)$ does not exist and e.g. $\limsup_{t\to\infty} v(t) \geq -\liminf_{t\to\infty} v(t)$, then we can again choose a sequence $\{t_n\}_{n=1}^\infty$ (tending to infinity) such that

$$\lim_{n\to\infty} v(t_n) = \limsup_{t\to\infty} v(t),\ v'(t_n) = 0, \qquad n = 1, 2, \ldots. \tag{3.30}$$

Since $\limsup_{n\to\infty} \left|\int_{[0,\,t_n]} v(t-s)\,dh(s)\right| \leq \operatorname{var}(h; R^+)\limsup_{t\to\infty} v(t)$ and $\operatorname{var}(h; R^+) < 1$ it follows from (3.28) and (3.29) that $v'(t_n) < 0$ if n is large enough. But then (3.30) gives a contradiction, and we see that $\lim_{t\to\infty} v(t)$ exists. If this limit is negative, we get a contradiction in the same way as in the case when $A \in L^1(R^+)$ and if the limit is positive we argue in the same manner as when $kP \leq 1$ and $A \notin L^1(R^+)$. Hence we have established that

$$\lim_{t\to\infty} v(t) = 0. \tag{3.31}$$

We conclude from (2.1), (2.3), (2.5), (3.6), (3.7), (3.12), (3.27) and (3.31) that (2.9) holds in this case and by (3.13) and (3.14) it remains to show that

$$\lim_{t\to\infty} y(t) = 0.$$

To do this we observe that by (2.3), (2.5), (3.6), (3.27)–(3.29) and (3.31) we have $\lim_{t\to\infty} F(t) = 0$ if F is defined by

$$F(t) = y(t) - \int_0^t a(t-s)y(s)\,ds + \int_0^t A_0(t-s)y(s)\,ds, \qquad t \in R^+.$$

(Recall that $v'(t) + f(t) = y(t) - \int_0^t a(t-s)y(s)\,ds$ and that x and u, hence also y, are uniformly continuous.) Therefore it is sufficient to show that there exists a function $r_1 \in L^1(R^+)$ such that (" $\wedge$ " denotes Laplace transform)

$$\hat{r}_1(z) = (-\hat{a}(z) + \hat{A}_0(z))(1 - \hat{a}(z) + \hat{A}_0(z))^{-1}, \qquad \text{Re } z \geq 0. \tag{3.32}$$

Since $a \in L^1(R^+)$ and $1 - \hat{a}(z)(1 + \hat{A}_0(z))^{-1} \neq 0$, Re $z \geq 0$ (see (2.4) and (3.6)) and

$$\hat{r}_1(z) = (-\hat{a}(z)(1 - \hat{r}_2(z)) + \hat{r}_2(z))(1 - \hat{a}(z)(1 - \hat{r}_2(z)))^{-1}$$

by (3.32) where $\hat{r}_2(z) = \hat{A}_0(z)(1 + \hat{A}_0(z))^{-1}$, it suffices to prove that $r_2 \in L^1(R^+)$, see [10]. But by (2.7), (3.6) and (3.7) (note that $z\hat{h}(z) = \int_0^\infty e^{-zt}\, dh(t)$),

$$\hat{r}_2(z) = \hat{r}(z)(1 + z\hat{h}(z))^{-1}(1 + \hat{r}(z)(1 + z\hat{h}(z))^{-1} - \hat{r}(z))^{-1},$$

and since $r \in L^1(R^+)$, var$(h;\, R^+) < 1$ and $1 + \hat{A}_0(z) \neq 0$, Re $z \geq 0$ it is easy to conclude from Banach algebra arguments (see also [10]) that $r_2 \in L^1(R^+)$. This completes the proof of the Theorem.

