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On Some Generalized Difference Sequence Spaces

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Abstract

The main aim of this article is to introduce a new class of difference sequence spaces associated with a multiplier sequence which are isomorphic with the classical spaces c_0, c and ℓ_∞ respectively and investigate some algebraic and topological structures of the spaces.

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1 Introduction

Let w denote the space of all scalar sequences and any subspace of w is called a sequence space. Let ℓ_∞ , c and c_0 be the spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms respectively, normed by

$$\|x\|_\infty = \sup_k |x_k| \tag{1.1}$$

Let $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then for E a sequence space, the multiplier sequence space $E(\Lambda)$, associated with the multiplier sequence Λ is defined as

$$E(\Lambda) = \{(x_k) \in w : (\lambda_k x_k) \in E\}.$$

The scope for the studies on sequence spaces was extended by using the notion of associated multiplier sequences. Goes and Goes [3] defined the differentiated sequence space dE and integrated sequence space $\int E$ for a given sequence space E, using the multiplier sequences (k^{-1}) and (k) respectively. A multiplier sequence can be used to accelerate the convergence of the sequences in some spaces. In some sense, it can be viewed as a catalyst, which is used to accelerate the process of chemical reaction. Sometimes the associated multiplier sequence delays the rate of convergence of a sequence. Thus it also covers a larger class of sequences for study. In the present article we shall consider a general multiplier sequence $\Lambda = (\lambda_k)$ of non-zero scalars.

The notion of difference sequence spaces was introduce by Kizmaz [4]. The notion was further generalized by Et and Colak [1] by introducing the spaces $\ell_\infty(\Delta^s)$, $c(\Delta^s)$ and $c_0(\Delta^s)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [6], who studied the spaces $\ell_\infty(\Delta_r)$, $c(\Delta_r)$ and $c_0(\Delta_r)$. Tripathy, Esi and Tripathy [7] generalized the above notions and unified these as follows:

Let r, s be non-negative integers, then for Z a given sequence space we have

$$Z(\Delta_r^s) = \{x = (x_k) \in w : (\Delta_r^s x_k) \in Z\},$$

where $\Delta_r^s x = (\Delta_r^s x_k) = (\Delta_r^{s-1} x_k - \Delta_r^{s-1} x_{k+r})$ and $\Delta_r^0 x_k = x_k$ for all $k \in N$ and which is equivalent to the binomial representation

$$\Delta_r^s x_k = \sum_{v=0}^{s} (-1)^v \binom{s}{v} x_{k+rv}.$$

Let r, s be non-negative integers and $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then for Z, a given sequence space we define the following sequence spaces:

$$Z(\Delta_{(r)}^s, \Lambda) = \{x = (x_k) \in w : (\Delta_{(r)}^s \lambda_k x_k) \in Z\},\ for\ \ Z = \ell_\infty,\ c\ \ and\ \ c_0$$

where $(\Delta_{(r)}^s \lambda_k x_k) = (\Delta_{(r)}^{s-1} \lambda_k x_k - \Delta_{(r)}^{s-1} \lambda_{k-r} x_{k-r})$ and $\Delta_{(r)}^0 \lambda_k x_k = \lambda_k x_k$ for all $k \in N$ and which is equivalent to the binomial representation

$$\Delta_{(r)}^s \lambda_k x_k = \sum_{v=0}^{s} (-1)^v \binom{s}{v} \lambda_{k-rv} x_{k-rv}.$$

In this expansion it is important to note that we take $\lambda_{k-rv} = 0$ and $x_{k-rv} = 0$ for non-positive values of $k - rv$.

For $s = 1$ and $\lambda_k = 1$ for all $k \in N$, we get the spaces $\ell_\infty(\Delta_r)$, $c(\Delta_r)$ and $c_0(\Delta_r)$. For $r = 1$ and $\lambda_k = 1$ for all $k \in N$, we get the spaces $\ell_\infty(\Delta^s)$, $c(\Delta^s)$ and $c_0(\Delta^s)$. For $r = s = 1$ and $\lambda_k = 1$ for all $k \in N$, we get the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$.

Similarly we can define the spaces $Z(\Delta_r^s, \Lambda)$, for $Z = \ell_\infty$, c and c_0.

