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ON SOME EXTENTION PROPERTIES OF VON NEUMANN ALGEBRAS

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1. In the theory of the structure of von Neumann algebras, a great deal of discussions has been devoted to the algebraic invariants. Recently, J. Schwartz [6], has shown a new behavior of the hyperfinite factor by introducing a new property, called property P, and proved the existence of new algebraic type of continuous finite factor. Very recently, one of the authors has proved that property P is an algebraic invariant [2] and it makes us to have some interest that the main results of the paper [6], especially the key results [6: Cor. 6 and Lemma 7], can be deduced only from the point of view of the existence of a projection mapping of norm one.

Thus we shall investigate in the following the algebraic version of property P as an extension property of the commutant of a given von Neumann algebra. We shall also study this extension property as the property of the commutant itself. These properties will turn out to be algebraic invariants and it is proved that our extension property can be defined space-freely in a form very similar to the usual extension property of Banach space. Relationships between tensor products of von Neumann algebras and these properties are also studied.

2. In the following we denote by $\mathfrak{L}(\mathfrak{H})$, the algebra of all bounded linear operators on a Hilbert space $\mathfrak{H}$. We shall define the extension property of a von Neumann algebra $\mathfrak{A}$ as follows.

DEFINITION 2.1. A von Neumann algebra $\mathfrak{A}$, acting on a Hilbert space $\mathfrak{H}$, has extension property, if there exists a projection of norm one from $\mathfrak{L}(\mathfrak{H})$ to $\mathfrak{A}$.

If we consider this extension property as the property of the commutant $\mathfrak{A}'$, of $\mathfrak{A}$, we get an algebraic version of the property P in Schwartz [6].

DEFINITION 2.2. A von Neumann algebra $\mathfrak{A}$ is called to have abstract property P, abbreviated hereafter property AP, if the commutant $\mathfrak{A}'$ has extension property in the above Definition 2.1.

As we said already in the introduction, the crucial point in the result of [6] is simply that property P implies the above property AP and the group von Neumann algebra $A(\Phi)$ of a free group Φ having two generators has not property AP.

We shall show in the following steps that both of these properties are algebraic invariants.

LEMMA 2.3. *If $\mathfrak{A}$ has extension property then tensor product $\mathfrak{A}\otimes\mathfrak{L}(\mathfrak{K})$ has the same property.*

PROOF. Let π be a projection of norm one from $\mathfrak{L}(\mathfrak{H})$ to $\mathfrak{A}$, and $(T_{\iota\kappa})_{\iota,\kappa\in I}$ be the matrix representation of a bounded operator T on $\mathfrak{H}\otimes\mathfrak{K}$ according to the decomposition of $\mathfrak{H}\otimes\mathfrak{K}$ into $\sum_{\iota\in I}\mathfrak{H}_\iota$ $(\mathfrak{H}_\iota\cong\mathfrak{H})$ $T_{\iota\kappa}$ are all bounded operators on $\mathfrak{H}$. For arbitrary finite subset J of I, we denote by E_J the projection to the subspace $\sum_{\iota\in J}\mathfrak{H}_\iota$. Let $\pi_J(T)=(a_{\iota\kappa}(J)\,\pi(T_{\iota\kappa}))_{\iota,\kappa\in I}$ where

$$a_{\iota\kappa}(J)=\begin{cases}1 & \text{for } (\iota,\kappa)\in J\times J\\ 0 & \text{for } (\iota,\kappa)\notin J\times J\,.\end{cases}$$

