# 51. *On Some Homogeneous Boundary Value Problems Bounded Below*

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**§1. Introduction.** Let $\Omega$ be a compact oriented Riemannian $n$-space with smooth boundary $\Gamma$. Let $A$ be a linear partial differential operator on $\Omega$ of order $2m$. We assume $A$ is strongly elliptic, that is, there is a constant $C>0$ such that, for any $x$ in $\Omega$ and for any non zero vector $\xi$ cotangent to $\Omega$ at $x$, we have

$$C^{-1}|\xi|^{2m} \leq \mathrm{Re}\, \sigma_{2m}(A)(x, \xi) \leq C|\xi|^{2m},$$

where $\sigma_{2m}(A)$ is the principal symbol of $A$. We consider normal systems $\{B_r\}_{r\in R}$, $R=(r_1, r_2, \cdots, r_m)$, of $m$ boundary operators $B_{r_j}$. $r_j$ is the order of $B_{r_j}$. We assume $r_j<2m$ for any $j=0, 1, \cdots, m$. The problem to be considered is

**Problem 1.** Characterize those couples $\{A, \{B_r\}_{r\in R}\}$ which give, with some constants $1/2\geq\varepsilon\geq 0$, $C, \beta>0$, the estimate

$$\mathrm{Re}((A+\beta)u, u)_{L^2(\Omega)} \geq C\|u\|^2_{H^{m-\varepsilon}(\Omega)} \tag{1}$$

for all $u$ in $H^{2m}_B(\Omega)=\{u \in H^{2m}(\Omega)\,;\, B_r u|_\Gamma=0, \text{ for any } r \in R\}$.

Here $H^s(\Omega)$ denotes the Sobolev space on $\Omega$ of order $s$, $\| \quad \|_{H^s(\Omega)}$ is its norm and $( \ ,\ )_{L^2(\Omega)}$ is the inner product in $L^2(\Omega)$.

If $1/2>\varepsilon\geq 0$, the problem was treated in far stronger form in [3]. In this note we concern with the case $\varepsilon=1/2$. So the problem is

**Problem 1′.** Characterize those couples $\{A, \{B_r\}_{r\in R}\}$ which give, with some constants $C, \beta>0$, the estimate

$$\mathrm{Re}((A+\beta)u, u)_{L^2(\Omega)} \geq C\|u\|^2_{H^{m-1/2}(\Omega)} \tag{2}$$

for all $u$ in $H^{2m}_B(\Omega)$.

We assume the following hypothesis (H) that was proved in the case $0\leq\varepsilon<1/2$ necessary for the estimate (1) to hold. (See [3] and [6].)
(H) The set $R$ coincides with one of the $R_j$'s defined by $R_j=(0, 1, \cdots \cdots, m-j-1, m, m+1, \cdots, m+j-1)$, $1\leq j\leq m$. Under this hypothesis we give a necessary and sufficient condition for the estimate (2) to hold.

Proofs are omitted. Detailed discussions will be published elsewhere.*)

**§2. Results.** We denote by $\nu$ the interior unit normal to $\Gamma$ and

*) This work was done during the author's stay in Paris. He expresses his hearty thanks to Professor J. L. Lions for his constant encouragement.

by $D_n$ the normal derivative $-i\frac{\partial}{\partial\nu}$ multiplied by $-i=-\sqrt{-1}$. $S_j$ is the complement of $R_j$ in the set $\{0,1,2,\cdots,2m-1\}$. Then $B_r$, $r\in R_j$ can be written as

$$B_r=D_n^r-\sum_{\substack{\rho\in S_j\\ \rho<r}}B_{r-\rho}^r D_n^\rho,$$

where $B_{r-\rho}^r$ is a pseudo-differential operator on $\Gamma$ of order $\leq r-\rho$. Let $\Lambda=(1-\Delta')^{1/2}$ where $\Delta'$ is the Laplace-Beltrami operator associated with the metric on $\Gamma$. Then $\Lambda^k$ is an isomorphism from $H^s(\Gamma)$ to $H^{s-k}(\Gamma)$. $A^*$ denotes the formal adjoint of $A$.

We choose and fix $\alpha$ so large that we can solve uniquely the problem:

$$\begin{gathered}(A+A^*+2\alpha)v=0\\ D_n^k v|_\Gamma=\Lambda^k\phi_k,\quad m-1\geq k\geq m-j,\\ D_n^k v|_\Gamma=0,\quad m-j-1\geq k\geq 0,\end{gathered}$$

and obtain the estimates, for any $s\in\boldsymbol{R}$,

$$C^{-1}\sum_{k=m-j}^{m-1}\|\phi_k\|^2_{H^{s-1/2}(\Gamma)}\leq\|v\|^2_{H^s(\Omega)}\leq C\sum_{k=m-j}^{m-1}\|\phi_k\|^2_{H^{s-1/2}(\Gamma)}. \tag{5}$$

Here and hereafter we denote by $C$ different constants $>0$ in different occurrences.

