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On Some Identities for *k*-Fibonacci Sequence

Paula Catarino[1]

Department of Mathematics
University of Trás-os-Montes e Alto Douro
5001-801 Vila Real, Portugal

Abstract

We obtain some identities for *k*-Fibonacci numbers by using its Binet's formula. Also, another expression for the general term of the sequence, using the ordinary generating function, is provided.

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1 Introduction

The well-known Fibonacci sequence is one of the sequences of numbers that have been studied over several years. Many papers and research work are dedicated to Fibonacci sequence, such as the work of Hoggatt in [13] and Vorobiov, in [6], among others. Also relating with Fibonacci sequence, in Falcón and Plaza, [12], we find some properties for *k*-Fibonacci numbers obtained from elementary matrix algebra and its identities including generating function and divisibility properties appears in Bolat and Köse, in [1]. The Fibonacci sequence is one of the sequence that is defined recursively as well as the sequences of Pell, Pell-Lucas and Modified Pell numbers. More recently, P. Catarino and P. Vasco did some research about the sequences of numbers that arising from these

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[1] Member of CM-UTAD (Research Centre of Mathematics of UTAD) and Collaborator of LabDCT-CIDTFF (Research Centre of "Didactics and Technology in Education of Trainers"- Pole of UTAD).

sequences: for any positive real number k, the k-Pell sequence $\{P_{k,n}\}_n$, k-Pell-Lucas sequence $\{Q_{k,n}\}_n$ and Modified k-Pell sequence $\{q_{k,n}\}_n$, that are also defined by recursive recurrences. In these cases, for $n \geq 1$, we have, respectively, the following: $P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}$, $P_{k,0} = 0$, $P_{k,1} = 1$; $Q_{k,n+1} = 2Q_{k,n} + kQ_{k,n-1}$, $Q_{k,0} = Q_{k,1} = 2$; $q_{k,n+1} = 2q_{k,n} + kq_{k,n-1}$, $q_{k,0} = q_{k,1} = 1$. For more detail about these sequences, see the works of Catarino [7], [11] and Catarino and Vasco [8], [9] and [10].

The Binet's formula is also well known for several of these sequences. Claude Levesque, in [2] finds the general Binet's formula for a general m^{th} order linear recurrence. We have, $P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}$, $Q_{k,n} = r_1^n + r_2^n$, $q_{k,n} = \frac{r_1^n + r_2^n}{r_1 + r_2} = \frac{r_1^n + r_2^n}{2}$, where $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$ are the roots of the characteristic equation of the sequences $\{P_{k,n}\}_n$, $\{Q_{k,n}\}_n$ and $\{q_{k,n}\}_n$, respectively. As a curiosity, for $k = 1$, we obtain the silver ratio which is related with the Pell number sequence. Silver ratio is the limiting ratio of consecutive Pell numbers. Sometimes some basic properties come from the Binet's formula. For example, for the sequence of Jacobsthal number, Koken and Bozkurt, in [4], deduce some properties and the Binet's formula, using matrix method. In [5], Yilmaz and Bozkurt study some more properties related with k- Jacobsthal numbers as well as Jhala *et al*. in [3], Catarino in [7] and Catarino and Vasco in [9] and [10] did a similar study for k-Pell, k-Pell-Lucas and Modified k-Pell sequences and many properties are proved by easy arguments for the k-Pell, k-Pell-Lucas and Modified k-Pell number. In this paper, we consider the k-Fibonacci sequence and many identities are proved by easy arguments for the k-Fibonacci number.

2 The *k*-Fibonacci sequence and some properties

The following sequence was defined by Falcón and Plaza for any positive real number k. The k-Fibonacci sequence say $\{F_{k,n}\}_{n \in \mathbb{N}}$ is defined recurrently by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \text{ for } n \geq 1, \tag{1}$$

with the initial conditions given by

$$F_{k,0} = 0, \ F_{k,1} = 1. \tag{2}$$

Consider the following characteristic equation, associated to the recurrence relation (1),

$$\alpha^2 - k\alpha - 1 = 0, \tag{3}$$

with two distinct roots α_1 and α_2. Note that the roots of the equation (3) are $\alpha_1 = \frac{k + \sqrt{k^2+4}}{2}$ e $\alpha_2 = \frac{k - \sqrt{k^2+4}}{2}$, where k is a real positive number. Since $\sqrt{k^2+4} > 1$, then $\alpha_2 < 0$ and so $\alpha_2 < 0 < \alpha_1$, $|\alpha_2| < \alpha_1$. Also we obtain that $\alpha_1 + \alpha_2 = k$, $\alpha_1 - \alpha_2 = \sqrt{k^2+4}$ and $\alpha_1 \alpha_2 = -1$.

