ON SOME INEQUALITIES FOR THE INCOMPLETE GAMMA FUNCTION

HORST ALZER

ABSTRACT. Let $p \neq 1$ be a positive real number. We determine all real numbers $\alpha = \alpha(p)$ and $\beta = \beta(p)$ such that the inequalities

$$[1 - e^{-\beta x^p}]^{1/p} < \frac{1}{\Gamma(1+1/p)} \int_0^x e^{-t^p}\, dt < [1 - e^{-\alpha x^p}]^{1/p}$$

are valid for all $x > 0$. And, we determine all real numbers a and b such that

$$-\log(1 - e^{-ax}) \leq \int_x^\infty \frac{e^{-t}}{t}\, dt \leq -\log(1 - e^{-bx})$$

hold for all $x > 0$.

1. INTRODUCTION

In 1955, J. T. Chu [1] presented sharp upper and lower bounds for the error function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$. He proved that the inequalities

$$(1.1) \qquad [1 - e^{-rx^2}]^{1/2} \leq \mathrm{erf}(x) \leq [1 - e^{-sx^2}]^{1/2}$$

are valid for all $x \geq 0$ if and only if $0 \leq r \leq 1$ and $s \geq 4/\pi$. The right-hand inequality of (1.1) (with $s = 4/\pi$) was proved independently by J. D. Williams (1946) and G. Pólya (1949); see [1].

An interesting survey on inequalities involving the complementary error function $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$ and related functions is given in [4, pp. 177–181]. In particular, one can find inequalities for Mills' ratio $e^{x^2/2} \int_x^\infty e^{-t^2/2}\, dt$, derived by several authors.

In 1959, W. Gautschi [3] provided upper and lower bounds for the more general expression

$$(1.2) \qquad I_p(x) = e^{x^p} \int_x^\infty e^{-t^p}\, dt.$$

He established that the double-inequality

$$(1.3) \qquad \frac{1}{2}[(x^p + 2)^{1/p} - x] < I_p(x) \leq c_p[(x^p + 1/c_p)^{1/p} - x]$$

(with $c_p = [\Gamma(1+1/p)]^{p/(p-1)}$) holds for all real numbers $p > 1$ and $x \geq 0$. It has been pointed out in [3] that the integral in (1.2) for $p = 3$ occurs in heat transfer problems, and for $p = 4$ in the study of electrical discharge through gases. We note

that the integral $\int_x^\infty e^{-t^p}\,dt$ can be expressed in terms of the incomplete gamma function

$$\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\,dt,$$

namely,

$$\int_x^\infty e^{-t^p}\,dt = \frac{1}{p}\Gamma\left(\frac{1}{p},x^p\right).$$

Gautschi [3] showed that the inequalities (1.3) can be used to derive bounds for the exponential integral $E_1(x) = \Gamma(0,x)$. Indeed, if p tends to ∞, then (1.3) leads to

$$(1.4) \qquad \frac{1}{2}\log\left(1+\frac{2}{x}\right) \le e^x E_1(x) \le \log\left(1+\frac{1}{x}\right) \qquad (0 < x < \infty).$$

It is the main purpose of this paper to establish new inequalities for $\int_0^x e^{-t^p}\,dt$ and $\int_x^\infty e^{-t^p}\,dt$. In Section 2 we present sharp upper and lower bounds for

$$\frac{1}{\Gamma(1+1/p)}\int_0^x e^{-t^p}\,dt \quad \text{and} \quad \frac{1}{\Gamma(1+1/p)}\int_x^\infty e^{-t^p}\,dt,$$

which are valid not only for $p > 1$, but also for $p \in (0,1)$. In particular, we obtain an extension of Chu's double-inequality (1.1). Moreover, we provide sharp inequalities for the exponential integral $E_1(x)$. Finally, in Section 3 we compare our bounds with those given in (1.3) and (1.4).

2. Main results

First, we generalize the inequalities (1.1).