$\pi_J(T)$ is a bounded operator on $\mathfrak{H}\otimes\mathfrak{K}$. It is clear that π_J is a linear self-adjoint projection mapping from $\mathfrak{L}(\mathfrak{H}\otimes\mathfrak{K})$ to $E_J[\mathfrak{A}\otimes\mathfrak{L}(\mathfrak{K})]\,E_J$. Suppose T is a positive operator on $\mathfrak{L}(\mathfrak{H}\otimes\mathfrak{K})$, then E_JTE_J is also positive and by [8: Theorem 1] and [3: Theorem 5] one can easily see that $\pi_J(T)\geqq 0$. Moreover the positiveness of the operator $\|T\|E_J-E_JTE_J$, implies

$$\pi_J(\|T\|E_J)-\pi_J(E_JTE_J)=\|T\|E_J-\pi_J(T)\geqq 0\,,$$

and

$$\|\pi_J(T)\|\leqq\|\,\|T\|E_J\,\|=\|T\|\,.$$

Moreover, by Theorem 1 in [8], a simple calculation shows that

$\pi_J(E_JSE_JTE_JS'E_J)=E_JSE_J\pi_J(T)\,E_JS'E_J$ for arbitrary $S,S'\in\mathfrak{A}\otimes\mathfrak{L}(\mathfrak{K})$ and $T\in\mathfrak{L}(\mathfrak{H}\otimes\mathfrak{K})$.

Therefore

$$\pi_J((T-\pi_J(T))^*(T-\pi_J(T)))\geqq 0 \quad \text{implies,}$$

$\pi_J(T^*T) - \pi_J(T)^* \pi_J(T) \geqq 0$. Hence for an arbitrary bounded operator T on $\mathfrak{H}\otimes\mathfrak{K}$. We have

$$\|\pi_J(T)\|^2 = \|\pi_J(T)^*\pi_J(T)\| \leqq \|\pi_J(T^*T)\| \leqq \|T^*T\| = \|T\|^2.$$

Hence $\|\pi_J(T)\| \leqq \|T\|$. Thus for a fixed operator T, $\{\pi_J(T)\}_{J\subset I}$ is a bounded family of operators in $\mathfrak{A}\otimes\mathfrak{L}(\mathfrak{K})$. Put $\tilde{\pi}(T) = \lim_J \pi_J(T)$ (operator Banach limit of $\pi_J(T)$ cf. [6]). Then as shown in [6] $\tilde{\pi}(T)$ is an operator of $\mathfrak{A}\otimes\mathfrak{L}(\mathfrak{K})$ and $\|\tilde{\pi}(T)\| \leqq \|T\|$. Moreover, one can easily see that $\tilde{\pi}$ is a projection mapping from $\mathfrak{L}(\mathfrak{H}\otimes\mathfrak{K})$ to $\mathfrak{A}\otimes\mathfrak{L}(\mathfrak{K})$. q.e.d.

For a von Neumann algebra $\mathfrak{A}$ on $\mathfrak{H}$ and a projection E in $\mathfrak{L}(\mathfrak{H})$ we denote by $\mathfrak{A}_E$ the restricted von Neumann algebra of $\mathfrak{A}$ to $E\mathfrak{H}$. If $E\in\mathfrak{A}$, $\mathfrak{A}_E$ consists of all elements T in $\mathfrak{A}$ with $ETE=T$.

LEMMA 2.4. *If $\mathfrak{A}$ has extension property then $\mathfrak{A}_E$ $(E\in\mathfrak{A})$ has the same property.*

PROOF. Let π be a projection of norm one from $\mathfrak{L}(\mathfrak{H})$ to $\mathfrak{A}$, then the restriction of π to the algebra $E\mathfrak{L}(\mathfrak{H})E=\mathfrak{L}(E\mathfrak{H})$ is a required projection as we see from Theorem 1 in [8].

LEMMA 2.5. *Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{A}\otimes\mathfrak{B}$ be von Neumann algebras and their tensor product as von Neumann algebras. Let φ and ψ be normal states on $\mathfrak{A}$ and $\mathfrak{B}$, then there exist linear mappings of norm one L_ψ and R_φ from $\mathfrak{A}\otimes\mathfrak{B}$ to $\mathfrak{A}$ and $\mathfrak{B}$ such that $L_\psi(T\otimes I)=T$ and $R_\varphi(I\otimes S)=S$, for $T\in\mathfrak{A}$ and $S\in\mathfrak{B}$.*

