UDC 517.9

B. Altay, F. Başar (Inönü Üniv., Turkey)

SOME EULER SEQUENCE SPACES OF NONABSOLUTE TYPE

ДЕЯКІ ПРОСТОРИ ПОСЛІДОВНОСТЕЙ ЕЙЛЕРА НЕАБСОЛЮТНОГО ТИПУ

In the present paper, the Euler sequence spaces e_0^r and e_c^r of nonabsolute type which are the BK-spaces including the spaces c_0 and c have been introduced and proved that the spaces e_0^r and e_c^r are linearly i somorphic to the spaces c_0 and c, respectively. Furthemore, some inclusion theorems have been given. Additionally, the α-, β-, γ- and continuous duals of the spaces e_0^r and e_c^r have been computed and their basis have been constructed. Finally, the necessary and sufficient conditions on an infinite matrix belonging to the classes $\left(e_c^r : \ell_p\right)$ and $\left(e_c^r : c\right)$ have been determined and the characterizations of some other classes of infinite matrices have also been derived by means of a given basic lemma, where $1 \le p \le \infty$.

Введено поняття просторів послідовностей Ейлера e_0^r та e_c^r неабсолютного типу — BK-просторів, що містять простори c_0 та c. Доведено, що простори e_0^r та e_c^r лінійно ізоморфні відповідно до просторів c_0 та c. Наведено деякі теореми про включення. Крім того, обчислено α-, β-, γ- та неперервні простори, дуальні до просторів e_0^r та e_c^r, і побудовано базиси цих просторів. Визначено необхідні та достатні умови належності нескінченної матриці до класів $\left(e_c^r : \ell_p\right)$ та $\left(e_c^r : c\right)$. Отримано характеристики деяких інших класів нескінченних матриць з використанням наведеної в роботі основної леми для випадку $1 \le p \le \infty$.

1. Preliminaries, background and notation. By w, we shall denote the space of all real or complex valued sequences. Any vector subspace of w is called a *sequence space*. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also, by bs, cs, ℓ_1 and ℓ_p, we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series, respectively; here, $1 < p < \infty$.

A sequence space λ with a linear topology is called a K-*space* provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$; here, $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. A K-space λ is called an FK-*space* provided λ is a complete linear metric space. An FK-*space* whose topology is normable is called a BK-*space* (see [1, p. 272, 273]).

Let λ, μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $k, n \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ; here,

$$(Ax)_n = \sum_k a_{nk} x_k, \quad n \in \mathbb{N}. \tag{1.1}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda : \mu)$, we denote the class of all matrices A such that $A :$

$\lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right-hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to l if Ax converges to l which is called as the A-limit of x.

The *matrix domain* λ_A of an infinite matrix A is a sequence space λ is defined by

$$\lambda_A = \{x = (x_k) \in w : Ax \in \lambda\} \tag{1.2}$$

which is a sequence space. The new sequence space λ_A generated by the limitation matrix A from the space λ either includes the space λ or is included by the space λ, in general, i.e., the space λ_A is the expansion of the contraction of the original space λ.

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has been employed by Wang [2], Ng and Lee [3], Malkowsky [4], and Başar and Altay [5]. They introduced the sequence spaces $(\ell_p)_{N_q}$ in [2], $(\ell_\infty)_{C_1} = X_\infty$ and $(\ell_p)_{C_1} = X_p$ in [3], $(\ell_\infty)_{R^t} = r^t_\infty$, $c_{R^t} = r^t_c$ and $(c_0)_{R^t} = r^t_0$ in [4] and $(\ell_p)_\Delta = bv_p$ in [5]; here, N_q, C_1 and R^t denote the Nörlund, arithmetic and Riesz means, respectively, and Δ also denotes the band matrix defining the difference operator and $1 \le p < \infty$. Quite recently, Aydın and Başar have studied the sequence spaces $(c_0)_{A^r} = a^r_0$, $c_{A^r} = a^r_c$ in [6], $(\ell_p)_{A^r} = a^r_p$, $(\ell_\infty)_{A^r} = a^r_\infty$ in [7], $a^r_0(\Delta) = (a^r_0)_\Delta$, $a^r_c(\Delta) = (a^r_c)_\Delta$ in [8] and extended the sequence spaces a^r_0, a^r_c to the paranormed spaces $a^r_0(p)$, $a^r_c(p)$ in [9]; here, A^r denotes the matrix $A^r = (a^r_{nk})$ defined by

$$a^r_{nk} = \begin{cases} \dfrac{1 + r^k}{n+1}, & 0 \le k \le n, \\ 0, & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$ and any fixed $r \in \mathbb{R}$. In the present paper, following [2 – 9], we introduce the Euler sequence spaces e^r_0 and e^r_c of nonabsolute type and derive some results related to those sequence spaces. Furthermore, we have constructed the basis and computed the α-, β-, γ-, and continuous duals of the spaces e^r_0 and e^r_c. Finally, we have essentially characterized the matrix classes $(e^r_c : \ell_p)$, $(e^r_c : c)$ and also derived the characterizations of some other classes by means of a given basic lemma, where $1 \le p \le \infty$. Besides, we have stated and proved a Steinhaus type theorem concerning with the disjointness of the classes $(e^r_\infty : c)$ and $(e^r_c : c)_s$.