Theorem 1. *Let $p \ne 1$ be a positive real number, and let $\alpha = \alpha(p)$ and $\beta = \beta(p)$ be given by*

$$\alpha = 1, \quad \beta = [\Gamma(1+1/p)]^{-p}, \quad \text{if } 0 < p < 1,$$

and

$$\alpha = [\Gamma(1+1/p)]^{-p}, \quad \beta = 1, \quad \text{if } p > 1.$$

Then we have for all positive real x:

$$(2.1) \qquad [1 - e^{-\beta x^p}]^{1/p} < \frac{1}{\Gamma(1+1/p)}\int_0^x e^{-t^p}\,dt < [1 - e^{-\alpha x^p}]^{1/p}.$$

Proof. We have to show that the functions

$$F_p(x) = \int_0^x e^{-t^p}\,dt - \Gamma(1+1/p)[1 - e^{-x^p}]^{1/p}$$

and

$$G_p(x) = -\int_0^x e^{-t^p}\,dt + \Gamma(1+1/p)[1 - e^{-ax^p}]^{1/p} \qquad (a = [\Gamma(1+1/p)]^{-p})$$

are both positive on $(0,\infty)$, if $p > 1$, and are both negative on $(0,\infty)$, if $0 < p < 1$.

First, we determine the sign of $F_p(x)$. Differentiation yields

$$e^{x^p}\frac{\partial}{\partial x}F_p(x) = 1 - \Gamma(1+1/p)[L(z(x))]^{(1-p)/p},$$

where

$$L(z) = (z-1)/\log(z) \quad \text{and} \quad z(x) = e^{-x^p}.$$

Setting $f_p(x) = e^{x^p}\frac{\partial}{\partial x}F_p(x)$, we obtain

$$(2.2) \qquad \frac{\partial}{\partial x}f_p(x) = \frac{p-1}{p}\Gamma(1+1/p)(L(z(x)))^{-2+1/p}\frac{d}{dx}z(x)\frac{d}{dz}L(z)|_{z=z(x)}.$$

Since

$$\frac{d}{dz}L(z) = [\log(z) - 1 + 1/z]/(\log(z))^2 > 0 \qquad (0 < z \neq 1)$$

and

$$\frac{d}{dx}z(x) < 0,$$

we conclude from (2.2) that

$$\frac{\partial}{\partial x}f_p(x) < 0, \quad \text{if } p > 1,$$

and

$$\frac{\partial}{\partial x}f_p(x) > 0, \quad \text{if } 0 < p < 1.$$

If $p > 1$, then we have

$$f_p(0) = 1 - \Gamma(1+1/p) > 0 \quad \text{and} \quad \lim_{x\to\infty} f_p(x) = -\infty,$$

which implies that there exists a number $x_0 > 0$ such that $f_p(x) > 0$ for $x \in (0, x_0)$ and $f_p(x) < 0$ for $x \in (x_0, \infty)$. Hence, the function $x \mapsto F_p(x)$ is strictly increasing on $[0, x_0]$ and strictly decreasing on $[x_0, \infty)$. Since $F_p(0) = \lim_{x\to\infty} F_p(x) = 0$, we obtain $F_p(x) > 0$ for all $x > 0$.

If $0 < p < 1$, then we have

$$f_p(0) = 1 - \Gamma(1+1/p) < 0 \quad \text{and} \quad \lim_{x\to\infty} f_p(x) = 1.$$

This implies that there exists a number $x_1 > 0$ such that $x \mapsto F_p(x)$ is strictly decreasing on $[0, x_1]$ and strictly increasing on $[x_1, \infty)$. From $F_p(0) = \lim_{x\to\infty} F_p(x) = 0$ we conclude that $F_p(x) < 0$ for all $x > 0$.

Next, we consider $G_p(x)$. Differentiation leads to

$$(2.3) \qquad e^{x^p}\frac{\partial}{\partial x}G_p(x) = -1 + (y(x))^{1-1/a}[L(y(x))]^{(1-p)/p},$$

where

$$L(y) = (y-1)/\log(y) \quad \text{and} \quad y(x) = e^{-ax^p}$$

with $a = a(p) = [\Gamma(1+1/p)]^{-p}$. To determine the sign of $\frac{\partial}{\partial x}G_p(x)$ we need the inequalities

$$(2.4) \qquad 0 < \left(1 - \frac{1}{a(p)}\right)\frac{p}{p-1} < \frac{1}{2} \quad \text{for } 0 < p \neq 1.$$

The left-hand inequality of (2.4) is obviously true. A simple calculation reveals that the second inequality of (2.4) is equivalent to

$$(2.5) \qquad (1-x)\left[\Gamma(x+1) - \left(\frac{x+1}{2}\right)^x\right] > 0 \quad \text{for } 0 < x \neq 1.$$