4. Proof of the Proposition. Assume that (2.10) holds. Then the Laplace transforms of a and A are $\hat{a}(z) = ce^{-t_0 z}(z + c)^{-1}$, $\hat{A}(z) = (1 - \hat{a}(z)\hat{\alpha}(z))/z$ where $\hat{\alpha}$ denotes the Laplace-Stieltjes transform of the measure α. It is therefore sufficient by (2.6), to show that $(1 - \hat{\alpha}(z))\hat{a}(z)(1 - \hat{a}(z))^{-1}$ is the Laplace transform of a function in $L^1(R^+)$ with norm <1. But if $\hat{a}(z)(1 - \hat{a}(z))^{-1} = \hat{q}(z)$, then ($q(t) = 0$ if $t < 0$)

$$\begin{aligned}\int_0^\infty \left| q(t) - \int_{[0,\,t]} q(t-s)\, d\alpha(s) \right| dt &\leq \int_{R^+} \int_{-\infty}^\infty |q(t) - q(t-s)|\, dt\, d\alpha(s) \\ &\leq \text{var}(q;\, R) \int_{R^+} s\, d\alpha(s).\end{aligned} \tag{4.1}$$

Consequently we must show that

$$\text{var}(q;\, R) \leq c(1 + c(1 + ct_0/2)(\pi/2)^{1/2}(t_0 + t_0^3(2 + ct_0)^2/12)^{1/2}). \tag{4.2}$$

But it follows from the definition of q that

$$z\hat{q}(z) = ce^{-t_0 z}(1 - \hat{r}_1(z)) \tag{4.3}$$

where

$$\hat{r}_1(z) = \hat{q}_1(z)(1 + \hat{q}_1(z))^{-1}, \qquad \hat{q}_1(z) = c/z - e^{-t_0 z}c/z. \tag{4.4}$$

From Plancherel's theorem and Hölder's inequality we have

$$\begin{aligned}\int_0^\infty |r_1(t)|\, dt &\leq \left(\int_0^\infty (1 + t^2)^{-1}\, dt\right)^{1/2} \left(\int_{-\infty}^\infty (1 + t^2)|r_1(t)|^2\, dt\right)^{1/2} \\ &= 2^{-1}\left(\int_{-\infty}^\infty (|\hat{r}_1(ix)|^2 + |\hat{r}_1'(ix)|^2)\, dx\right)^{1/2}.\end{aligned} \tag{4.5}$$

It is easy to see that $|1 + \hat{q}_1(ix)|^{-1} \leq 1 + t_0 c/2$, and therefore it follows from Plancherel's theorem and (4.4) that

$$\int_{-\infty}^\infty (|\hat{r}_1(ix)|^2 + |\hat{r}_1'(ix)|^2)\, dx \leq 2\pi(c^2 t_0(1 + ct_0/2)^2 + c^2 t_0^3(1 + ct_0/2)^4/3).$$

If we use this inequality together with (4.3) and (4.5), then we obtain (4.2) and the proof of the first part of the Proposition is completed.

Assume next that (2.11) holds. In this case we must clearly show that

$$z\hat{h}(z) = (2^{-1}c^2(z+c)^{-2} + 2^{-1}c(z+c)^{-1} - c^2(z+c)^{-2}\hat{\alpha}(z)) \times (1 - 2^{-1}c^2(z+c)^{-2} - 2^{-1}c(z+c)^{-1})^{-1}$$

is the Laplace transform of a function in $L^1(R^+)$ with norm <1. Some calculations show that

$$z\hat{h}(z) = (c^2(z^2 + 3cz/2)^{-1}(1 - \hat{\alpha}(z)) + 2^{-1}c(z + 3c/2)^{-1}).$$

From this equation we see that it is obviously sufficient to have (recall that $\alpha(R^+) = 1$ and see (4.1))

$$2c/3 \int_{R^+} s\, d\alpha(s) + 1/3 < 1.$$

This completes the proof of the second part of the Proposition.

To establish the assertion in the case when (2.12) holds, we proceed in the same manner as in the case (2.10) and it is clearly sufficient to show that

$$\operatorname{var}(q; R) \le c\left[1 + \varphi\left(c \int_{R^+} t\, d\beta(t)\right)\right],$$

(q is defined above). But it follows from the definition of a that $zq(z) = c\beta(z) \times (1 + c(1 - \beta(z))/z)^{-1}$ and since $(1 - \beta(z))/z$ is the Laplace transform of a nonnegative, nonincreasing function with L^1-norm $\int_{R^+} t\, d\beta(t)$ the desired result follows from [6, Thm. 1] and the fact that $\beta(R^+) = 1$. This completes the proof of the Proposition.

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