2. The Euler sequence spaces e^r_0 and e^r_c of nonabsolute type. Altay, Başar and Mursaleen [10] have recently studied Euler sequence spaces e^r_p and e^r_∞ consisting of the sequences whose E^r-transforms are in the spaces ℓ_p and ℓ_∞, respectively; here, $1 \le p < \infty$ and E^r denotes the Euler means of order r defined by the matrix $E^r = (e^r_{nk})$,

$$e^r_{nk} = \begin{cases} \dbinom{n}{k}(1-r)^{n-k} r^k, & 0 \le k \le n, \\ 0, & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. It is known that the method E^r is regular for $0 < r < 1$ (see [11, p. 57]), and we assume unless stated otherwise that $0 < r < 1$. Mursaleen, Başar and Altay have given the characterizations of the matrix classes related to the spaces e_p^r and e_∞^r, and emphasized some geometric properties, for example, Banach – Saks and weak Banach – Saks properties, etc., of the space e_p^r in [12]. Continuing on this way, we introduce the Euler sequence spaces

$$e_0^r = \left\{ x = (x_k) \in w\colon \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k = 0 \right\}$$

and

$$e_c^r = \left\{ x = (x_k) \in w\colon \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k \text{ exists} \right\}.$$

With the notation of (1.2), we may redefine the spaces e_0^r and e_c^r as follows:

$$e_0^r = (c_0)_{E^r} \quad \text{and} \quad e_c^r = c_{E^r}. \tag{2.1}$$

It is trivial that $e_0^r \subset e_c^r$. If λ is any normed sequence space, then we call the matrix domain λ_{E^r} as the *Euler sequence space*. Define the sequence $y = \{y_k(r)\}$, which will be frequently used, as the E^r-transform of a sequence $x = (x_k)$, i.e.,

$$y_k(r) = \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j x_j, \quad k \in \mathbb{N}. \tag{2.2}$$

We now may begin with the following theorem which is essential in the text:

Theorem 2.1. *The sets* e_0^r *and* e_c^r *are the linear spaces with the coordinatewise addition and scalar multiplication which are the BK-spaces with the norm* $\|x\|_{e_0^r} = \|x\|_{e_c^r} = \left\| E^r x \right\|_{\ell_\infty}$.

Proof. The first part of the theorem is a routine verification and so we omit it. Furthermore, since (2.1) holds and c_0, c are the BK-spaces with respect to their natural norm (see [13, p. 217, 218]), and the matrix $E^r = \left(e_{nk}^r\right)$ is normal, i.e., $e_{nn}^r \neq 0$, $e_{nk}^r = 0$, $k > n$, for all $k, n \in \mathbb{N}$, Theorem 4.3.2 of Wilansky [14, p. 61] gives the fact that the spaces e_0^r, e_c^r are the BK-spaces.

The theorem is proved.

Therefore, one can easily check that the absolute property does not hold on the spaces e_0^r and e_c^r, since $\|x\|_{e_0^r} \neq \||x|\|_{e_0^r}$ and $\|x\|_{e_c^r} \neq \||x|\|_{e_c^r}$ for at least one sequence in the spaces e_0^r and e_c^r, where $|x| = (|x_k|)$. This says that e_0^r and e_c^r are the sequence spaces of nonabsolute type.

Theorem 2.2. *The Euler sequence spaces* e_0^r *and* e_c^r *of nonabsolute type are linearly isomorphic to the spaces* c_0 *and* c, *respectively, i.e.,* $e_0^r \cong c_0$ *and* $e_c^r \cong c$.

Proof. To prove this, we should show the existence of a linear bijection between the spaces e_0^r and c_0. Consider the transformation T defined, with the notation of (2.2), from e_0^r to c_0 by $x \mapsto y = Tx$. The linearity of T is clear. Further, it is trivial that $x = \theta$ whenever $Tx = \theta$ and hence T is injective, where $\theta = (0, 0, 0, \ldots)$.

Let $y \in c_0$ and define the sequence $x = \{x_k(r)\}$ by

$$x_k(r) = \sum_{j=0}^{k} \binom{k}{j} (r-1)^{k-j} r^{-k} y_j, \quad k \in \mathbb{N}.$$

Then, we have

$$\lim_{n\to\infty} \left(E^r x\right)_n = \lim_{n\to\infty} \left[\sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k \sum_{j=0}^{k} \binom{k}{j} (r-1)^{k-j} r^{-k} y_j \right] = \lim_{n\to\infty} y_n = 0$$

which says us that $x \in e_0^r$. Additionally, we observe that

$$\| x \|_{e_0^r} = \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k \sum_{j=0}^{k} \binom{k}{j} (r-1)^{k-j} r^{-k} y_j \right| =$$

$$= \sup_{n\in\mathbb{N}} | y_n | = \| y \|_{c_0} < \infty.$$

Consequently, we see from here that T is surjective and is norm preserving. Hence, T is a linear bijection which therefore says us that the spaces e_0^r and c_0 are linearly isomorphic, as was desired.

It is clear here that if the spaces e_0^r and c_0 are respectively replaced by the spaces e_c^r and c, then we obtain the fact that $e_c^r \cong c$. This completes the proof.

We now may give our two theorems on the inclusion relations concerning with the spaces e_0^r and e_c^r.

Theorem 2.3. *Although the inclusions* $c_0 \subset e_0^r$ *and* $c \subset e_c^r$ *strictly hold, neither of the spaces* e_0^r *and* ℓ_∞ *includes the other one.*

Proof. Let us take any $s \in c_0$. Then, bearing in mind the regularity of the Euler means of order r, we immediately observe that $E^r s \in c_0$ which means that $s \in e_0^r$. Hence, the inclusion $c_0 \subset e_0^r$ holds. Furthermore, let us consider the sequence $u = \{u_k(r)\}$ defined by $u_k(r) = (-r)^{-k}$ for all $k \in \mathbb{N}$. Then, since $E^r u = \left\{(-r)^k\right\} \in c_0$, u is in e_0^r but not in c_0. By the similar discussion, one can see that the inclusion $c \subset e_c^r$ also holds.

To establish the second part of theorem, consider that sequence $u = \{u_k(r)\}$ defined above, and $x = e = (1, 1, 1, \ldots)$. Then, u is in e_0^r but not in ℓ_∞ and x is in ℓ_∞ but not in e_0^r. Hence, the sequence spaces e_0^r and ℓ_∞ overlap but neither contains the other. This completes the proof.

Theorem 2.4. *If* $0 < t < r < 1$, *then* $e_0^r \subset e_0^t$ *and* $e_c^r \subset e_c^t$.

