ON SOME INFINITELY PRESENTED ASSOCIATIVE ALGEBRAS

Dedicated to the memory of Hanna Neumann

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We prove here that if F is a finitely generated free associative algebra over the field $\mathfrak{k}$ and R is an ideal of F, then F/R^2 is finitely presented if and only if F/R has finite $\mathfrak{k}$ dimension. Amitsur, [1, p. 136] asked whether a finitely generated $\mathfrak{k}$ algebra which is embeddable in matrices over a commutative $\mathfrak{k}$ algebra is necessarily finitely presented. Let $R = F'$, the commutator ideal of F, then [4, theorem 6], F/F'^2 is embeddable and thus provides a negative answer to his question. Another such example can be found in Small [6]. We also show that there are uncountably many two generator $\mathfrak{k}$ algebras which satisfy a polynomial identity yet are not embeddable in any algebra of $n \times n$ matrices over a commutative $\mathfrak{k}$ algebra.

We begin by recalling the elements of the free differential calculus for associative algebras. Details can be found in [4].

Let F be the free $\mathfrak{k}$ algebra, over the field $\mathfrak{k}$, freely generated by the set $\{p_\alpha; \alpha \in A\}$. Let U, V be two ideals of F and let T be a free $F/V - F/U$ bimodule with basis $\{t_\alpha; \alpha \in A\}$. We define a $\mathfrak{k}$ derivation $\delta: F \to T$ by declaring $1\delta = 0$ and $p_\alpha \delta = t_\alpha$. This is enough to define δ on all of F since δ is $\mathfrak{k}$ linear and, for f_1, f_2 in F

$$(1) \qquad (f_1 f_2)\delta = (f_1\delta)(f_2 + U) + (f_1 + V)(f_2\delta).$$

In fact it is easily verified inductively that if $m = p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_k}$ is a monomial of F, then

$$(2) \qquad m\delta = \sum_{i=1}^{k} (p_{\alpha_1} \cdots p_{\alpha_{i-1}} + V) t_{\alpha_i} (p_{\alpha_{i+1}} \cdots p_{\alpha_k} + U).$$

(With the convention that the empty monomial is the identity of F.)

One checks that the ideal VU is the kernel of δ and hence that δ induces a derivation $D: F/VU \to T$. Now, left and right multiplication by F define a

$F/V - F/U$ bimodule structure on $(U \cap V)/VU$ and, using (1), it follows readily that D restricted to $(U \cap V)/VU$ is a bimodule homomorphism. Theorem 3 of [4] then states

(3) $$D: \frac{U \cap V}{VU} \to T \text{ is a bimodule monomorphism.}$$

THEOREM 1. *Let F be a free $\mathfrak{k}$ algebra generated by a finite set $\{p_\alpha; \alpha \in A\}$ and let R be a nonzero ideal of F. Then R^2/R^3 is a finitely generated F/R bimodule if and only if F/R has finite $\mathfrak{k}$ dimension.*

If F/R has finite dimension, then [3, proposition 2, Corollary], R is a finitely generated right ideal so that R/R^2 is a finitely generated right F/R module. R/R^2 is then again finite dimensional and hence so is F/R^2. Using [3, proposition 2, Corollary] again, R^2 is a finitely generated right ideal, and, a fortiori, R^2/R^3 is a finitely generated F/R bimodule.

Suppose now that F/R has infinite dimension. Then, [3, Theorem 3, Corollary] R is not a finitely generated right ideal and, since R is a free right F module [2, theorem 3.5], there exist elements $e_i \in F$ with $R = \oplus_{i=1}^{\infty} e_i F$. We now use the embedding (3) with $U = R^2$, $V = R$. We consider T as a $(F/R)^{opp} \otimes_{\mathfrak{k}} F/R^2$ module with $(F/R)^{opp}$ the opposite algebra of F/R. Thus we write bta as $t(b \otimes a)$ with b now considered as an element of $(F/R)^{opp}$.

If $r = r_1 r_2$, with r_1, r_2 in R then, by (1),

$$r\delta = (r_1\delta)(r_2 + R^2) + (r_1 + R)(r_2\delta) = (r_1\delta)(r_2 + R^2),$$

and thus every element of $R^2\delta$ has its coefficients in $(F/R)^{opp} \otimes_{\mathfrak{k}} R/R^2$. Let now $R_n = \oplus_{i=1}^{n} e_i F$, $S = (F/R)^{opp} \otimes_{\mathfrak{k}} R/R^2$ and $S_n = (F/R)^{opp} \otimes_{\mathfrak{k}} (R_n + R^2)/R^2$. S_n is a right ideal of $(F/R)^{opp} \otimes_{\mathfrak{k}} F/R^2$ and hence the set T_n of elements of T whose coefficients are in S_n is a submodule of T. Since $\bigcup_n S_n = S$ and $R^2\delta \subseteq TS$ it follows that $R^2\delta = \bigcup_n (R^2\delta \cap T_n)$.

Suppose now that every element of R has degree at least d. Then all the monomials of F of degree at most $d - 1$ are $\mathfrak{k}$ independent modulo R, and hence $\mathfrak{k}$ independent modulo R^2. It follows that the vectors $t_\alpha(m_i + R \otimes m_k + R^2)$ with m_i, m_j monomials of degree at most $d-1$, may be taken as part of a basis for T.

