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ON SOME L_1-FINITE TYPE (HYPER)SURFACES IN $\mathbb{R}^{n+1}$

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ABSTRACT. We say that an isometric immersed hypersurface $x : M^n \to \mathbb{R}^{n+1}$ is of L_k-finite type (L_k-f.t.) if $x = \sum_{i=0}^{p} x_i$ for some positive integer $p < \infty$, $x_i : M \to \mathbb{R}^{n+1}$ is smooth and $L_k x_i = \lambda_i x_i, \lambda_i \in \mathbb{R}, 0 \leq i \leq p$, $L_k f = \mathrm{tr} P_k \circ \nabla^2 f$ for $f \in C^\infty(M)$, where P_k is the kth Newton transformation, $\nabla^2 f$ is the Hessian of f, $L_k x = (L_k x^1, \ldots, L_k x^{n+1})$, $x = (x^1, \ldots, x^{n+1})$. In this article we study the following (hyper)surfaces in $\mathbb{R}^{n+1}$ from the view point of L_1-finiteness type: totally umbilic ones, generalized cylinders $S^m(r) \times \mathbb{R}^{n-m}$, ruled surfaces in $\mathbb{R}^{n+1}$ and some revolution surfaces in $\mathbb{R}^3$.

1. Introduction

Finite type submanifolds have been introduced in the late seventies by B. Y. Chen [2] to find the best possible estimate of the total mean curvature of a compact submanifold of a Euclidean space and to find a notion of degree for submanifolds of Euclidean space. Since then the subject has had a rapid development and so many mathematicians contribute to it, see the excellent survey of B. Y. Chen [3]. We recall that an isometrically immersed submanifold $x : M^n \to \mathbb{R}^{n+k}$ is said to be of finite type if x has a finite decomposition as $x = \sum_{i=0}^{p} x_i$, for some positive integer $p < +\infty$, such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}, 0 \leq i \leq p$, where Δ is the Laplacian operator of M. If all λ_i,s are mutually different, then M^n is said to be of p-type. If in particular one of λ_i,s is zero, then M is said to be of null p-type.

As it is well known, the Laplacian operator of an isometrically immersed hypersurface $M^n \subset \mathbb{R}^{n+1}$ is an (intrinsic) 2nd-order linear differential operator which arises naturally as the linearized operator of the first variation of the mean curvature for normal variation of the hypersurface. From this point of view, the Laplacian operator Δ can be seen as the first one of n operators $L_0 = \Delta, L_1, \ldots, L_{n-1}$, where L_k stands for the linearized operator of the first variation of the $(k+1)$th mean curvature (the $(k+1)$th elementary symmetric function in terms of the principal curvatures of M) arising from normal variation of the hypersurface. These operators are given by $L_k(f) = \mathrm{tr}(P_k \circ \nabla^2 f)$ for

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any real smooth function f on M, where P_k denotes the kth Newton transformation associated to the 2nd fundamental form of the hypersurface and $\nabla^2 f$ is the Hessian of f. Although in general the operators L_k are not elliptic, they still share nice properties with the Laplacian operator of M, see [1]. So, it seems natural to consider isometrically immersed hypersurfaces $x : M^n \to \mathbb{R}^{n+1}$ in Euclidean space satisfying the condition $x = \sum_{i=0}^{p} x_i$, for some positive integer p such that $L_k x_i = \lambda_i x_i, \lambda_i \in \mathbb{R}, 0 \leq i \leq p,$. We call such hypersurfaces, L_k-finite type. This is an important and interesting generalization of finite type hypersurfaces, and with this paper, we begin to study such hypersurfaces. Here we consider totally umbilic hypersurfaces, generalized cylinders, ruled surfaces and some revolution hypersurfaces from the point of view of L_1- finiteness type.

We should mention that similar to Proposition 1 of [5] we have the following fact. If M is an L_1-k-type surface, and if $p(T) = \prod_{i=1}^{k}(T - \lambda_i)$, then $p(L_1)(x - x_0) = 0$. The polynomial p is called the minimal polynomial of M.

2. Main result

Proposition 2.1. *If $x : M^n \to \mathbb{R}^{n+1}$ is a totally umbilic isometric immersion, then either it is of L_k-null 1-type for every $k, 0 \leq k \leq n-1$, or it is of L_k-1-type for $1 \leq k \leq n-1$.*

Proof. If M is totally geodesic, then its shape operator S is $S \equiv 0 \Rightarrow H_k = 0$ for $0 \leq k \leq n \Rightarrow L_k x = c_k H_{k+1} N = 0$, H_k is the kth mean curvature of M, see [1], so M is of L_k-null 1-type for every k, $0 \leq k \leq n-1$.