Proof. Let us take $x = (x_k) \in e_0^r$. Then, for all $k \in \mathbb{N}$, we observe that

$$z_k = \sum_{i=0}^{k} e_{ki}^t x_i = \sum_{i=0}^{k} e_{ki}^t \left(\sum_{j=0}^{i} e_{ij}^{1/r} y_j \right) = \sum_{j=0}^{k} e_{kj}^{t/r} y_j.$$

Since $0 < \dfrac{t}{r} < 1$, the method $E^{t/r}$ is regular which implies that $z = (z_k) \in c_0$ whenever $y = (y_k) \in c_0$ and we thus see that $x = (x_k) \in e_0^t$. This means that the inclusion $e_0^r \subset e_0^t$ holds.

Now, one can show in the similar way that the inclusion $e_c^r \subset e_c^t$ also holds and so we leave the detail to the reader.

3. The basis for the spaces e_0^r and e_c^r. In the present section, we give two sequences of the points of the spaces e_0^r and e_c^r which form the basis for the spaces e_0^r and e_c^r.

Firstly, we define the Schauder basis of a normed space. If a normed sequence space λ contains a sequence (b_n) with the property that, for every $x \in \lambda$, there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \| x - (\alpha_0 b_0 + \alpha_1 b_1 + \ldots + \alpha_n b_n) \| = 0,$$

then (b_n) is called a *Schauder basis* (or briefly *basis*) for λ. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x = \sum \alpha_k b_k$.

Theorem 3.1. *Define the sequence* $b^{(k)}(r) = \{b_n^{(k)}(r)\}_{n\in\mathbb{N}}$ *of the elements of the space* e_0^r *by*

$$b_n^{(k)}(r) = \begin{cases} 0, & 0 \le n < k, \\ \binom{n}{k}(r-1)^{n-k} r^{-n}, & n \ge k, \end{cases} \tag{3.1}$$

for every fixed $k \in \mathbb{N}$. *Then:*

(i) *The sequence* $\{b^{(k)}(r)\}_{k\in\mathbb{N}}$ *is a basis for the space* e_0^r *and any* $x \in e_0^r$ *has a unique representation of the form*

$$x = \sum_k \lambda_k(r) b^{(k)}(r). \tag{3.2}$$

(ii) *The set* $\{e, b^{(k)}(r)\}$ *is a basis for the space* e_c^r *and any* $x \in e_c^r$ *has a unique representation of the form*

$$x = le + \sum_k [\lambda_k(r) - l] b^{(k)}(r), \tag{3.3}$$

where $\lambda_k(r) = (E^r x)_k$ *for all* $k \in \mathbb{N}$ *and*

$$l = \lim_{k\to\infty} (E^r x)_k. \tag{3.4}$$

Proof. (i) It is clear that $\{b^{(k)}(r)\} \subset e_0^r$, since

$$E^r b^{(k)}(r) = e^{(k)} \in c_0, \quad k = 0, 1, 2, \ldots, \tag{3.5}$$

where $e^{(k)}$ is the sequence whose only nonzero term is a 1 in k-th place for each $k \in \mathbb{N}$.

Let $x \in e_0^r$ be given. For every nonnegative integer m, we put

$$x^{[m]} = \sum_{k=0}^{m} \lambda_k(r) b^{(k)}(r). \tag{3.6}$$

Then, by applying E^r to (3.6) with (3.5), we obtain that

$$E^r x^{[m]} = \sum_{k=0}^{m} \lambda_k(r) E^r b^{(k)}(r) = \sum_{k=0}^{m} \left(E^r x\right)_k e^{(k)}$$

and

$$\left\{E^r\left(x - x^{[m]}\right)\right\}_i = \begin{cases} 0, & 0 \le i \le m, \\ \left(E^r x\right)_i, & i > m, \end{cases} \qquad i, m \in \mathbb{N}.$$

Given $\varepsilon > 0$, then there is an integer m_0 such that

$$\left|\left(E^r x\right)_m\right| < \frac{\varepsilon}{2}$$

for all $m \ge m_0$. Hence,

$$\left\| x - x^{[m]} \right\|_{e_0^r} = \sup_{n \ge m} \left|\left(E^r x\right)_n\right| \le \sup_{n \ge m_0} \left|\left(E^r x\right)_n\right| \le \frac{\varepsilon}{2} < \varepsilon$$

for all $m \ge m_0$ which proves that $x \in e_0^r$ is represented as in (3.2).

Let us show the uniqueness of the representation for $x \in e_0^r$ given by (3.2). Suppose, on the contrary, that there exists a representation $x = \sum_k \mu_k(r) b^{(k)}(r)$. Since the linear transformation T, from e_0^r to c_0, used in the proof of Theorem 2.2 is continuous, at this stage we have

$$\left(E^r x\right)_n = \sum_k \mu_k(r) \left\{E^r b^{(k)}(r)\right\}_n = \sum_k \mu_k(r) e_n^{(k)} = \mu_n(r), \quad n \in \mathbb{N},$$

which contradicts the fact that $\left(E^r x\right)_n = \lambda_n(r)$ for all $n \in \mathbb{N}$. Hence, the representation (3.2) of $x \in e_0^r$ is unique. Thus, the proof of the first part of theorem is completed.

(ii) Since $\left\{b^{(k)}(r)\right\} \subset e_0^r$ and $e \in c$, the inclusion $\left\{e, b^{(k)}(r)\right\} \subset e_c^r$ trivially holds. Let us take $x \in e_c^r$. Then, there uniquely exists an l satisfying (3.4). We thus have the fact that $u \in e_0^r$ whenever we set $u = x - le$. Therefore, we deduce by the part (i) of the present theorem that the representation of u is unique. This leads us to the fact that the representation of x given by (3.3) is unique and this step concludes the proof.

4. The α-, β-, γ- and continuous duals of the spaces e_0^r and e_c^r. In this section, we state and prove the theorems determining the α-, β-, γ- and continuous duals of the sequence spaces e_0^r and e_c^r of nonabsolute type.