PROOF. We proceed along with L_ψ. For an arbitrary element $\sum_{i=1}^n T_i\otimes S_i$ in the algebraic tensor produt $\mathfrak{B}\odot\mathfrak{B}$, we put

$$L'_\psi\left(\sum_{i=1}^n T_i\otimes S_i\right) = \sum_{i=1}^n \langle S_i, \psi\rangle T_i.$$

Take an element T in $\mathfrak{A}\otimes\mathfrak{B}$ and suppose that $T_\alpha = \sum_{i=1}^{n_\alpha} T_i^\alpha \otimes S_i^\alpha$ converges σ-weakly to T with $\|T_\alpha\| \leqq \|T\|$. Then for a σ-weakly continuous linear functional φ on $\mathfrak{A}$, we have

$$\lim_\alpha \langle \sum_{i=1}^{n_\alpha} T_i^\alpha \otimes S_i^\alpha, \varphi\otimes\psi\rangle = \lim_\alpha \langle L'_\psi(T_\alpha), \varphi\rangle = \langle T, \varphi\otimes\psi\rangle$$

and

$$|<T, \varphi\otimes\psi>| \leqq \overline{\lim_{\alpha}} \|T_\alpha\|\|\varphi\| < \|T\|\|\varphi\| .$$

Hene $f_{T,\psi}(\varphi) = <T, \varphi\otimes\psi>$ is a bounded linear functional on $\mathfrak{A}_*$, the space of all σ-weakly continuous linear fun.tionals on $\mathfrak{A}$. Therefore there exists an element $L_\psi(T)\in\mathfrak{A}$ with the property

$$<L_\psi(T), \varphi> = f_{T,\psi}(\varphi) \quad \text{for all } \varphi\in\mathfrak{A}_* \text{ (cf. [1 : p. 40]).}$$

Thus we can define the mapping L_ψ from $\mathfrak{A}\otimes\mathfrak{B}$ to $\mathfrak{A}$ which is clearly seen as an extension of the above defined mapping, and

$$1 = \|L_\psi(I_{\mathfrak{H}}\otimes I_{\mathfrak{K}})\| \leqq \|L_\psi\| = \sup_{\|T\|\leqq 1}\|L_\psi(T)\| = \sup_{\substack{\|T\|\leqq 1\\ \|\varphi\|\leqq 1}} |<L_\varphi(T), \varphi>|$$
$$= \sup_{\substack{\|T\|\leqq 1\\ \|\varphi\|\leqq 1}} |<T, \varphi\otimes\psi>| \leqq 1 .$$

The argument for R_φ is almost similar.

An isomorphism from a von Neumann algebra $\mathfrak{A}$ on $\mathfrak{H}$ to $\mathfrak{A}\otimes I_{\mathfrak{K}}$ on $\mathfrak{H}\otimes\mathfrak{K}$ is called ampliation, where $I_{\mathfrak{K}}$ means the identity operator on $\mathfrak{K}$.

LEMMA 2.6. *A von Neumann algebra $\mathfrak{A}$ has extension property if and only if its ampliation $\mathfrak{A}\otimes I_{\mathfrak{K}}$ has extension property.*

PROOF. Let $\mathfrak{A}$ (on $\mathfrak{H}$) have extension property, i.e. there exists a projection π of norm one from $\mathfrak{L}(\mathfrak{H})$ to $\mathfrak{A}$. Let θ be the ampliation, then one easily sees that for a normal state ψ on $\mathfrak{L}_{\mathfrak{K}}$, $\theta\cdot\pi\cdot L_\psi$ is a projection of norm one from $\mathfrak{L}(\mathfrak{H}\otimes\mathfrak{K}) = \mathfrak{L}(\mathfrak{B})\otimes\mathfrak{L}(\mathfrak{K})$ to $\mathfrak{A}\otimes I_{\mathfrak{K}}$.

Conversely, if $\mathfrak{A}\otimes I_{\mathfrak{K}}$ has extention property and π is a projection of norm one from $\mathfrak{L}(\mathfrak{H}\otimes\mathfrak{K})$, the mapping $\theta^{-1}\cdot\pi\cdot\theta$ is a projection of norm one from $\mathfrak{L}(\mathfrak{H})$ to $\mathfrak{A}$.

