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ON SOME INTUITIONISTIC MODAL LOGICS

This is an abstract of my paper *On some intuitionistic modal logics* submitted to **Publ. RIMS, Kyoto Univ.**.

Some modal logics based on logics weaker than the classical logic have been studied by Fitch [4], Prior [7], Bull [1], [2], [3], Prawitz [6] etc. Here we treat modal logics based on the intuitionistic propositional logic, which call intuitionistic modal logics (abbreviated as IML's).

Let H be the intuitionistic propositional logic formulated in the Hilbert-style. The rules of inference of H are modus ponens and the rule of substitution. The IML L_0 is obtained from H by adding the following three axioms,

$\Box p \supset p$,
$\Box p \supset \Box\Box p$,
$\Box(p \supset q) \supset (\Box p \supset \Box q)$,

and the rule of necessitation, i.e., from A infer $\Box A$. It is clear that L_0 with the law of excluded middle becomes $S4$. Now, consider the following axioms.

$A_1 :\ \neg\Box p \supset \Box\neg\Box p$,
$A_2 :\ (\Box p \supset \Box q) \supset \Box(\Box p \supset \Box q)$,
$A_3 :\ \Box(\Box p \vee q) \supset (\Box p \vee \Box q)$,
$A_4 :\ \Box p \vee \Box\neg\Box p$.

The logic L_0 with the axiom A_i is denoted by L_i for $i = 1, 2, 3, 4$. The logic L_3 with A_1 (or A_2) is denoted by L_{31} (or L_{32}). It is easy to see that $S4$ with any A_i is equal to $S5$.

We identity a logic L with the set of formulas provable in L.

THEOREM 1.

(i) *For* $J = 1, 2, 3, 31, 32,\ L_0 \subsetneq L_J \subsetneq L_4$.
(ii) $L_1 \subsetneq L_2 \subsetneq L_{32}$ *and* $L_3 \subsetneq L_{31} \subsetneq L_{32}$.

For IML's, we introduce a kind of Kripke models, which we call I models. A triple $(M, \leqslant, R)$ is an I frame, if

(i) M is a nonempty set with a partial order $\leqslant$,
(ii) R is a reflexive and transitive relation on M such that $x \leqslant y$ implies xRy each $x, y \in M$.

For any formula A and an element $a \in M$, a valuation $W(A, a) \in \{t, f\}$ is defined in the same way as a valuation on a Kripke model for the intuitionistic propositional logic. For instance,

$W(A \supset B, a) = T$ if and only if for any b such that
$a \leqslant b, W(A, b) = f$ or $W(B, b) = t$.

Moreover, we claim that

$W(\Box A, a) = t$ if and only if for any b such that
$aRb\ W(A, b) = t$.

A quadruple $(M, \leqslant, R, W)$ is an I model if $(M, \leqslant, R)$ is an I frame and W is a valuation on it. A formula A is valid in an I frame $(M, \leqslant, R)$ if $W(A, a) = t$ for any valuation W on $(M, \leqslant, R)$ and any element $a \in M$.

For any binary relation R, we write $x \sim_R y$ if xRy and yRx hold. In what follows we omit the subscript letter R. Now define I frames of type J for $J = 0, 1, 2, 3, 31, 32, 4$ as follows.

(0) Any I frame is of type 0.

(1) An I frame $(M, \leqslant, R)$ is of type 1 when for each $x, y \in M$, if xRy then there is an element y' in M such that $x \leqslant y'$ and yRy'.

(2) An I frame $(M, \leqslant, R)$ is type 2 when for each $x, y \in M$, if xRy then there is an element y' in M such that $x \leqslant y'$ and $y \sim y'$.

(3) An I frame $(M, \leqslant, R)$ is of type 3 when for each $x, y \in M$, if xRy then there is an element x' in M such that $x \sim x'$ and $x' \leqslant y$.

($3i$) An I frame is of type $3i$ if it is both of type 3 and of type i, for $i = 1, 2$.

(4) An I frame $(M, \leqslant, R)$ is of type 4 if R is symmetric.

Theorem 2. *A formula is provable in L_J if and only if it is valid in any I frame of type J, for $J = 0, 1, 2, 3, 31, 32, 4$.*

An IML L_J has the finite model property if for any formula A not provable in L_J there is a finite I frame of type J in which A is not valid.

Theorem 3. *For $J = 0, 2, 3, 32, 4$, L_J has the finite model property.*

In [5], another kind of Kripke models is introduced and discussed.

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