For the sequence spaces λ and μ, define the set $S(\lambda, \mu)$ by

$$S(\lambda, \mu) = \left\{z = (z_k) \in w \colon xz = (x_k z_k) \in \mu \quad \text{for all} \quad x \in \lambda\right\}. \tag{4.1}$$

With the notation of (4.1), the α-, β- and γ-duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ, are defined by

$$\lambda^\alpha = S(\lambda, \ell_1), \quad \lambda^\beta = S(\lambda, cs), \quad \text{and} \quad \lambda^\gamma = S(\lambda, bs).$$

It is well known that

$$\left(\ell_p\right)^{\beta} = \ell_q \quad \text{and} \quad (\ell_\infty)^{\beta} = \ell_1, \tag{4.2}$$

where $1 \le p < \infty$ and $p^{-1} + q^{-1} = 1$ (see [15, p. 68, 69]). We shall throughout denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

The continuous dual of a normed space X is defined as the space of all bounded linear functionals on X and is denoted by X^*.

We shall begin with quoting the lemmas, due to Stieglitz and Tietz [16], which are needed in proving Theorems 4.1 – 4.3.

Lemma 4.1. *$A \in (c_0 : \ell_1) = (c : \ell_1)$ if and only if*

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} a_{nk} \right| < \infty.$$

Lemma 4.2. *$A \in (c_0 : c)$ if and only if*

$$\lim_{n\to\infty} a_{nk} = \alpha_k, \quad k \in \mathbb{N}, \tag{4.3}$$

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}| < \infty. \tag{4.4}$$

Lemma 4.3. *$A \in (c_0 : \ell_\infty)$ if and only if* (4.4) *holds.*

Theorem 4.1. *The α-dual of the spaces e_0^r and e_c^r is*

$$b_r = \left\{ a = (a_k) \in w \colon \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} \binom{n}{k} (r-1)^{n-k} r^{-n} a_n \right| < \infty \right\}.$$

Proof. Let $a = (a_n) \in w$ and define the matrix B^r whose rows are the product of the rows of the matrix $E^{1/r}$ and the sequence $a = (a_n)$. Bearing in mind the relation (2.2), we immediately derive that

$$a_n x_n = \sum_{k=0}^{n} \binom{n}{k} (r-1)^{n-k} r^{-n} a_n y_k = \left(B^r y\right)_n, \quad n \in \mathbb{N}. \tag{4.5}$$

We therefore observe by (4.5) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in e_0^r$ or e_c^r if and only if $B^r y \in \ell_1$ whenever $y \in c_0$ of c. Then we derive by Lemma 4.1 that

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} \binom{n}{k} (r-1)^{n-k} r^{-n} a_n \right| < \infty$$

which yields the consequence that $\left\{e_0^r\right\}^{\alpha} = \left\{e_c^r\right\}^{\alpha} = b_r$.

Theorem 4.2. *Define the sets d_1^r, d_2^r, and d_3^r by*

$$d_1^r = \left\{ a = (a_k) \in w \colon \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \right| < \infty \right\},$$

$$d_2^r = \left\{ a = (a_k) \in w \colon \sum_{j=k}^{\infty} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \ \text{exists for each} \ k \in \mathbb{N} \right\},$$

and

$$d_3^r = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \ \ exists \right\}.$$

Then $\{e_0^r\}^\beta = d_1^r \cap d_2^r$ *and* $\{e_c^r\}^\beta = d_1^r \cap d_2^r \cap d_3^r$.

Proof. Because of the proof may also be obtained for the space e_c^r in the similar way, we omit it and give the proof only for the space e_0^r. Consider the equation

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} \left[\sum_{j=0}^{k} \binom{k}{j} (r-1)^{k-j} r^{-k} y_j \right] a_k =$$

$$= \sum_{k=0}^{n} \left[\sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j \right] y_k = \left(T^r y \right)_n, \tag{4.6}$$

where $T^r = \left(t_{nk}^r\right)$ is defined by

$$t_{nk}^r = \begin{cases} \sum_{j=k}^{n} \binom{j}{k} (r-1)^{j-k} r^{-j} a_j, & 0 \le k \le n, \\ 0, & k > n, \end{cases} \qquad k,\ n \in \mathbb{N}. \tag{4.7}$$

Thus, we deduce from Lemma 4.2 with (4.6) that $ax = (a_x x_k) \in cs$ whenever $x = (x_k) \in e_0^r$ if and only if $T^r y \in c$ whenever $y = (y_k) \in c_0$. Therefore, we derive from (4.3) and (4.4) that

$$\lim_{n\to\infty} t_{nk}^r \ \text{ exists for each } \ k \in \mathbb{N} \ \text{ and } \ \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| t_{nk}^r \right| < \infty \tag{4.8}$$

which shows that $\{e_0^r\}^\beta = d_1^r \cap d_2^r$.

Theorem 4.3. *The* γ*-dual of the spaces* e_0^r *and* e_c^r *is* d_1^r.

Proof. It is of course that the present theorem may be proved by the technique used in the proof of Theorems 4.1 and 4.2, above. But we prefer here following the classical way and give the proof for the space e_0^r.

Let $a = (a_k) \in d_1^r$ and $x = (x_k) \in e_0^r$. Consider the following equality:

$$\left| \sum_{k=0}^{n} a_k x_k \right| = \left| \sum_{k=0}^{n} a_k \left[\sum_{j=0}^{k} \binom{k}{j} (r-1)^{k-j} r^{-k} y_j \right] \right| = \left| \sum_{k=0}^{n} t_{nk}^r y_k \right| \le \sum_{k=0}^{n} \left| t_{nk}^r \right| |y_k|$$

which gives us by taking supremum over $n \in \mathbb{N}$ that

$$\sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} a_k x_k \right| \le \sup_{n\in\mathbb{N}} \left(\sum_{k=0}^{n} \left| t_{nk}^r \right| |y_k| \right) \le \|y\|_{c_0} \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| t_{nk}^r \right| \right) \le \infty.$$

This means that $a = (a_k) \in \{e_0^r\}^\gamma$. Hence,

$$d_1^r \subset \{e_0^r\}^\gamma. \tag{4.9}$$

Conversely, let $a = (a_k) \in \{e_0^r\}^\gamma$ and $x \in e_0^r$. Then, one can easily see that $\left\{ \sum_{k=0}^{n} t_{nk}^r y_k \right\}_{n\in\mathbb{N}} \in \ell_\infty$ whenever $(a_k x_k) \in bs$. This shows that the triangle matrix $T^r = \left(t_{nk}^r\right)$, defined by (4.7), is in the class $(c_0 : \ell_\infty)$. Hence, the condition

$$\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|t_{nk}^{r}\right| < \infty$$

is satisfied which yields that $a=(a_k)\in d_1^r$. That is to say that

$$\left\{e_0^r\right\}^{\gamma} \subset d_1^r. \tag{4.10}$$

Therefore by combining the inclusions (4.9) and (4.10), we deduce that the γ-dual of the space e_0^r is d_1^r and this completes the proof.