THEOREM 2.7. *Extension property is invariant under isomorphism.*

PROOF. Let $\mathfrak{A}, \mathfrak{B}$ are isomorphic von Neumann algebras on $\mathfrak{H}_1$, $\mathfrak{H}_2$ respectively. If $\mathfrak{A}$ has extension property, its ampliation $\mathfrak{A}\otimes I_{\mathfrak{K}}$ has the same property by the above Lemma 2.7. Let $\mathfrak{K}$ be a Hilbert space whose dimension is high enough for the dimension of $\mathfrak{H}_1$ and $\mathfrak{H}_2$. Then the theory of spacial invariants tells us that $\mathfrak{A}\otimes I_{\mathfrak{K}}$ is spacially isomorphic to $\mathfrak{B}\otimes I_{\mathfrak{K}}$ (cf. [9 : Theorem 2]). Therefore $\mathfrak{B}\otimes I_{\mathfrak{K}}$ has extension property, and $\mathfrak{H}$ has the same property.

THEOREM 2.8. *Property AP is invariant under isomorphism.*

PROOF. Let $\mathfrak{A}$ and $\mathfrak{B}$ be mutually isomorphic von Neumann algebras and $\mathfrak{A}$ has property AP. By [1: Chap. I, §4, 4, Théorème 3], it is sufficient to consider only the following cases; $\mathfrak{B}=\mathfrak{A}\otimes I_{\mathfrak{K}}$ or $\mathfrak{B}=\mathfrak{A}_E$ *where* E is a projection of $\mathfrak{A}'$. However both case are treated already in Lemma 2.3 and Lemma 2.4.
q. e. d.

All von Neumann algebras of type I have extension property and property AP and a continuous hyperfinite factor has also both properties. In fact, in case of algebras of type I, we may assume without loss of generality that their commutants are commutative and since commutative von Neumann algebras have usual extension properties as Banach spaces, they have property AP. A continuous hyperfinite factor has property P, hence property AP. And as we shall see later (Theorem 4.2), generally speaking, extension property and property AP are seen equivalent properties for semi-finite von Neumann algebras in the sense that if the algebra has the one it has always the other.

One interesting conclusion of this observation is the following fact which is proved at first by Sakai [11]. As it is well known there exists a projection of norm one from a continuous hyperfinite factor to any von Neumann subalgebras of it, so every subfactor of a continuous hyperfinite factor have extension property. On the other hand there exists a factor of type II_1 which does not have extension property ([6]). Hence we can see that the following question raised in Sakai's lecture note ([5; §3. 85, question 3]) is negative; can we construct for any finite factor $\mathfrak{B}$, a hyperfinite factor $\mathfrak{A}$, such that $\mathfrak{B}\subset\mathfrak{A}$?.

3. For the tensor products of von Neumann algebras, if $\mathfrak{A}$ and $\mathfrak{B}$ have property P then $\mathfrak{A}\otimes\mathfrak{B}$ has the same property [2]. We shall show in the following that our extension property and property AP are almost compatible with the tensor products of von Neumann algebras.

LEMMA 3.1. *Let ψ (resp. φ) be a normal state on $\mathfrak{L}(\mathfrak{K})$ (resp. $\mathfrak{L}(\mathfrak{H})$), then the mapping L_ψ (resp. R_φ) maps $(\mathfrak{A}\otimes\mathfrak{B})'$ onto $\mathfrak{A}'$ (resp. $\mathfrak{B}'$).*

PROOF. We notice at first that the definition of the mapping L_ψ shows the relation

$$S[L_\psi(T)]S' = L_\psi((S\otimes I)\cdot T\cdot(S'\otimes I))$$

for every operators $S, S'\in\mathfrak{L}(\mathfrak{H})$ and $T\in\mathfrak{L}(\mathfrak{H}\otimes\mathfrak{K})$. Hence if $S\in\mathfrak{A}$ and

$T \in (\mathfrak{A} \otimes \mathfrak{B})'$ we have

$$S[L_\psi(T)] = L_\psi((S \otimes I) \cdot T) = L_\psi(T \cdot (S \otimes I)) = [L_\psi(T)]\, S,$$

which shows that $L_\psi(T)$ belongs to $\mathfrak{A}'$. It it clear that L_ψ is an onto-mapping.
q. e. d.