Now we fix $B=\{B_r\}_{r\in R_j}$. We decompose any $u$ in $H_B^{2m}(\Omega)$ into sum of two functions $v$ and $w$:

$$u=v+w, \tag{6}$$

where

$$(A+A^*+2\beta)v=0 \text{ on } \Omega,\quad D_n^k v|_\Gamma=D_n^k u|_\Gamma,\quad 0\leq k\leq m-1, \tag{7}$$

and $D_n^k w|_\Gamma=0$, $0\leq k\leq m-1$. This implies that $D_n^k v|_\Gamma=0$ for $0\leq k\leq m-j-1$. We set $D_n^k u|_\Gamma=\Lambda^k\varphi_k$, $m-j\leq k\leq m-1$. Let $H_B^s(\Omega)$ be the closure of $H_B^{2m}(\Omega)$ in $H^s(\Omega)$. Then $H_B^m(\Omega)=\{u\in H^m(\Omega): D_n^k u|_\Gamma=0,\ 0\leq k\leq m-j-1\}$. The decomposition (6) is a topological decomposition of $H_B^m(\Omega)$. (See [5].) Now we take any $u$ in $H_B^{2m}(\Omega)$. Then using the boundary condition $B_r u|_\Gamma=0$ and the decomposition (6), we can find pseudo-differential operators $H_{p,q}$ on $\Gamma$ of order $2m-1$, $m-j\leq p$, $q\leq m-1$, such that

$$\begin{aligned}&\mathrm{Re}((A+\beta)u,u)_{L^2(\Omega)}\\ &\quad=\mathrm{Re}((A+\beta)w,w)_{L^2(\Omega)}+\sum_{p,q=m-j}^{m-1}(H_{pq}(\beta)\varphi_q,\varphi_p)_{L^2(\Gamma)}.\end{aligned} \tag{8}$$

(See [2].)

Let $T$ be the 1 dimensional circle$=\boldsymbol{R}/2\pi\boldsymbol{Z}$. We consider the elliptic operator $\tilde{A}=A+D_s^{2m}$, $s\in T$, on $\Omega\times T$ and boundary operators $\{B_r\}_{r\in R_j}$ on $\Gamma\times T$. $H_B^s(\Omega\times T)$ denotes the closure in $H^s(\Omega\times T)$ of $H_B^{2m}(\Omega\times T)=\{f\in H^{2m}(\Omega\times T): B_r f|_{\Gamma\times T}=0,\ r\in R_j\}$. Decomposition corresponding to (6) holds for functions in $H_B^{2m}(\Omega\times T)$, that is, for any $f$ in $H_B^{2m}(\Omega\times T)$,

$$(9)\qquad f=g+h,\quad (\tilde{A}+\tilde{A}^*+2\beta)g=0 \text{ on } \Omega\times T,$$
$$D_n^k g|_{\Gamma\times T}=D_n^k f|_{\Gamma\times T},\quad 0\le k\le m-1.$$

We set $D_n^k f|_{\Gamma\times T}=\tilde{\Lambda}^k\phi_k$, $m-j\le k\le m-1$, where $\tilde{\Lambda}=(1-\Delta'+D_s^2)^{1/2}$. Just as we did above, we can find pseudo-differential operators $\tilde{H}_{pq}(\beta)$ on $\Gamma\times T$ of order $2m-1$ such that for any $f$ in $H_B^{2m}(\Omega\times T)$

$$(10)\qquad \operatorname{Re}((\tilde{A}+\beta)f,f)_{L^2(\Omega\times T)}$$
$$=\operatorname{Re}((\tilde{A}+\beta)h,h)_{L^2(\Omega\times T)}+\sum_{p,q=m-j}^{m-1}(\tilde{H}_{pq}(\beta)\phi_q,\phi_p)_{L^2(\Gamma\times T)}.$$

Our first result is

**Theorem 1.** *Each of the following four propositions are equivalent to the other:*

(i) *There are some* $\beta_1$, $C_1>0$ *such that the estimate* (2) *holds for any* $u\in H_B^{2m}(\Omega)$.