Theorem 4.4. *$\left\{e_c^r\right\}^*$ and $\left\{e_0^r\right\}^*$ are isometrically isomorphic to ℓ_1.*

Proof. We only give the proof for the space e_c^r. Suppose that $f\in\left\{e_c^r\right\}^*$. Since by the part (ii) of Theorem 3.1, $\left\{e, b^k(r)\right\}$ is a basis for the space e_c^r and any element $x\in e_c^r$ can be expressed as in the form of (3.3). By the linearity and the continuity of f, we get from (3.3) that

$$f(x) = lf(e) + \sum_k[\lambda_k(r)-l]f\left\{b^{(k)}(r)\right\}$$

for all $x\in e_c^r$. Define the sequence $x=\{x_k(r)\}\in e_c^r$ such that $\|x\|_{e_c^r}=1$ by

$$x_k(r) = \begin{cases} \displaystyle\sum_{j=0}^{k}\binom{k}{j}(r-1)^{k-j}r^{-k}\operatorname{sgn} f\left(b^{(j)}(r)\right), & 0\le k\le n, \\ \displaystyle\sum_{j=0}^{n}\binom{k}{j}(r-1)^{k-j}r^{-k}\operatorname{sgn} f\left(b^{(j)}(r)\right), & k>n. \end{cases}$$

Therefore, we have

$$|f(x)| = \sum_{k=0}^{n}\left|f\left(b^{(k)}(r)\right)\right| \le \|f\|. \tag{4.11}$$

It follows from the inequality (4.11) that

$$\sum_k\left|f\left(b^{(k)}(r)\right)\right| = \sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|f\left(b^{(k)}(r)\right)\right| \le \|f\|.$$

Write $f(x) = al+\sum_k a_k\lambda_k(r)$, where $a = f(e)-\sum_k f\left(b^{(k)}(r)\right)$, $a_k = f\left(b^{(k)}(r)\right)$, the series $\sum_k f\left(b^{(k)}(r)\right)$ being absolutely convergent. Since $\left|\lim_{k\to\infty}\left(E^r x\right)_k\right| \le$ $\le \|x\|_{e_c^r}$, we have

$$|f(x)| \le \|x\|_{e_c^r}\left(|a|+\sum_k|a_k|\right),$$

whence

$$\|f\| \le |a| + \sum_k|a_k|. \tag{4.12}$$

Also, for $\|x\|_{e_c^r}=1$, we have

$$|f(x)| \leq \|f\|,$$

so we define for any $n \geq 0$,

$$x_k(r) = \begin{cases} \sum_{j=0}^{k} \binom{k}{j}(r-1)^{k-j} r^{-k} \operatorname{sgn} a_j, & 0 \leq k \leq n, \\ \sum_{j=0}^{n} \binom{k}{j}(r-1)^{k-j} r^{-k} \operatorname{sgn} a_j + \sum_{j=n+1}^{k} \binom{k}{j}(r-1)^{k-j} r^{-k} \operatorname{sgn} a, & k > n. \end{cases}$$

Then $x \in e_c^r$ with $\|x\|_{e_c^r} = 1$, $\lim_{k\to\infty}\left(E^r x\right)_k = \operatorname{sgn} a$ and so

$$|f(x)| = \left| |a| + \sum_{k=0}^{n} |a_k| + \sum_{k=n+1}^{\infty} a_k \operatorname{sgn} a \right| \leq \|f\|. \tag{4.13}$$

Since $(a_k) \in \ell_1$ we have $\sum_{k=n+1}^{\infty} a_k \to 0$ as $n \to \infty$, and thus we obtain by letting $n \to \infty$ in (4.13) that

$$|a| + \sum_k |a_k| \leq \|f\|. \tag{4.14}$$

Combining the results (4.12) and (4.14) we see that

$$\|f\| = |a| + \sum_k |a_k|$$

which is the norm on ℓ_1.

Now, let $T\colon \left\{e_c^r\right\}^* \to \ell_1$ be defined by $f \mapsto (a, a_0, a_1, \ldots)$. Then, we have

$$|T(f)| = |a| + |a_0| + |a_1| + \ldots = |f|.$$

$\|T(f)\|$ being the ℓ_1-norm. Thus, T is norm preserving. T is obviously surjective and linear, and hence is an isomorphism from $\left\{e_c^r\right\}^\gamma$ to ℓ_1. This completes the proof.

5. Some matrix mappings related to the Euler sequence spaces. In this section, we characterize the matrix mappings from e_c^r into some of the known sequence spaces and into the Euler, difference, Riesz, Cesàro sequence spaces. We directly prove the theorems characterizing the classes $\left(e_c^r : \ell_p\right)$, $\left(e_c^r : c\right)$ and derive the other characterizations from them by means of a given basic lemma, where $1 \leq p \leq \infty$. Furthermore, we give a Steinhaus-type theorem which asserts that the classes $\left(e_\infty^r : c\right)$ and $\left(e_c^r : c\right)_s$ are disjoint.

We shall write throughout for brevity that

$$a(n,k) = \sum_{j=0}^{n} a_{jk} \quad \text{and} \quad \tilde{a}_{nk} = \sum_{j=k}^{n} \binom{j}{k}(r-1)^{j-k} r^{-j} a_{nj}$$

for all $k, n \in \mathbb{N}$. We will also use the similar notations with other letters and use the convention that any term with negative subscript is equal to naught. We shall begin with two lemmas which are needed in the proof of our theorems.