2 Main Results

In this section we study the spaces $Z(\Delta_{(r)}^s, \Lambda)$ and $Z(\Delta_r^s, \Lambda)$, for $Z = \ell_\infty$, c and c_0 for some linear algebraic and topological structure.

Proposition 2.1. *(i) The spaces* $Z(\Delta^s_{(r)}, \Lambda)$ *and* $Z(\Delta^s_r, \Lambda)$, *for* $Z = \ell_\infty$, c *and* c_0 *are linear.*
(ii) $c_0(\Delta^s_{(r)}, \Lambda) \subset c(\Delta^s_{(r)}, \Lambda) \subset \ell_\infty(\Delta^s_{(r)}, \Lambda)$.
(iii) $c_0(\Delta^s_r, \Lambda) \subset c(\Delta^s_r, \Lambda) \subset \ell_\infty(\Delta^s_r, \Lambda)$.

Proof. Proofs are routine verification and thus omitted

Proposition 2.2. *For* $Z = \ell_\infty$, c *and* c_0
(i) $Z(\Delta^s_r, \Lambda)$ *are normed linear spaces, normed by*

$$\|x\| = \sum_{k=1}^{rs} |\lambda_k x_k| + \sup_k |\Delta^s_r \lambda_k x_k| \tag{2.1}$$

(ii) $Z(\Delta^s_{(r)}, \Lambda)$ *are normed linear spaces, normed by*

$$\|x\|' = \sup_k |\Delta^s_{(r)} \lambda_k x_k| \tag{2.2}$$

Proof. (i) For $x = \theta$, we have $\|x\| = 0$. Conversely, let $\|x\| = 0$. Then using (2.1), we have

$$\sum_{k=1}^{rs} |\lambda_k x_k| + \sup_k |\Delta^s_r \lambda_k x_k| = 0 \tag{2.3}$$

It follows that $\sum\limits_{k=1}^{rs} |\lambda_k x_k| = 0$. Hence $x_k = 0$, for $k = 1, 2, \ldots, rs$, since (λ_k) is a sequence of non-zero scalars (2.4)
Again from (2.3) we have $\sup\limits_k |\Delta^s_r \lambda_k x_k| = 0$. It follows that $\Delta^s_r \lambda_k x_k = 0$, for all $k \geq 1$. Let $k = 1$, then $\Delta^s_r \lambda_1 x_1 = \sum\limits_{v=0}^{s} (-1)^v \binom{s}{v} \lambda_{1+rv} x_{1+rv} = 0$ and so $x_{1+rs} = 0$ using (2.4). Similarly taking $k = 2$, we have $x_{2+rs} = 0$. Proceeding in this way $x_k = 0$ for all $k \geq 1$. Hence $x = \theta$. Again it is easy to show that $\|x + y\| \leq \|x\| + \|y\|$ and for any scalar α, $\|\alpha x\| = |\alpha|\ \|x\|$. This completes the proof.

(ii) For this part we only prove that $\|x\|' = 0$ implies $x = \theta$. Proof of other properties are similar with part (i).
Let $\|x\|' = 0$. Then using (2.2), we have $\sup\limits_k |\Delta^s_{(r)} \lambda_k x_k| = 0$. It follows that $\Delta^s_{(r)} \lambda_k x_k = 0$ for all $k \geq 1$. Let $k = 1$, then $\Delta^s_{(r)} \lambda_1 x_1 = \sum\limits_{v=0}^{s} (-1)^v \binom{s}{v} \lambda_{1-rv} x_{1-rv} = 0$ and so $x_1 = 0$, since (λ_k) is a sequence of non-zero scalars and by putting $x_{1-rv} = 0$ for $v = 1, 2, \ldots, s$. Similarly taking $k = 2$ we have $x_2 = 0$. Proceeding in this way $x_k = 0$ for all $k \geq 1$. Hence $x = \theta$. This completes the proof.

Proposition 2.3. *For $Z = \ell_\infty$, c and c_0*
(i) $Z(\Delta^i_{(r)}, \Lambda) \subset Z(\Delta^s_{(r)}, \Lambda)$, $(i = 0, 1, 2, \ldots, s-1)$ *and the inclusions are proper.*
(ii) $Z(\Delta^i_r, \Lambda) \subset Z(\Delta^s_r, \Lambda)$, $(i = 0, 1, 2, \ldots, s-1)$ *and the inclusions are proper.*

Proof. Proofs are easy and so omitted.