THEOREM 3.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be von Neumann algebras, then $\mathfrak{A} \otimes \mathfrak{B}$ has property AP if and only if both $\mathfrak{A}$ and $\mathfrak{B}$ have property AP.*

PROOF. Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ have property *AP*. By Lemma 2.3 there exists a projection π_1 (resp. π_2) of norm one from $\mathfrak{L}(\mathfrak{H} \otimes \mathfrak{K})$ to $\mathfrak{A}' \otimes \mathfrak{L}(\mathfrak{K})$ (resp. $\mathfrak{L}(\mathfrak{H}) \otimes \mathfrak{B}'$). Put $\pi = \pi_1 \pi_2$. We assert that π is a projection of norm one from $\mathfrak{L}(\mathfrak{H} \otimes \mathfrak{K})$ to

$$\mathfrak{A} \otimes \mathfrak{L}(\mathfrak{K}) \cap \mathfrak{L}(\mathfrak{H}) \otimes \mathfrak{B}' = (\mathfrak{A} \otimes \mathfrak{B})'.$$

In fact, take an operator $T \in \mathfrak{L}(\mathfrak{H} \otimes \mathfrak{K})$ and $S \in I\mathfrak{H} \otimes \mathfrak{B}$. By Theorem 1 in [8], the identity

$$S\pi(T) = S\pi_1(\pi_2(T)) = \pi_1(S\pi_2(T)) = \pi_1(\pi_2(T)S) = \pi_1(\pi_2(T))S = \pi(T)S$$

holds since $S \in \mathfrak{A}' \otimes \mathfrak{L}(\mathfrak{K})$ and $\pi_2(T) \in (I\mathfrak{H} \otimes \mathfrak{B})'$. Hence

$$\pi(T) \in \mathfrak{A}' \otimes \mathfrak{L}(\mathfrak{K}) \cap \mathfrak{L}(\mathfrak{H}) \otimes \mathfrak{B} = (\mathfrak{A} \otimes \mathfrak{B})'.$$

Clearly $\pi(T) = T$ for arbitrary $T \in (\mathfrak{A} \otimes \mathfrak{B})'$ and

$$1 = \|\pi(I)\| \leqq \|\pi\| \leqq \|\pi_1\| \cdot \|\pi_2\| \leqq 1.$$

Thus $\mathfrak{A} \otimes \mathfrak{B}$ has property *AP*.

Conversely, let $\mathfrak{A} \otimes \mathfrak{B}$ have property *AP*, then there exists a projection π of norm one from $\mathfrak{L}(\mathfrak{H} \otimes \mathfrak{K})$ to $(\mathfrak{A} \otimes \mathfrak{B})'$. Let θ_1 (resp. θ_2) be the ampliation of $\mathfrak{L}(\mathfrak{H})$ (resp. $\mathfrak{L}(\mathfrak{K})$) to $\mathfrak{L}(\mathfrak{H}) \otimes I_{\mathfrak{K}}$ (resp. $I\mathfrak{H} \otimes \mathfrak{L}(\mathfrak{K})$). Then from Lemma 3.1 we can easily see that the mapping $L_\psi \cdot \pi \cdot \theta_1$ (resp. $R_\varphi \cdot \pi \cdot \theta_2$) is a projection of norm one from $\mathfrak{L}(\mathfrak{H})$ (resp. $\mathfrak{L}(\mathfrak{K})$) to $\mathfrak{A}'$(resp. $\mathfrak{B}'$) where ψ (resp. φ) is a normal state of $\mathfrak{L}(\mathfrak{K})$ (resp. $\mathfrak{L}(\mathfrak{H})$).

On the contrary, the result for extension property is not completely satisfactory one.