As a curiosity, for $k = 1$ we obtain that $\alpha_1 = \frac{1+\sqrt{5}}{2}$ is the golden number which is related with the Fibonacci number sequence $\{F_n\}_{n \in \mathbb{N}}$. Golden number is the limiting ratio of consecutive Fibonacci numbers. The following formula was established by Bolat and Köse in [1].

Proposition 1 (Binet's formula) [Proposition 2 of [1]] *The n^{th} k-Fibonacci number is given by*

$$F_{k,n} = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} \tag{4}$$

where α_1, α_2 are the roots of the characteristic equation (3) and $\alpha_1 > \alpha_2$.

Proposition 2 (Catalan's identity)

$$F_{k,n-r}F_{k,n+r} - F_{k,n}^2 = (-1)^{n+1-r}F_{k,r}^2 \tag{5}$$

Proof: Using the Binet's formula (4) and the fact that $\alpha_1\alpha_2 = -1$, we get

$$\begin{aligned} F_{k,n-r}F_{k,n+r} - F_{k,n}^2 &= \left(\frac{\alpha_1^{n-r} - \alpha_2^{n-r}}{\alpha_1-\alpha_2}\right)\left(\frac{\alpha_1^{n+r} - \alpha_2^{n+r}}{\alpha_1-\alpha_2}\right) - \left(\frac{\alpha_1^n - \alpha_2^n}{\alpha_1-\alpha_2}\right)^2 \\ &= (-1)^{n+1-r}\left(\frac{\alpha_1^r-\alpha_2^r}{\alpha_1-\alpha_2}\right)^2 \\ &= (-1)^{n+1-r}F_{k,r}^2, \end{aligned}$$

that is, the identity required.

■

Note that for $r = 1$ in Catalan's identity obtained, we get the Cassini's identity for the k-Fibonacci numbers sequence. In fact, the equation (5) for $r = 1$, yields

$$F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = (-1)^n F_{k,1}^2$$

and using one of the initial conditions of this sequence we proved the following result.

Proposition 3 (Cassini's identity)

$$F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = (-1)^n \tag{6}$$

■

The d'Ocagne's identity can also obtained using the Binet's formula as it was done in Jhala *et al.*, in [3] for the *k*-Jacobsthal sequence, Catarino in [7] for the *k*-Pell numbers and Catarino and Vasco in [9], [10] for *k*-Pell-Lucas and Modified *k*-Pell numbers, respectively. Hence we have

Proposition 4 (d'Ocagne's identity) *If m > n then*

$$F_{k,m}F_{k,n+1} - F_{k,m+1}F_{k,n} = (-1)^n F_{k,m-n}. \tag{7}$$

Proof: Once more, using the Binet's formula (4), the fact that $\alpha_1\alpha_2 = -1$ and $m > n$, we get

$$F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = \left(\frac{\alpha_1^m - \alpha_2^m}{\alpha_1-\alpha_2}\right)\left(\frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1-\alpha_2}\right) - \left(\frac{\alpha_1^{m+1} - \alpha_2^{m+1}}{\alpha_1-\alpha_2}\right)\left(\frac{\alpha_1^n - \alpha_2^n}{\alpha_1-\alpha_2}\right)$$

$$= (\alpha_1\alpha_2)^n\left(\frac{(\alpha_1-\alpha_2)(\alpha_1^{m-n}-\alpha_2^{m-n})}{(\alpha_1-\alpha_2)^2}\right)$$
$$= (\alpha_1\alpha_2)^n\left(\frac{\alpha_1^{m-n}-\alpha_2^{m-n}}{\alpha_1-\alpha_2}\right)$$
$$= (-1)^n F_{k,m-n}.$$

■

Again using the Binet's formula (4) we obtain other property of the k-Fibonacci number sequence which is stated in the following proposition.