Lemma 5.1 [14, p. 57]. *The matrix mappings between the BK-spaces are continuous.*

Lemma 5.2 [14, p. 128]. *$A \in (c : \ell_p)$ if and only if*

$$\sup_{F \in \mathcal{F}} \sum_n \left| \sum_{k \in F} a_{nk} \right|^p < \infty, \quad 1 \le p < \infty. \tag{5.1}$$

Theorem 5.1. *$A \in (e_c^r : \ell_p)$ if and only if*
(i) *for* $1 \le p < \infty$,

$$\sup_{F \in \mathcal{F}} \sum_n \left| \sum_{k \in F} \tilde{a}_{nk} \right|^p < \infty, \tag{5.2}$$

$$\tilde{a}_{nk} \quad \textit{exists for all} \quad k,\ n \in \mathbb{N}, \tag{5.3}$$

$$\sum_k \tilde{a}_{nk} \quad \textit{converges for all} \quad n \in \mathbb{N}, \tag{5.4}$$

$$\sup_{m \in \mathbb{N}} \sum_{k=0}^{m} \left| \sum_{j=k}^{m} \binom{j}{k} (r-1)^{j-k} r^{-j} a_{nj} \right| < \infty, \quad n \in \mathbb{N}; \tag{5.5}$$

(ii) *for* $p = \infty$, (5.3) *and* (5.5) *hold, and*

$$\sup_{n \in \mathbb{N}} \sum_k |\tilde{a}_{nk}| < \infty. \tag{5.6}$$

Proof. Suppose conditions (5.2) – (5.5) hold and take any $x \in e_c^r$. Then, $\{a_{nk}\}_{k \in \mathbb{N}} \in \{e_c^r\}^\beta$ for all $n \in \mathbb{N}$ and this implies that Ax exists. Let us define the matrix $B = (b_{nk})$ with $b_{nk} = \tilde{a}_{nk}$ for all $k,\ n \in \mathbb{N}$. Then, since (5.1) is satisfied for that matrix B, we have $B \in (c : \ell_p)$. Let us now consider the following equality obtained from the m-th partial sum of the series $\sum_k a_{nk} x_k$:

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} \sum_{j=k}^{m} \binom{j}{k} (r-1)^{j-k} r^{-j} a_{nj} y_k, \quad m,\ n \in \mathbb{N}. \tag{5.7}$$

Therefore, we derive from (5.7) as $m \to \infty$ that

$$\sum_k a_{nk} x_k = \sum_k \tilde{a}_{nk} y_k, \quad n \in \mathbb{N}, \tag{5.8}$$

which yields by taking ℓ_p-norm that

$$\| Ax \|_{\ell_p} = \| By \|_{\ell_p} < \infty.$$

This means that $A \in (e_c^r : \ell_p)$.

Conversely, suppose that $A \in (e_c^r : \ell_p)$. Then, since e_c^r and ℓ_p are the *BK*-spaces, we have from Lemma 5.1 that there exists some real constant $K > 0$ such that

$$\| Ax \|_{\ell_p} \le K \| x \|_{e_c^r} \tag{5.9}$$

for all $x \in e_c^r$. Since inequality (5.9) also holds for the sequence $x = (x_k) = \sum_{k \in F} b^{(k)}(r)$ belonging to the space e_c^r, where $b^{(k)}(r) = \{b_n^{(k)}(r)\}$ is defined by (3.1), we thus have for any $F \in \mathcal{F}$ that

$$\|Ax\|_{\ell_p} = \left(\sum_n \left|\sum_{k\in F}\tilde{a}_{nk}\right|^p\right)^{1/p} \le K\|x\|_{e_c^r} = K$$

which shows the necessity of (5.2).

Since A is applicable to the space e_c^r by the hypothesis, the necessity of conditions (5.3) – (5.5) is trivial. This completes the proof of the part (i) of theorem.

Since the part (ii) may also be proved in the similar way that of the part (i), we leave the detailed proof to the reader.

Theorem 5.2. $A \in (e_c^r : c)$ *if and only if* (5.3), (5.5), *and* (5.6) *hold,*

$$\lim_{n\to\infty}\tilde{a}_{nk} = \alpha_k \quad \text{for each} \quad k\in\mathbb{N} \tag{5.10}$$

and

$$\lim_{n\to\infty}\sum_k \tilde{a}_{nk} = \alpha. \tag{5.11}$$

Proof. Suppose that A satisfies conditions (5.3), (5.5), (5.6), (5.10), and (5.11). Let us take any $x=(x_k)$ in e_c^r such that $x_k \to l$ as $k\to\infty$. Then Ax exists and it is trivial that the sequence $y=(y_k)$ connected with the sequence $x=(x_k)$ by relation (2.2) is in c such that $y_k\to l$ as $k\to\infty$. At this stage, we observe from (5.10) and (5.6) that

$$\sum_{j=0}^{k}|\alpha_j| \le \sup_{n\in\mathbb{N}}\sum_j|\tilde{a}_{nj}| < \infty$$

holds for every $k\in\mathbb{N}$. This leads us to the consequence that $(\alpha_k)\in\ell_1$. Considering (5.8), let us write

$$\sum_k a_{nk}x_k = \sum_k \tilde{a}_{nk}(y_k-l) + l\sum_k\tilde{a}_{nk}, \quad n\in\mathbb{N}. \tag{5.12}$$

In this situation, by letting $n\to\infty$ in (5.12), we observe that the first term on the right-hand side tends to $\sum_k \alpha_k(y_k-l)$ by (5.6) and (5.10), and the second term tends to $l\alpha$ by (5.11). Now, under the light of these facts, we obtain from (5.12) as $n\to\infty$ that

$$(Ax)_n \to \sum_k \alpha_k(y_k-l) + l\alpha \tag{5.13}$$

and this shows that $A\in(e_c^r:c)$.