THEOREM 3.3. *If $\mathfrak{A} \otimes \mathfrak{B}$ has extension property, then both $\mathfrak{A}$ and $\mathfrak{B}$*

have extension property. The converse holds for semi-finite von Neumann algebras.

PROOF. Let π be a projection of norm one from $\mathfrak{L}(\mathfrak{H}\otimes\mathfrak{K})$ to $\mathfrak{A}\otimes\mathfrak{B}$. Using the same notations as in the proof of Theorem 3.2, it is seen without difficulty that the mappings $L_\psi\cdot\pi\cdot\theta_1$ and $R_\varphi\cdot\pi\cdot\theta_2$ are projections of norm one to $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Next, suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are semi-finite and both have extention property. This means that $\mathfrak{A}'$ and $\mathfrak{B}'$ have property AP, hence $\mathfrak{A}'\otimes\mathfrak{B}'$ has also property AP by the above theorem. Thus $(\mathfrak{A}'\otimes\mathfrak{B}')'=\mathfrak{A}\otimes\mathfrak{B}$ has extension property.

It is to be noticed that these results simplify considerably well the discussions in Schwartz [7] for finding the third factor of type III. In fact, all we must do in this case is to find a factor of type III having property AP. Once it is found, we can easily construct from this factor, say $\mathfrak{A}$, the triplet of factors of type III which are not mutually non-isomorphic in the following way. Let $\mathfrak{B}$ be continuous hyperfinite factor and $A(\Phi)$ the group von Neumann algebra of the free group Φ having two generators. Then $\mathfrak{A}\otimes\mathfrak{B}$, $\mathfrak{A}\otimes\mathfrak{B}\otimes A(\Phi)$ and a factor of type III constructed in Pukanszky [4], which has not property L are required triplet since $\mathfrak{A}\otimes\mathfrak{B}$ has both property AP and L ([5 ; p. 3. 83]) while $\mathfrak{A}\otimes\mathfrak{B}\otimes A(\Phi)$ has property L([5; p. 3. 83]) but not property AP.

4. In this section, at first we shall show that the extension property of von Neumann algebra can be characterized as an extension property free from the underlying Hilbert space. Let A be a C^*-algebra and φ a state of A. By Φ_φ, we shall denote the canonical representation of A induced by φ and its representation space will be written by $\mathfrak{H}_\varphi$. We shall mean by Γ_φ the canonical linear mapping from A to a dense subspace of $\mathfrak{H}_\varphi$ (cf. [1]).

THEOREM 4.1. *Extension property of a von Neumann algebra $\mathfrak{A}$ is characterized as the following general extension property:*

(*) *For any C^*-algebra A containing $\mathfrak{A}$, there exists a projection of norm one from A to $\mathfrak{A}$.*

PROOF. Let $\mathfrak{A}$ have extension property and $\{\varphi_\alpha\}$ be a total family of normal states of $\mathfrak{A}$. Let $\tilde{\varphi}_\alpha$ be the state extension of φ_α to A and consider the representations $\Phi=\sum_\alpha\Phi_{\varphi_\alpha}$ of $\mathfrak{A}$ and $\Phi=\sum_\alpha\Phi_{\tilde{\varphi}_\alpha}$ of A. Then one sees without difficulty that the restriction of $\tilde{\Phi}(\mathfrak{A})$ to the subspace $\sum_\alpha\oplus[\Phi_{\tilde{\varphi}_\alpha}(\mathfrak{A})\Gamma_{\tilde{\varphi}_\alpha}(I)]$ is spacially isomorphic to a von Neumann algebra $\Phi(\mathfrak{A})$ by an isomorphism θ, where I means the unit of $\mathfrak{A}$. Let E be the projection to the subspace