To establish (2.5) we define for $x > 0$:

$$g(x) = \log \Gamma(x+1) - x \log \frac{x+1}{2}.$$

Then we have

$$\frac{d^2}{dx^2} g(x) = \frac{d}{dx} \psi(x+1) - \frac{x+2}{(x+1)^2} = \sum_{n=2}^{\infty} \frac{1}{(x+n)^2} - \frac{1}{x+1}$$

$$< \int_1^{\infty} \frac{dt}{(x+t)^2} - \frac{1}{x+1} = 0.$$

Thus, g is strictly concave on $[0, \infty)$. Since $g(0) = g(1) = 0$, we conclude that g is positive on $(0, 1)$ and negative on $(1, \infty)$. This implies (2.5).

Let $0 < r < 1/2$; we define for $y \in (0, 1)$:

$$h_r(y) = y^r \log(y)/(y-1).$$

Then we get

$$(y-1)^2 y^{1-r} \frac{\partial}{\partial y} h_r(y) = [(r-1)y - r] \log(y) + y - 1$$

$$= \varphi_r(y), \quad \text{say.}$$

Since

$$\frac{\partial^2}{\partial y^2} \varphi_r(y) = \frac{r-1}{y^2} \left[y - \frac{r}{1-r} \right],$$

it follows that φ_r is strictly convex on $(0, \frac{r}{1-r})$ and strictly concave on $(\frac{r}{1-r}, 1)$. From $\lim_{y \to 0} \varphi_r(y) = \infty$,

$$\varphi_r(1) = \frac{\partial}{\partial y} \varphi_r(y)|_{y=1} = 0 \quad \text{and} \quad \frac{\partial^2}{\partial y^2} \varphi_r(y)|_{y=1} = 2r - 1 < 0,$$

we conclude that there exists a number $y_0 \in (0, 1)$ such that φ_r is positive on $(0, y_0)$ and negative on $(y_0, 1)$. This implies that $y \mapsto h_r(y)$ is strictly increasing on $(0, y_0)$ and strictly decreasing on $(y_0, 1)$. Since $\lim_{y \to 0} h_r(y) = 0$ and $\lim_{y \to 1} h_r(y) = 1$, we conclude that there exists a number $y_1 \in (0, 1)$ such that $h_r(y) < 1$ for $y \in (0, y_1)$ and $h_r(y) > 1$ for $y \in (y_1, 1)$. The function $y(x) = e^{-ax^p}$ is strictly decreasing on $[0, \infty)$. Since $y(0) = 1$ and $\lim_{x \to \infty} y(x) = 0$, there exists a number $x^* > 0$ such that

$$y_1 < y(x) < 1 \quad \text{for } x \in (0, x^*),$$

and

$$0 < y(x) < y_1 \quad \text{for } x \in (x^*, \infty).$$

Hence, we have:

If $0 < x < x^*$, then $h_r(y(x)) > 1$, and, if $x^* < x$, then $h_r(y(x)) < 1$. We set $r = (1 - \frac{1}{a(p)}) \frac{p}{p-1}$; then we obtain from (2.3) that

$$h_r(y(x)) = \left[1 + e^{x^p} \frac{\partial}{\partial x} G_p(x) \right]^{p/(p-1)}.$$

Therefore, if $p > 1$, then

$$\frac{\partial}{\partial x} G_p(x) > 0 \quad \text{for } x \in (0, x^*) \quad \text{and} \quad \frac{\partial}{\partial x} G_p(x) < 0 \quad \text{for } x \in (x^*, \infty);$$

and, if $0 < p < 1$, then

$$\frac{\partial}{\partial x}G_p(x) < 0 \quad \text{for } x \in (0, x^*) \quad \text{and} \quad \frac{\partial}{\partial x}G_p(x) > 0 \quad \text{for } x \in (x^*, \infty).$$

Since $G_p(0) = \lim_{x \to \infty} G_p(x) = 0$, we conclude that

$$G_p(x) > 0 \quad \text{for } x \in (0, \infty), \text{ if } p > 1,$$

and

$$G_p(x) < 0 \quad \text{for } x \in (0, \infty), \text{ if } 0 < p < 1.$$

This completes the proof of Theorem 1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark. It is natural to ask whether the double-inequality (2.1) can be refined by replacing α by a positive number which is smaller than

$$\max\{1, [\Gamma(1 + 1/p)]^{-p}\} = \begin{cases} 1, & \text{if } 0 < p < 1, \\ [\Gamma(1 + 1/p)]^{-p}, & \text{if } p > 1, \end{cases}$$

or by replacing β by a number which is greater than

$$\min\{1, [\Gamma(1 + 1/p)]^{-p}\} = \begin{cases} [\Gamma(1 + 1/p)]^{-p}, & \text{if } 0 < p < 1, \\ 1, & \text{if } p > 1. \end{cases}$$