(ii) *There are some* $\beta_2$, $C_2>0$, *such that the estimate*

$$(11)\qquad \operatorname{Re}((\tilde{A}+\beta_2)f,f)_{L^2(\Omega\times T)}\ge C_2\|f\|^2_{H^{m-1/2}(\Omega\times T)}$$

*holds for any* $f$ *in* $H_B^{2m}(\Omega\times T)$.

(iii) *There are some constants* $\beta_3$, $C_3>0$ *such that the estimate*

$$(12)\qquad \sum_{p,q=m-j}^{m-1}(H_{pq}(\beta_3)\varphi_q,\varphi_p)_{L^2(\Gamma)}\ge C_3\sum_{p=m-j}^{m-1}\|\varphi_p\|^2_{H^{m-1}(\Gamma)}$$

*holds for any* $\varphi_{m-j},\varphi_{m-j+1},\cdots,\varphi_{m-1}\in H^{m-1/2}(\Gamma)$.

(iv) *There are some constants* $\gamma,\beta_4,C_4>0$ *such that the estimate*

$$(13)\qquad \sum_{p,q=m-j}^{m-1}(\tilde{H}_{pq}(\gamma)\phi_q,\phi_p)_{L^2(\Gamma\times T)}+\beta_4\sum_{p=m-j}^{m-1}\|\phi_p\|^2_{H^{-1/2}(\Gamma\times T)}$$
$$\ge C_4\sum_{p=m-j}^{m-1}\|\phi_p\|^2_{H^{m-1}(\Gamma\times T)}$$

*holds for any* $\phi_{m-j},\phi_{m-j+1},\cdots,\phi_{m-1}$ *in* $H^{m-1/2}(\Gamma\times T)$.

**Remark 1.** In the case $0\le\varepsilon<1/2$ the estimate holds with some $\beta$, $C>0$, if and only if, with some $\gamma$, $\beta$, $C>0$, the estimate

$$(14)\qquad \sum_{p,q=m-j}^{m-1}(H_{pq}(\gamma)\varphi_q,\varphi_p)_{L^2(\Gamma)}+\beta\sum_{p=m-j}^{m-1}\|\varphi_p\|^2_{H^{-1/2}(\Gamma)}$$
$$\ge C\sum_{p=m-j}^{m-1}\|\varphi_p\|^2_{H^{m-1/2-\varepsilon}(\Gamma)}$$

holds for any $\varphi_{m-j},\cdots,\varphi_{m-1}$ in $H^{m-1/2}(\Gamma)$.

We consider pseudo-differential operators $\tilde{H}_{pq}(\gamma)$, $m-j\le p$, $q\le m-1$, of order $2m-1$ defined on $\Gamma\times T$ and satisfying the property (iv) of Theorem 1.

The property (iv) of Theorem 1 can be localized.

**Theorem 2.** *Assume that there exists a family of finite number of real functions* $\{\mu_k(x)\}_{k=1}^N$ *in* $\mathcal{D}$ $(\Gamma\times T)$ *satisfying*

(i) $\sum\mu_k(x,s)^2=1$,

(ii) *for any* $\phi_{m-j},\phi_{m-j+1},\cdots,\phi_{m-1}\in\mathcal{D}$ $(\Gamma\times T)$ *and for any* $k$ *the following estimate holds*:

$$(15)\qquad \sum_{p,q=m-j}^{m-1} (\tilde{H}_{pq}(\gamma)\mu_k\phi_q, \mu_k\phi_p)_{L^2(\Gamma\times T)} + \beta \sum_{p=m-j}^{m-1} \|\mu_k\phi_p\|^2_{H^{-1/2}(\Gamma\times T)} \geq C \sum_{p=m-j}^{m-1} \|\mu_k\phi_p\|^2_{H^{m-1}(\Gamma\times T)}.$$

*Then for any $\phi_{m-j}, \phi_{m-j+1}, \cdots, \phi_{m-1} \in \mathcal{D}\ (\Gamma\times T)$ the estimate* (13) *holds with some $\beta_4$, $C_4$ and $\gamma_4>0$.*

Let $\Omega$ be any open set (not necessarily connected) in $\boldsymbol{R}^n$. Let $Q_{rs}$, $m-j\leq r$, $s\leq m-1$, be pseudo-differential operators of order 1 defined in $\Omega$. $q_{rs}(x,\xi)\sim\sum_{j=0}^{\infty} q^j_{rs}(x,\xi)$ denote the symbol of $Q_{rs}$. We assume the matrix $(q^0_{rs}(x,\xi))_{rs}$ of the principal symbols of $Q_{rs}$ is Hermitian. Then we have