Let $w = w(p_{\alpha_1}, \cdots, p_{\alpha_k})$ of degree d be an element of least degree in R. If m is a monomial of F occurring in w with coefficient k_m and $m = m_1 p_2 m_2$, then from the above remark and equation (2), the basis element $t_\alpha(m_1 + R \otimes m_2 + R^2)$ occurs in $w\delta$ with coefficient exactly k_m. In particular if q is a monomial of degree d occurring in w and $q = q' p_\gamma$ then $t_\alpha(q' + R \otimes 1)$ occurs in $w\delta$ with coefficient k_q. Further if another term $t_\beta(q' + R \otimes m + R^2)$ occurs in $w\delta$ with nonzero coefficient then $\beta \neq \alpha$. It now follows that if, with $f \in F$, $(wf)\delta = (w\delta)(f + R^2)$

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is in T_n then $t_\alpha(q' + R \otimes f + R^2) \in T_n$, and hence that $f \in R_n + R^2$. Thus if $R^2\delta \subseteq T_n$ then $R = R_n + R^2$. This however cannot happen since $R/R^2 \cong R \otimes_F F/R = \oplus_{i=1}^{\infty}(e_i + R^2)F/R$. Thus $\{R^2\delta \cap T_n\}$ is infinite and $R^2\delta$ is not finitely generated as a $F/R - F/R^2$ module. By (3), neither is R^2/R^3. Since R annihilates R^2/R^3 from the right, R^2/R^3 is an F/R bimodule and is clearly still not finitely generated when considered as such. This proves the theorem.

The assertion in our opening sentence now follows easily: if F/R has finite $\mathfrak{k}$ dimension then, as in the first part of the proof of the theorem, R^2 is finitely generated even as a right ideal and hence F/R^2 is finitely presented. Conversely if F/R^2 is finitely presented, then R^2 is a finitely generated F bimodule. It follows that R^2/R^3 is a finitely generated F/R bimodule and hence, by the theorem, F/R has finite $\mathfrak{k}$ dimension.

Theorem 1 was motivated by the following observations; Let $\mathfrak{k}$ be a countable field and let F be the free $\mathfrak{k}$ algebra on $\{x, y\}$. Let $R = F'$ the commutator ideal of F. Then R is generated, qua F bimoule by $xy - yx$ and, using (3) with $U = V = R$, we see that R/R^2 is a one generator subbimodule of a free F/R bimodule. Since $(F/R)^{opp} \otimes F/R \simeq F/R \otimes F/R$ is isomorphic to a (commutative) polynomial algebra on four variables, it has no zero divisors hence R/R^2 is itself a free F/R bimodule. So $R/R^2 \simeq F/R \otimes_{\mathfrak{k}} F/R$. In particular R/R^2 is both right and left F/R free (this is true for any R) and multiplication in F induces an F/R bimodule isomorphism $R/R^2 \otimes_{F/R} R/R^2 \simeq R^2/R^3$. Thus

$$R^2/R^3 \simeq (F/R \otimes_{\mathfrak{k}} F/R) \otimes_{F/R} (F/R \otimes_{\mathfrak{k}} F/R) \simeq F/R \otimes_{\mathfrak{k}} F/R \otimes \mathfrak{k}\, F/R.$$

Clearly, then, R^2/R^3 is a free F/R bimodule of infinite rank. It follows readily that R^2/R^3 contains uncountably many submodules and hence that F/R^3 contains uncountably many ideals. Since F/R^3 is finitely generated, F/R^3 has uncountably many non-isomorphic epimorphic images. Further [4, theorem 8] each of these images satisfies all the polynomial identities of the algebra of 3×3 matrices over $\mathfrak{k}$.

Recall now that a $\mathfrak{k}$ algebra B is said to be embeddable in matrices if, for some n, it is a subalgebra of the algebra of $n \times n$ matrices over some commutative $\mathfrak{k}$ algebra A. If B is embeddable and finitely generated then we may choose A to also be finitely generated [5]. By the Hilbert basis theorem there are only countably many finitely generated commutative $\mathfrak{k}$ algebras. Hence only countably many finitely generated $\mathfrak{k}$ algebras are embeddable in matrices. Thus we have

THEOREM 2. *Let $\mathfrak{k}$ be a countable field. There are uncountably many non-isomorphic two generator $\mathfrak{k}$ algebras B with $B'^3 = 0$ which are not embeddable in matrices. Each B satisfies all the identities of 3×3 matrices over $\mathfrak{k}$.*

An example of this type was first discovered by Small [5].

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References

[1] S. A. Amitsur, 'A noncommutative Hilbert basis theorem and subrings of matrices', *Trans. Amer. Math. Soc.* 149 (1970), 133–142.

[2] P. M. Cohn, 'On a generalization of the Euclidean algorithm', *Proc. Cambridge Phil. Soc.* 57 (1961), 18–30.

[3] J. Lewin, 'Free modules over free algebras and free group algebras: The Schreier technique', *Trans. Amer. Math. Soc.* 145, (1969) 455–465.

[4] J. Lewin, 'A matrix representation for associative algebras I.' (to appear in *Trans. Amer. Math. Soc.*).

[5] L. Small, 'An example in *P. I.* rings', *J. Algebra* 17, (1971) 434–436.

[6] L. Small, 'Ideals in finitely generated *PI*-Algebras', *Ring theory* (Academic Press) 1972, 347–352.

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