Proposition 5

$$\lim_{n\to\infty}\frac{F_{k,n}}{F_{k,n-1}} = \alpha_1. \quad (8)$$

Proof: We have that

$$\lim_{n\to\infty}\frac{F_{k,n}}{F_{k,n-1}} = \lim_{n\to\infty}\left(\frac{\alpha_1^n-\alpha_2^n}{\alpha_1-\alpha_2}\right)\left(\frac{\alpha_1-\alpha_2}{\alpha_1^{n-1}-\alpha_2^{n-1}}\right) = \lim_{n\to\infty}\left(\frac{\alpha_1^n-\alpha_2^n}{\alpha_1^{n-1}-\alpha_2^{n-1}}\right). \quad (9)$$

Using the ratio $\frac{\alpha_2}{\alpha_1}$ and since $\left|\frac{\alpha_2}{\alpha_1}\right| < 1$, then $\lim_{n\to\infty}\left(\frac{\alpha_2}{\alpha_1}\right)^n = 0$. Next we use this fact writing (9) with an equivalent form using this ratio, obtaining

$$\lim_{n\to\infty}\frac{F_{k,n}}{F_{k,n-1}} = \lim_{n\to\infty}\frac{1-\left(\frac{\alpha_2}{\alpha_1}\right)^n}{\frac{1}{\alpha_1}-\left(\frac{\alpha_2}{\alpha_1}\right)^n\frac{1}{\alpha_2}} = \lim_{n\to\infty}\frac{1}{\frac{1}{\alpha_1}} = \alpha_1.$$

■

Also, we easily can show the following result using basic tools of calculus of limits and the (8).

Proposition 6

$$\lim_{n\to\infty}\frac{F_{k,n-1}}{F_{k,n}} = \frac{1}{\alpha_1}. \quad (10)$$

■

3 Generating functions for the *k*-Fibonacci sequences

The generating functions for the *k*-Fibonacci sequences are given by Bolat and Röse in Proposition13 in [1]. As we know the *k*-Fibonacci sequences (and other sequences) can be considered as the coefficients of the power series of the corresponding generating function. Bolat and Röse in this Proposition listed some properties of the generating functions for the *k*-Fibonacci sequences. One of the equalities given in [1] is the following

$$\sum_{n=0}^{\infty} F_{k,n}x^n = \frac{x}{1-kx-x^2}. \quad (11)$$

Recall that for a sequence $(a_n)_n$, if $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = L$, where L is a positive real number, then, considering the power series $\sum_{n=0}^{\infty} a_n x^k$, its radius of convergence R is equal to $\frac{1}{L}$. So, for the k-Fibonacci sequence, using (8) and then (10) we know that she can be written as a power series with radius of convergence equal to

$\frac{1}{\alpha_1}$. Next we give another expression for the general term of the *k*-Fibonacci sequence using the ordinary generating function.

Proposition 7

Let us consider $f(x) = \sum_{n=0}^{\infty} F_{k,n} x^n$, *for* $x \in \left]-\frac{1}{\alpha_1}, \frac{1}{\alpha_1}\right[$. *Then we have that*

$$F_{k,n} = \frac{f^{(n)}(0)}{n!}, \tag{12}$$

where $f^{(n)}(x)$ *denotes the nth order derivative of the function* f.

Proof: We have from (11) that $f(0) = F_{k,0} = 0$ and

$$f'(x) = \sum_{n=1}^{\infty} nF_{k,n}\, x^{n-1} = 1 + \sum_{n=2}^{\infty} F_{k,n}\, x^{n-1}.$$

Also,

$$f^{(2)}(x) = \sum_{n=2}^{\infty} n(n-1)F_{k,n}\, x^{n-2} = 2.1.F_{k,2} + \sum_{n=3}^{\infty} n(n-1)F_{k,n}\, x^{n-2};$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$f^{(l)}(x) = \sum_{n=l}^{\infty} n(n-1)\ldots(n-(l-1))F_{k,n}\, x^{n-l}.$$

So, $f^{(l)}(x) = l(l-1)\ldots\big(l-(l-1)\big)F_{k,l} + \sum_{n=l+1}^{\infty} n(n-1)\ldots(n-(l-1))F_{k,n}\, x^{n-l}$ and then we get $f^{(l)}(x) = l!\,F_{k,l} + \sum_{n=l+1}^{\infty} n(n-1)\ldots(n-(l-1))F_{k,n}\, x^{n-l}$. Therefore, $f^{(l)}(0) = l!\,F_{k,l}$ or $F_{k,l} = \frac{f^{(l)}(0)}{l!}$. Hence, for all $n \geq 1$, we have that $F_{k,n} = \frac{f^{(n)}(0)}{n!}$, as required.

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