**Theorem 3.** *The following two properties are equivalent:*

(i) *For any compact set $K$ in $\Omega$, there are constants $C_0$ and $C_1>0$ such that, for any $\phi_{m-j}, \phi_{m-j+1}, \cdots, \phi_{m-1} \in \mathcal{D}(K)$,*

$$(16)\quad \mathrm{Re} \sum_{r,s=m-j}^{m-1} (Q_{rs}\phi_s, \phi_r)_{L^2(\Omega)} + C_1 \sum_{r=m-j}^{m-1} \|\phi_r\|^2_{H^{-1/2}(\Omega)} \geq C_0 \sum_{r=m-j}^{m-1} \|\phi_r\|^2_{H^0(\Omega)}.$$

(ii) *For any compact set $K_1$ in $\Omega$, there exist constant $C>0$, integer $N>0$ and a function $\varepsilon(\xi)$ with $\varepsilon(\xi)\to 0$ when $|\xi|\to\infty$ such that, for any $x\in K_1$, $\psi_{m-j}, \cdots, \psi_{m-1} \in \mathcal{D}(\boldsymbol{R}^n)$,*

$$(17)\quad \mathrm{Re} \sum_{r,s=m-j}^{m-1} \sum_{|\alpha|+|\beta|\leq 2} \frac{|\xi|^{(|\beta|-|\alpha|)/2}}{\alpha!\beta!} q^{0(\beta)}_{rs(\alpha)}(x,\xi) \int_{\boldsymbol{R}^n} (iD_y)^\beta \psi_s(y) \overline{(-iy)^\alpha \psi_r(y)}\, dy$$
$$+ \mathrm{Re} \sum_{r,s=m-j}^{m-1} q^1_{rs}(x,\xi) \int_{\boldsymbol{R}^n} \psi_s(y)\overline{\psi_r(y)}\, dy$$
$$+ \varepsilon(\xi) \sum_{|\alpha|+|\beta|\leq N} \sum_{r=m-j}^{m-1} \int_{\boldsymbol{R}^n} |D^\alpha_y y^\beta \psi_r(y)|\, dy$$
$$\geq C \sum_{r=m-j}^{m-1} \int_{\boldsymbol{R}^n} |\psi_r(y)|^2\, dy,$$

*where $q^{0(\beta)}_{(\alpha)}(x,\xi) = D^\alpha_x D^\beta_\xi q^0(x,\xi)$.*

**Remark 2.** The estimate (14) holds for any $\varphi_{m-j}, \cdots, \varphi_{m-1} \in H^{m-1/2}(\Gamma)$ if and only if the matrix defined by the principal symbols $\sigma_{2m-1}(H_{pq}(\beta))(x',\xi')$ is uniformly positive definite. Thus we can prove the result in [3] without the assumption that $\sigma_{2m}(A)(x,\xi)$ is real.

To prove Theorem 3 we use the following theorem which is interesting in itself.

**Theorem 4.***) *Let $K$ be any compact set in an open set $\Omega$ in $\boldsymbol{R}^n$ and let $P$ be a peudo-differential operator of order $\rho$ defined on $\Omega$, whose symbol is denoted by $p(x,\xi)$. Assume $\varphi\in\mathcal{D}(\Omega)$ is identically* 1 *in some neighbourhood of $K$. Then for any $N>0$, there is a constant $C>0$ such that for any $x\in K$, $\xi\in\boldsymbol{R}^n$ with $|\xi|\geq 1$, and $\phi, \varphi$ in $\mathcal{D}(\boldsymbol{R}^n)$,*

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*) During the preparation of this article the author had a chance to know that A. P. Calderón also had obtained, independently, a result similar to Theorem 4 in a little stronger form.

$$
\begin{aligned}
&|\xi|^{n/2}\int_{\Omega}(P\varphi v_1)(y)\overline{\varphi v_2(y)}dy \\
&\quad-\sum_{|\alpha|,|\beta|<N}\frac{|\xi|^{(|\beta|-|\alpha|)/2}}{\alpha!\beta!}p^{(\beta)}_{(\alpha)}(x,\xi)\int(iD_y)^{\beta}\psi(y)\overline{(-iy)^{\alpha}\phi(y)}dy \\
&\quad\le C|\xi|^{-N/2+2|\beta|+n}\|\psi\|_{H^{3N/2}}\|(1+|y|)^N\phi\|_{H^0(\boldsymbol{R}^n)},
\end{aligned}
$$

*where* $v_1(y)=\psi((y-x)|\xi|^{1/2})e^{iy\cdot\xi}$ *and* $v_2(y)=\phi((y-x)|\xi|^{1/2})e^{iy\cdot\xi}$.

Proofs of Theorems 3 and 4 are omitted here. They are similar to the discussion in [4].

## References

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