Remark 2.4. *It is obvious that for any sequence $x = (x_k)$, $x \in Z(\Delta^s_r, \Lambda)$ if and only if $x \in Z(\Delta^s_{(r)}, \Lambda)$. Hence from the above Proposition we can conclude that Z is a subspace of both $Z(\Delta^s_r)$ and $Z(\Delta^s_{(r)})$. Now if we compare norm (1.1) with norms (2.1) and (2.2), (2.2) looks quite natural as norm on a generalized space of Z. Keeping these in mind this new operator $\Delta^s_{(r)}$ is introduced. The fruitfulness of introducing this operator will be more visible in Proposition 2.8. and Proposition 2.9. Again it is clear that norms $\|.\|$ and $\|.\|'$ are equivalent.*

Theorem 2.5. *For $Z = \ell_\infty$, c and c_0*
(i) $Z(\Delta^s_{(r)}, \Lambda)$ *are Banach spaces, normed by* $\|.\|'$.
(ii) $Z(\Delta^s_r, \Lambda)$ *are Banach spaces, normed by* $\|.\|$.

Proof. We give the proof of part (i) only. Proof of part (ii) follows on applying similar arguments.

Let (x^i) be a Cauchy sequence in $Z(\Delta^s_{(r)}, \Lambda)$, where $(x^i) = (x^i_k) = (x^i_1, x^i_2, \ldots)$ for each $i \geq 1$. Then for a given $\epsilon > 0$, there exists a positive integer n_0 such that

$$\|x^i - x^j\|' = \sup_k |\Delta^s_{(r)} \lambda_k (x^i_k - x^j_k)| < \epsilon,$$

for all $i, j \geq n_0$. It follows that $|\Delta^s_{(r)} \lambda_k (x^i_k - x^j_k)| < \epsilon$, for all $i, j \geq n_0$ and for all $k \geq 1$. This implies that $(\Delta^s_{(r)} \lambda_k x^i_k)$ is a Cauchy sequence in C for all $k \geq 1$ and so it is convergent in C for all $k \geq 1$.

Let $\lim_{i \to \infty} \Delta^s_{(r)} \lambda_k x^i_k = y_k$, say for each $k \geq 1$. Considering $k = 1, 2, \ldots, rs, \ldots$, we can easily conclude that $\lim_{i \to \infty} x^i_k = x_k$, exists for each $k \geq 1$. Now we can have

$$\lim_{j \longrightarrow \infty} |\Delta^s_{(r)} \lambda_k (x^i_k - x^j_k)| < \epsilon$$

for all $i \geq n_0$ and $k \geq 1$ and hence

$$\sup_k |\Delta^s_{(r)} \lambda_k (x^i_k - x_k) < \epsilon$$

for all $i \geq n_0$. This implies that $(x^i - x) \in Z(\Delta^s_{(r)}, \Lambda)$. Since $Z(\Delta^s_{(r)}, \Lambda)$ is a linear space, $x = x^i - (x^i - x) \in Z(\Delta^s_{(r)}, \Lambda)$. Hence $Z(\Delta^s_{(r)}, \Lambda)$ is complete.

From the above proof we can easily conclude that $\|x^i - x\|' \longrightarrow 0$ implies $|x^i_k - x_k| \longrightarrow 0$ as $i \longrightarrow \infty$, for each $k \geq 1$. Hence we have the following Proposition.

Proposition 2.6. *For $Z = \ell_\infty$, c and c_0, $Z(\Delta^s_{(r)}, \Lambda)$ and $Z(\Delta^s_r, \Lambda)$ are BK-spaces.*

Proposition 2.7. *(i) The spaces $c_0(\Delta^s_{(r)}, \Lambda)$ and $c(\Delta^s_{(r)}, \Lambda)$ are nowhere dense subsets of $\ell_\infty(\Delta^s_{(r)}, \Lambda)$.*
(ii) The spaces $c_0(\Delta^s_r, \Lambda)$ and $c(\Delta^s_r, \Lambda)$ are nowhere dense subsets of $\ell_\infty(\Delta^s_r, \Lambda)$.

Proof. We give the proof of part (i). Proof of part (ii) follows on applying similar arguments.