We show that the answer is "no"! Let $\alpha > 0$ be a real number such that the right-hand inequality of (2.1) holds for all $x > 0$. This implies that the function

$$\widetilde{F}_p(x) = \int_0^x e^{-t^p}\, dt - \Gamma(1 + 1/p)[1 - e^{-\alpha x^p}]^{1/p}$$

is negative on $(0, \infty)$. Since $\widetilde{F}_p(0) = 0$, we obtain

$$\frac{\partial}{\partial x}\widetilde{F}_p(x)|_{x=0} = 1 - \alpha^{1/p}\Gamma(1 + 1/p) \le 0,$$

which leads to $\alpha \ge [\Gamma(1 + 1/p)]^{-p}$. If $\alpha \in (0, 1)$, then we conclude from

$$\lim_{x \to \infty} e^{x^p} \frac{\partial}{\partial x}\widetilde{F}_p(x) = -\infty$$

that there exists a number $\overline{x} > 0$ such that $x \mapsto \widetilde{F}_p(x)$ is negative and strictly decreasing on $[\overline{x}, \infty)$. This contradicts $\lim_{x \to \infty} \widetilde{F}_p(x) = 0$. Thus, we have $\alpha \ge \max\{1, [\Gamma(1 + 1/p)]^{-p}\}$.

Next, we suppose that $\beta > 0$ is a real number such that the first inequality of (2.1) is valid for all $x > 0$. This implies

$$\widetilde{G}_p(x) = -\int_0^x e^{-t^p}\, dt + \Gamma(1 + 1/p)[1 - e^{-\beta x^p}]^{1/p} < 0$$

for all $x > 0$. Since $\widetilde{G}_p(0) = 0$, we obtain

$$\frac{\partial}{\partial x}\widetilde{G}_p(x)|_{x=0} = \beta^{1/p}\Gamma(1 + 1/p) - 1 \le 0,$$

which yields $\beta \le [\Gamma(1 + 1/p)]^{-p}$. If $\beta > 1$, then we get

$$\lim_{x \to \infty} e^{x^p} \frac{\partial}{\partial x}\widetilde{G}_p(x) = -1.$$

This implies that there exists a number $\tilde{x} > 0$ such that $x \mapsto \widetilde{G}_p(x)$ is negative and strictly decreasing on $[\tilde{x}, \infty)$. This contradicts $\lim_{x \to \infty} \widetilde{G}_p(x) = 0$. Hence, we get $\beta \leq \min\{1, [\Gamma(1 + 1/p)]^{-p}\}$.

As an immediate consequence of Theorem 1, the Remark, and the representation $\int_x^\infty e^{-t^p} dt = \Gamma(1 + 1/p) - \int_0^x e^{-t^p} dt$, we obtain the following sharp bounds for the ratio $\int_x^\infty e^{-t^p} dt / \int_0^\infty e^{-t^p} dt$.

Corollary. *Let $p \neq 1$ be a positive real number. The inequalities*

$$(2.6) \qquad 1 - [1 - e^{-\alpha x^p}]^{1/p} < \frac{1}{\Gamma(1 + 1/p)} \int_x^\infty e^{-t^p} dt < 1 - [1 - e^{-\beta x^p}]^{1/p}$$

are valid for all positive x if and only if

$$\alpha \geq \max\{1, [\Gamma(1 + 1/p)]^{-p}\} \quad and \quad 0 \leq \beta \leq \min\{1, [\Gamma(1 + 1/p)]^{-p}\}.$$

Now, we provide new upper and lower bounds for the exponential integral $E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt$.

Theorem 2. *The inequalities*

$$(2.7) \qquad\qquad -\log(1 - e^{-ax}) \leq E_1(x) \leq -\log(1 - e^{-bx})$$

are valid for all positive real x if and only if

$$a \geq e^C \quad and \quad 0 < b \leq 1,$$

where $C = 0.5772\ldots$ is Euler's constant.