Conversely, suppose that $A\in(e_c^r:c)$. Then, since the inclusion $c\subset\ell_\infty$ holds, the necessities of (5.3), (5.5) and (5.6) are immediately obtained from the part (ii) of Theorem 5.1. To prove the necessity of (5.10), consider the sequence $x = b^{(k)}(r) = \left\{b_n^{(k)}(r)\right\}_{n\in\mathbb{N}}$ in e_c^r, defined by (3.1), for every fixed $k\in\mathbb{N}$. Since Ax exists and is in c for every $x\in e_c^r$, one can easily see that $Ab^{(k)}(r)=\{\tilde{a}_{nk}\}_{n\in\mathbb{N}}\in c$ for each $k\in\mathbb{N}$ which shows the necessity of (5.10).

Similarly, by putting $x=e$ in (5.8), we also obtain that $Ax=\left\{\sum_k\tilde{a}_{nk}\right\}_{n\in\mathbb{N}}$ which belongs to the space c and this shows the necessity of (5.11). This step concludes the proof.

Let us define the concept of s-multiplicativity of a limitation matrix. When there is some notion of limit or sum in the sequence spaces λ and μ, we shall say that the method $A \in (\lambda : \mu)$ is multiplicative s if every $x \in \lambda$ is A-summable to s times of $\lim x$, for any fixed real number s and denote the class of all s-multiplicative matrices by $(\lambda : \mu)_s$. It is of course that the class $\left(e_c^r : c\right)_s$ of s-multiplicative matrices reduces to the classes $\left(e_c^r : c_0\right)$ and $\left(e_c^r : c\right)_{reg}$ in the cases $s = 0$ and $s = 1$, respectively; here, $\left(e_c^r : c\right)_{reg}$ denotes the class of all matrix mappings A from e_c^r to c such that $A - \lim x = \lim x$ for all $x \in e_c^r$. Now, we may give the corollary to Theorem 5.2 without proof.

***Corollary* 5.1.** *$A \in \left(e_c^r : c\right)_s$ if and only if* (5.3), (5.5), (5.6) *hold,* (5.10) *and* (5.11) *also hold with $\alpha_k = 0$ for each $k \in \mathbb{N}$ and $\alpha = s$, respectively.*

The Steinhaus-type theorems were formulated by Maddox [17] as follows: Consider the class $(\lambda : \mu)_1$ of 1-multiplicative matrices and ν be a sequence space such that $\nu \supset \lambda$. Then the result of the form $(\lambda : \mu)_1 \cap (\nu : \mu) = \varnothing$ is called a theorem of Steinhaus type, where $\varnothing$ denotes the empty set. Now, we may give a Steinhaus-type theorem whose proof requires the following lemma:

Lemma 5.3 ([12], Corollary 2.5 (iii)). *$A \in \left(e_\infty^r : c\right)$ if and only if* (5.6), (5.10) *hold, and*

$$\lim_{n\to\infty} \sum_k |\tilde{a}_{nk}| = \sum_k \left| \lim_{n\to\infty} \tilde{a}_{nk} \right|, \tag{5.14}$$

$$\lim_{m\to\infty} \sum_k \left| \sum_{j=k}^{m} \binom{j}{k} (r-1)^{j-k} r^{-j} a_{nj} \right| = \sum_k |\tilde{a}_{nk}|, \quad n \in \mathbb{N}. \tag{5.15}$$

Theorem 5.3. *There is no matrix belonging to the classes both $\left(e_c^r : c\right)_s$ and $\left(e_\infty^r : c\right)$.*

Proof. Suppose that the classes $\left(e_c^r : c\right)_s$ and $\left(e_\infty^r : c\right)$ are not disjoint. Then there is at least matrix A satisfying the conditions of both Lemma 5.3 and Corollary 5.1. Combining condition (5.10) with (5.14), one can easily see that

$$\lim_{n\to\infty} \sum_k |\tilde{a}_{nk}| = 0$$

which contradicts condition (5.11). This completes the proof.

We now may present our basis lemma which is useful for obtaining the characterization of some new matrix classes from Theorems 5.1, 5.2 and Corollary 5.1.

Lemma 5.4 ([12], Lemma 2.6). *Let λ, μ be any two sequence spaces, A be an infinite matrix and B a triangle matrix. Then $A \in (\lambda : \mu_B)$ if and only if $BA \in (\lambda : \mu)$.*

It is trivial that Lemma 5.4 has several consequences, some of them give the necessary and sufficient conditions of matrix mappings between the Euler sequence spaces. Indeed, combining Lemma 5.4 with Theorems 5.1, 5.2 and Corollary 5.1, one can easily derive the following results:

***Corollary* 5.2.** *Let $A = (a_{nk})$ be an infinite matrix and define the matrix $C = (c_{nk})$ by*

$$c_{nk} = \sum_{j=0}^{n} \binom{n}{j} (1-t)^{n-j} t^j a_{jk}, \quad 0 < t < 1 \quad \text{and} \quad k, n \in \mathbb{N}.$$

Then the necessary and sufficient conditions in order for A *belongs to anyone of the classes* $\left(e_c^r : e_\infty^t\right)$, $\left(e_c^r : e_p^t\right)$, $\left(e_c^r : e_c^t\right)$ *and* $\left(e_c^r : e_c^t\right)_s$ *are obtained from the respective ones in Theorems* 5.1, 5.2 *and Corollary 5.1 by replacing the entries of the matrix* A *by those of the matrix* C.

***Corollary* 5.3.** *Let* $A = (a_{nk})$ *be an infinite matrix and* $t = (t_k)$ *be a sequence of positive numbers and define the matrix* $C = (c_{nk})$ *by*

$$c_{nk} = \frac{1}{T_n} \sum_{j=0}^{n} t_j a_{jk}, \quad k, n \in \mathbb{N},$$

where $T_n = \sum_{k=0}^{n} t_k$ *for all* $n \in \mathbb{N}$. *Then the necessary and sufficient conditions in order for* A *belongs to anyone of the classes* $\left(e_c^r : r_\infty^t\right)$, $\left(e_c^r : r_p^t\right)$, $\left(e_c^r : r_c^t\right)$ *and* $\left(e_c^r : r_c^t\right)_s$ *are obtained from the respective ones in Theorems* 5.1, 5.2 *and Corollary* 5.1 *by replacing the entries of the matrix* A *by those of the matrix* C; *here,* r_p^t *is defined in* [18] *as the space of all sequences whose* R^t-*transforms are in the space* ℓ_p *and is derived from the paranormed spaces* $r^t(p)$ *in the case* $p_k = p$ *for all* $k \in \mathbb{N}$, *and* r_∞^t, r_c^t *are obtained in the case* $p = e$ *from the paranormed spaces* $r_\infty^t(p)$, $r_c^t(p)$ *and are studied by Malkowsky* [4].