From proposition 2.1 (ii), we have $c_0(\Delta^s_{(r)}, \Lambda)$ and $c(\Delta^s_{(r)}, \Lambda)$ are proper subspaces of $\ell_\infty(\Delta^s_{(r)}, \Lambda)$. Again from Theorem 2.5 (i), it follows that $c_0(\Delta^s_{(r)}, \Lambda)$ and $c(\Delta^s_{(r)}, \Lambda)$ are closed subspaces of $\ell_\infty(\Delta^s_{(r)}, \Lambda)$. Hence the proof follows.

Proposition 2.8. *(i) The spaces $c_0(\Delta^s_{(r)}, \Lambda)$, $c(\Delta^s_{(r)}, \Lambda)$ and $\ell_\infty(\Delta^s_{(r)}, \Lambda)$ are topologically isomorphic to the spaces c_0, c and ℓ_∞ respectively.*
(ii) The spaces $Sc_0(\Delta^s_r, \Lambda)$, $Sc(\Delta^s_r, \Lambda)$ and $S\ell_\infty(\Delta^s_r, \Lambda)$ are topologically isomorphic to the spaces c_0, c and ℓ_∞ respectively, where $SZ(\Delta^s_r, \Lambda)$ is a subspace of $Z(\Delta^s_r, \Lambda)$ defined by

$$SZ(\Delta^s_r, \Lambda) = \{x = (x_k) : x \in Z(\Delta^s_r, \Lambda),\ x_1 = x_2 = \cdots = x_{rs} = 0\}$$

and normed by

$$\|x\| = \sup_k |\Delta^s_r \lambda_k x_k|.$$

Proof. We give the proof of part (i) and proof of part (ii) follows on applying similar arguments. For $Z = \ell_\infty$, c and c_0, consider the mapping

$$T : Z(\Delta^s_{(r)}, \Lambda) \longrightarrow Z,$$

defined by $Tx = y = (\Delta^s_{(r)} \lambda_k x_k)$, for every $x \in Z(\Delta^s_{(r)}, \Lambda)$ (2.5)
Then clearly T is a linear homeomorphism.

Proposition 2.9. *For $Z = \ell_\infty$, c and c_0, $Z(\Delta^s_{(r)}, \Lambda)$ and $Z(\Delta^s_r, \Lambda)$ are isometrically isomorphic with the spaces c_0, c and ℓ_∞ respectively.*

Proof. In view of Remark 2.4, we can define a mapping exactly similar with (2.5) on both the spaces $Z(\Delta^s_{(r)}, \Lambda)$ and $Z(\Delta^s_r, \Lambda)$. Then it is obvious that this mapping will be an isomorphic and isometry. This completes the proof.

Theorem 2.10. *The continuous dual of $Z(\Delta^s_{(r)}, \Lambda)$ and $Z(\Delta^s_r, \Lambda)$, for $Z = c$, c_0 is ℓ_1.*

Proof. Since continuous dual of c_0 and c is ℓ_1, proof follows from Proposition 2.9.

Theorem 2.11. *The spaces $Z(\Delta^s_{(r)}, \Lambda)$ and $Z(\Delta^s_r, \Lambda)$, for $Z = c$, c_0 are separable.*

Proof. Since ℓ_1 is separable, the proof follows from the fact that if the dual of a normed space is separable, then the space itself is separable.

Theorem 2.12. *(i) The spaces $Z(\Delta^s_{(r)}, \Lambda)$ and $Z(\Delta^s_r, \Lambda)$ for $Z = c$, c_0 are not reflexive.*
(ii) The spaces $Z(\Delta^s_{(r)}, \Lambda)$ and $Z(\Delta^s_r, \Lambda)$ for $Z = c$, c_0 are not Hilbert spaces.
(iii) The spaces $\ell_\infty(\Delta^s_{(r)}, \Lambda)$ and $\ell_\infty(\Delta^s_r, \Lambda)$ are not Hilbert spaces.
(iii) The spaces $\ell_\infty(\Delta^s_{(r)}, \Lambda)$ and $\ell_\infty(\Delta^s_r, \Lambda)$ are not reflexive.