$\sum_{\alpha} \oplus [\widetilde{\Phi}(\mathfrak{A})\, \Gamma_{\tilde{\varphi}_\alpha}(I)]$. We can see from Theorem 2.7 that $\Phi(\mathfrak{A})$ has extension property hence $E\widetilde{\Phi}(\mathfrak{A})E$ has extension property. There exists a projection π of norm one from $E\mathfrak{L}(\sum_{\alpha} \oplus \mathfrak{H}_{\tilde{\varphi}_\alpha})E = \mathfrak{L}(E(\sum_{\alpha} \oplus \mathfrak{H}_{\tilde{\varphi}_\alpha}))$ to $E\widetilde{\Phi}(\mathfrak{A})E$. Consider the mapping π_0 from A to $\mathfrak{A}$ defined by $\pi_0(a) = \Phi^{-1} \cdot \theta \cdot \pi(E\widetilde{\Phi}(a)E)$,

$$i.e.\quad A \xrightarrow{\widetilde{\Phi}} \widetilde{\Phi}(A) \longrightarrow E\widetilde{\Phi}(A)E \xrightarrow{\pi} E\widetilde{\Phi}(\mathfrak{A})E \xrightarrow{\theta} \Phi(\mathfrak{A}) \xrightarrow{\Phi^{-1}} \mathfrak{A}.$$

Clearly π_0 is a projection mapping with its norm not exceeding one. Since other implication is trivial, this concludes the proof. q. e. d.

We must notice that a slight modificaton of the above proof shows, of couse, that there exists always a projection of norm one from a C^*-algebra A to a C^*-subalgebra B of A which is isomrphic to $\mathfrak{A}$. Therefore, for example, any representing image of a contitnuous hyperfinite factor has extension property whatever the image becomes a von Neumann algebra or not on that acting space.

The theorem shows that our extension property of von Neumann algebras is very similar to the usual extension property of Banach spaces. Finally we shall show that extension property and property AP are equivalent for semi-finite von Neumann algebras in the sense that if the algebra has the the one it has always the other.

THEOREM 4.2. *Let $\mathfrak{A}$ be a semi-finite von Neumann algebra, then $\mathfrak{A}$ has extension property if and only if $\mathfrak{A}$ has property AP.*

PROOF. Let $\mathfrak{A}$ have extension property. Since $\mathfrak{A}$ is semi-finite, there exists a standard von Neumann algebra $\mathfrak{B}$ on $\mathfrak{H}$ such that $\mathfrak{A}$ is spacially isomorphic to $(\mathfrak{B}\otimes I_{\mathfrak{K}})_E$ where $E \in \mathfrak{B}'\otimes\mathfrak{L}(\mathfrak{K})$, and the mapping $\mathfrak{B}\otimes I_{\mathfrak{K}} \to (\mathfrak{B}\otimes I_{\mathfrak{K}})_E$ is an isomorphism. Since $\mathfrak{B}$ is standard, there exists a canonical involution mapping J in its acting space $\mathfrak{H}$ such as $J\mathfrak{B}J = \mathfrak{B}'$. By Theorem 2.7, $\mathfrak{B}$ has extension property. Let π be a projection of norm one from $\mathfrak{L}(\mathfrak{H})$ to $\mathfrak{B}$. Then an easy calculation shows that the mapping $T \in \mathfrak{L}(\mathfrak{H}) \to J\pi(JTJ)\,J$ is a projection of norm one to $\mathfrak{B}'$, and $\mathfrak{B}'$ has extension property. Therefore, by Lemma 2.3 and Lemma 2.4 $(\mathfrak{B}'\otimes\mathfrak{L}(\mathfrak{K}))_E$ has extension property, that is, $(\mathfrak{B}\otimes I_{\mathfrak{K}})_E$ has property AP.

Suppose $\mathfrak{A}$ has property AP, i.e. $\mathfrak{A}'$ has extension property. Then $\mathfrak{A}'$ has property AP by the above argument, that is, $(\mathfrak{A}')' = \mathfrak{A}$ has extension property. q. e. d.

The above theorem is likely to hold even in case of $\mathfrak{A}$ being of type III, however, we don't know the exact consequence. Of course, we can find an example of a factor of type III having both properties. For this purpose, it is sufficient to choose simply a known factor of type III having canonical involution mapping J and go along the same line as in the proof of the above theorem.

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