Since the spaces r_∞^t and r_p^t reduce in the case $t = e$ to the Cesàro sequence spaces X_∞ and X_p of nonabsolute type, respectively, Corollary 5.3 also includes the characterizations of the classes $\left(e_c^r : X_\infty\right)$ and $\left(e_c^r : X_p\right)$.

***Corollary* 5.4.** *Let* $A = (a_{nk})$ *be an infinite matrix and define the matrices* $C = (c_{nk})$ *and* $D = (d_{nk})$ *by* $c_{nk} = a_{nk} - a_{n+1,k}$ *and* $d_{nk} = a_{nk} - a_{n-1,k}$ *for all* $k, n \in \mathbb{N}$. *Then the necessary and sufficient conditions in order for* A *belongs to anyone of the classes* $\left(e_c^r : \ell_\infty(\Delta)\right)$, $\left(e_c^r : c(\Delta)\right)$, $\left(e_c^r : c(\Delta)\right)_s$ *and* $\left(e_c^r : bv_p\right)$ *are obtained from the respective ones in Theorem* 5.2, *Corollary* 5.1 *and Theorem* 5.1 *by replacing the entries of the matrix* A *by those of the matrices* C *and* D; *here,* $\ell_\infty(\Delta)$, $c(\Delta)$ *denote the difference spaces of all bounded, convergent sequences and introduced by Kızmaz* [19].

***Corollary* 5.5.** *Let* $A = (a_{nk})$ *be an infinite matrix and define the matrix* $C = (c_{nk})$ *by*

$$c_{nk} = \sum_{j=0}^{n} \frac{1 + t^j}{1 + n} a_{jk}, \quad 0 < t < 1,$$

for all $k, n \in \mathbb{N}$. *Then the necessary and sufficient conditions in order for* A *belongs to anyone of the classes* $\left(e_c^r : a_\infty^t\right)$, $\left(e_c^r : a_p^t\right)$, $\left(e_c^r : a_c^t\right)$ *and* $\left(e_c^r : a_c^t\right)_s$ *are obtained from the respective ones in Theorems* 5.1, 5.2 *and Corollary* 5.1 *by replacing the entries of the matrix* A *by those of the matrix* C.

***Corollary* 5.6.** *Let* $A = (a_{nk})$ *be an infinite matrix and define the matrix* $C = (c_{nk})$ *by* $c_{nk} = a(n, k)$ *for all* $k, n \in \mathbb{N}$. *Then the necessary and sufficient conditions in order for* A *belongs to anyone on the classes* $\left(e_c^r : bs\right)$, $\left(e_c^r : cs\right)$ *and* $\left(e_c^r : cs\right)_s$ *are obtained from the respective ones in Theorems* 5.1, 5.2 *and Corollary* 5.1 *by replacing the entries of the matrix* A *by those of the matrix* C.

1. *Choudhary B., Nanda S.* Functional analysis with applications. – New Delhi: John Wiley & Sons Inc., 1989.
2. *Wang C.-S.* On Nölund sequence spaces // Tamkang J. Math. – 1978. – **9**. – P. 269 – 274.
3. *Ng P.-N., Lee P.-Y.* Cesàro sequences spaces of non-absolute type // Comment. Math. Prace Mat. – 1978. – **20**, № 2. – P. 429 – 433.
4. *Malkowsky E.* Recent results in the theory of matrix transformations in sequence space // Mat. Vesnik. – 1997. – **49**. – P. 187 – 196.
5. *Başar F., Altay B.* On the space of sequences of p-bounded variation and related matrix mappings // Ukr. Math. J. – 2003. – **55**, № 1. – P. 136 – 147.
6. *Aydın C., Başar F.* On the new sequence spaces which include the spaces c_0 and c // Hokkaido Math. J. – 2004. – **33**, № 1. – P. 1 – 16.
7. *Aydın C., Başar F.* Some new sequence spaces which include the spaces ℓ_p and ℓ_∞ (under communication).
8. *Aydın C., Başar F.* Some new difference sequence spaces // Appl. Math. Comput. – 2004 (to appear).
9. *Aydın C., Başar F.* Some new paranormed sequence spaces // Inform. Sci. – 2004. – **160**. – P. 27 – 40.
10. *Altay B., Başar F., Mursaleen.* On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞. I (under communication).
11. *Powell R. E., Shah S.* M. Summability theory and its applications. – London: Van Nostrand Reinhold Company, 1972.
12. *Başar F., Altay B.* On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞. II (under communication).
13. *Maddox I. J.* Elements of functional analysis. – 2-nd ed. – Cambridge: Univ. Press, 1988.
14. *Wilansky A.* Summability through functional analysis // North-Holland Math. Stud. – 1984. – **85**.
15. *Kamthan P. K., Gupta M.* Sequence spaces and series. – New York; Basel: Marcel Dekker Inc., 1981.
16. *Stieglitz M., Tietz H.* Matrixtransformationen von Folgenräumen Eine Ergebnisübersict // Math. Z. – 1977. – **154**. – S. 1 – 16.
17. *Maddox I. J.* On theorems of Steinhaus type // J. London Math. Soc. – 1967. – **42**. – P. 239 – 244.
18. *Altay B., Başar F.* On the paranormed Riesz sequence spaces of non-absolute type // Sotutheast Asian Bull. Math. – 2002. – **26**, № 5. – P. 701 – 715.
19. *Kızmaz H.* On certain sequence spaces // Can. Math. Bull. – 1981. – **24**, № 2. – P. 169 – 176.

Received 11.09.2003