Proof. (i) Since ℓ_1 is not reflexive, the proof follows from the fact that if a normed space is reflexive then its dual is also reflexive.
(ii) Proof follows from the fact that every Hilbert space is reflexive.
(iii) We know that a closed subspace of a Hilbert space is Hilbert space. Here $Z(\Delta^s_{(r)}, \Lambda)$, $Z = c$, c_0 are closed subspaces of $\ell_\infty(\Delta^s_{(r)}, \Lambda)$ but both of them are not Hilbert spaces. So $\ell_\infty(\Delta^s_{(r)}, \Lambda)$ is not a Hilbert space.
By applying similar arguments we can argue that $\ell_\infty(\Delta^s_r, \Lambda)$ is not a Hilbert space.
(iv) We know that a closed subspace of a reflexive Banach space is reflexive. Here $Z(\Delta^s_{(r)}, \Lambda)$, $Z = c$, c_0 are closed subspaces of $\ell_\infty(\Delta^s_{(r)}, \Lambda)$ but both of them are not reflexive. So $\ell_\infty(\Delta^s_{(r)}, \Lambda)$ is not reflexive.
By applying similar arguments we can argue that $\ell_\infty(\Delta^s_r, \Lambda)$ is not reflexive.

Theorem 2.13. *The spaces $\ell_\infty(\Delta^s_{(r)}, \Lambda)$ and $\ell_\infty(\Delta^s_r, \Lambda)$ are not separable.*

Proof. We give the proof for the space $\ell_\infty(\Delta^s_{(r)}, \Lambda)$ only. For the other space $\ell_\infty(\Delta^s_r, \Lambda)$ it will follow on applying similar arguments. We can associate for every $y' \in [0, 1]$, a sequence $y = (y_i) \in \ell_\infty(\Delta^s_{(r)}, \Lambda)$ of zeros and ones, where $y' = \frac{y_1}{2} + \frac{y_2}{2^2} + \frac{y_3}{2^3} + \dots$. Since $[0, 1]$ is uncountable, so there are uncountably many sequences of zeros and ones. For any two different sequences x and y of $\ell_\infty(\Delta^s_{(r)}, \Lambda)$ we have

$$\begin{aligned} \|x - y\|' &= \sup_k |\Delta^s_{(r)} \lambda_k (x_k - y_k)| \\ &= \sup_k |(x_k - y_k)| = 1 \quad \text{by Proposition 2.9} \end{aligned}$$

If we let each of these sequences be the centers of neighbourhoods, say, of radius $\frac{1}{3}$, these neighbourhoods do not intersect and we have uncountably many of them. If D is any dense set in $\ell_\infty(\Delta^s_{(r)}, \Lambda)$, each of these non intersecting neighbourhoods must contain an element of D. Hence D can not be countable. Since D was an arbitrary dense set, this shows that $\ell_\infty(\Delta^s_{(r)}, \Lambda)$ can not have countable dense subset. Consequently, $\ell_\infty(\Delta^s_{(r)}, \Lambda)$ is not separable.

Theorem 2.14. *(i) Let $A \subset Z$. If A is convex, then $A(\Delta^s_{(r)}, \Lambda)$ is convex in $Z(\Delta^s_{(r)}, \Lambda)$.*
(ii) Let $A \subset Z$. If A is convex, then $A(\Delta^s_r, \Lambda)$ is convex in $Z(\Delta^s_r, \Lambda)$.

Proof. We give proof of part (i) only. The proof of part (ii) follows on applying similar arguments.
Let x, $y \in A(\Delta^s_{(r)}, \Lambda)$, then $(\Delta^s_{(r)}\lambda_k x_k)$, $(\Delta^s_{(r)}\lambda_k y_k) \in A$. Since $\Delta^s_{(r)}$ is linear, we have

$$\delta(\Delta^s_{(r)}\lambda_k x_k) + (1-\delta)(\Delta^s_{(r)}\lambda_k y_k) = \Delta^s_{(r)}(\delta(\lambda_k x_k) + (1-\delta)(\lambda_k y_k)), \quad 0 \le \delta \le 1$$

Since A is convex, $\delta(\Delta^s_{(r)}\lambda_k x_k) + (1-\delta)(\Delta^s_{(r)}\lambda_k y_k) \in A$.
Hence $\delta x + (1-\delta)y \in A(\Delta^s_{(r)}, \Lambda)$, $\quad 0 \le \delta \le 1$. This completes the proof.

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