Proof. The function $t \mapsto -\log(1 - e^{-tx})$ $(x > 0)$ is strictly decreasing on $(0, \infty)$. Therefore, it suffices to prove (2.7) with $a = e^C$ and $b = 1$. Let $p > 1$; from (2.6) with $\alpha = [\Gamma(1 + 1/p)]^{-p}$, $\beta = 1$, and x instead of x^p, we obtain

$$\Gamma(1/p)[1 - (1 - e^{-\alpha x})^{1/p}] < \int_x^\infty t^{-1+1/p} e^{-t} dt < \Gamma(1/p)[1 - (1 - e^{-x})^{1/p}].$$

If p tends to ∞, then we get

$$-\log(1 - e^{-ax}) \leq E_1(x) \leq -\log(1 - e^{-x})$$

with $a = e^C$.

We assume that there exists a real number $b > 1$ such that

$$E_1(x) \leq -\log(1 - e^{-bx})$$

holds for all $x > 0$. Since

$$e^x E_1(x) = \sum_{k=1}^n (-1)^{k-1}(k-1)!\, x^{-k} + r_n(x) \qquad (x > 0)$$

with

$$|r_n(x)| < n!\, x^{-n-1}$$

(see [2, pp. 673–674]), we obtain

$$(2.8) \qquad\qquad e^x x \log(1 - e^{-bx}) \leq -1 - x r_1(x).$$

If we let x tend to ∞, then inequality (2.8) implies $0 \leq -1$. Hence, we have $b \leq 1$.

Using the representation

$$E_1(x) = -C - \log(x) - \sum_{n=1}^\infty (-1)^n \frac{x^n}{n!\, n} \qquad (x > 0)$$

(see [2, p. 674]), we conclude from the left-hand inequality of (2.7) that

$$\log \frac{x}{1 - e^{-ax}} \le -C - \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n! \, n}.$$

If x tends to 0, then we obtain

$$\log(1/a) \le -C \quad \text{or} \quad a \ge e^C.$$

The proof of Theorem 2 is complete.   □

3. CONCLUDING REMARKS

In the final part of this paper we want to compare the bounds for the integrals $\int_x^{\infty} e^{-t^p} \, dt$ $(p > 1)$ and $\int_x^{\infty} \frac{e^{-t}}{t} \, dt$ which are given in (1.3), (2.6) and (1.4), (2.7), respectively. First, we consider the bounds for $\int_x^{\infty} e^{-t^p} \, dt$. We define

$$R_p(x) = \Gamma(1 + 1/p)\{1 - [1 - e^{-\alpha x^p}]^{1/p}\} - \frac{e^{-x^p}}{2}[(x^p + 2)^{1/p} - x]$$

with

$$\alpha = [\Gamma(1 + 1/p)]^{-p} \quad \text{and} \quad p > 1.$$

Then we have

$$R_p(0) = \Gamma(1 + 1/p) - 2^{-1+1/p} > 0,$$

$$\lim_{x \to \infty} R_p(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} e^{x^p} \frac{\partial}{\partial x} R_p(x) = 1.$$

This implies

$$R_p(x) > 0 \quad \text{for all sufficiently small } x > 0,$$

and

$$R_p(x) < 0 \quad \text{for all sufficiently large } x.$$

Let

$$S_p(x) = \Gamma(1 + 1/p)\{1 - [1 - e^{-x^p}]^{1/p}\} - ce^{-x^p}[(x^p + 1/c)^{1/p} - x]$$

with

$$c = [\Gamma(1 + 1/p)]^{p/(p-1)} \quad \text{and} \quad p > 1.$$

From $S_p(0) = 0$,

$$\lim_{x \to 0} \frac{\partial}{\partial x} S_p(x) = [\Gamma(1 + 1/p)]^{p/(p-1)} - \Gamma(1 + 1/p) < 0,$$

$$\lim_{x \to \infty} S_p(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} e^{x^p} \frac{\partial}{\partial x} S_p(x) = -\infty,$$

we conclude

$$S_p(x) < 0 \quad \text{for all sufficiently small } x > 0,$$

and

$$S_p(x) > 0 \quad \text{for all sufficiently large } x.$$

Hence, for small $x > 0$ the bounds for $\int_x^{\infty} e^{-t^p} \, dt$ $(p > 1)$ which are given in (2.6) are better than those presented in (1.3), whereas for large values of x the opposite is true.