If M is totally umbilic, but not totally geodesic, then its shape operator S is $S = \lambda id$, for some $\lambda \in \mathbb{R}, \lambda \neq 0, \prec x, x \succ = \lambda^2$, so $L_k x = c_k H_{k+1} N = c_k \binom{n}{k} \lambda^{k+1} N = c_k \binom{n}{k} \lambda^{k-1} x$, see [6, Proposition 4.36]. So M is of L_k-1-type for every k, $1 \leq k \leq n-1$. □

Proposition 2.2. *If the isometrically immersed hypersurface $M^n \subset \mathbb{R}^{n+1}$ is a generalized cylinder $S^m(r) \times \mathbb{R}^{n-m}$, then M is of L_k-null 1-type, if $k+1 > m$, and it is of L_k-null 2-type, if $k+1 < m$.*

Proof. Since $L_k x = c_k H_{k+1} N$ and the shape operator of M, S is

$$S = \text{diag}(\frac{1}{\sqrt{r}}, \ldots, \frac{1}{\sqrt{r}}, 0, \ldots, 0),$$

where the multiplicity of $\frac{1}{\sqrt{r}}$ is m. If $k+1 > m \Rightarrow H_{k+1} = 0$, so $L_k x = 0$ hence M is of L_k-null 1-type. If $k+1 < m \Rightarrow H_{k+1} = \binom{m}{k+1} \frac{1}{r^{\frac{k+1}{2}}}$, $N(x) = (x_1, \ldots, x_{m+1}, 0, \ldots, 0) \Rightarrow x = x_I + x_{II}$, $x_I = (x_1, \ldots, x_{m+1}) \times \{0\} \in \mathbb{R}^{n+1}$, $x_{II} = \{0\} \times (x_{m+2}, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$, $L_k x_I = c_k \binom{m}{k+1} \frac{1}{r^{\frac{k+1}{2}}} x_I$, $L_k x_{II} = 0$ hence M is of L_k-null 2-type. □

3. Ruled surfaces in $\mathbb{R}^{n+1}$

In this section we want to consider the ruled surfaces of $\mathbb{R}^{n+1}$. Here we follow [4] and give the result via Propositions 3.1 and 3.2.

Proposition 3.1. *If M is a cylinderical ruled surface in $\mathbb{R}^{n+1}$, then M is of L_1-finite type if and only if it is a cylinder over a curve of L_1-finite type.*

Proof. Let M be a cylinder over a curve γ in an affine hyperplane E^n which we can choose to have the equation $x_{n+1} = 0$. We can assume that γ is parameterized by its arc length, then a parametrization x of M is given by $x(s,t) = \gamma(s) + te_{n+1}$. The induced metric on x is $\left[\begin{smallmatrix} \langle\gamma',\gamma'\rangle & 0 \\ 0 & 1 \end{smallmatrix}\right]$, L_1 of M is of the form $L_1 = \mathrm{tr}(2H_1 I - S) \circ H$, where S is the shape operator of M which is of the form $\left[\begin{smallmatrix} \lambda & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $\lambda \in C^\infty(\mathbb{R})$, $H = (H_{ij})$ is the Hessian operator. To find L_1, it is enough to know H_{11}, H_{22}. By the formula of the metric one gets that $H_{11} = \frac{\partial^2}{\partial s^2}$, $H_{22} = \frac{\partial^2}{\partial t^2}$. By L_1' of γ, we mean $L_1\big|_\gamma$ and $L_1 x_{n+1} = L_1 t = 0$. Thus as in [4], one obtains that M is of L_1-finite type if and only if each component of $\gamma(s)$ is a finite sum of eigenfunctions of L_1, that is $\gamma(s) = \Gamma_0 + \sum_{i=1}^p \Gamma_i(s,t)$ where $L_1\Gamma_i = \lambda_i\Gamma_i$, it can be seen that Γ_i does not depend on t and $(L_1\big|_\gamma)\Gamma_i = L_1\Gamma_i = \lambda_i\Gamma_i$, $1 \le i \le p$, then M is of L_1-$(p+1)$-type, unless one of the eigenfunctions which appear in the decomposition of γ has eigenvalue 0, in which case M is of L_1-p-type. □

Proposition 3.2. *If M is a non cylinderical ruled surface in $\mathbb{R}^{n+1}$, then M is of L_1-null-1-type if and only if its Gaussian curvature is zero, i.e., M is flat.*

Proof. If M is not cylindrical, we can decompose M into open pieces such that on each piece we can find a parametrization x of the form $x(s,t) = \alpha(s) + t\beta(s)$, where α, β are curves in $\mathbb{R}^{n+1}$ such that $\langle\alpha',\beta\rangle = 0$, $\langle\beta,\beta\rangle = 1$ and $\langle\beta',\beta'\rangle = 1$, i.e., the induced metric on x is $\left[\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right]$ where $q = t^2 + 2\langle\alpha',\beta'\rangle t + \langle\alpha',\alpha'\rangle = \|\alpha' + t\beta'\|^2$. We have that

$$H_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} - \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}, \qquad 1 \le i,j,k \le 2,$$

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial s}, \quad \frac{\partial}{\partial x_2} = \frac{\partial}{\partial t},$$

$$\frac{\partial x}{\partial s} = \alpha'(s) + t\beta'(s), \quad \frac{\partial x}{\partial t} = \beta(s),$$

$$\frac{\partial^2 x}{\partial s^2} = \alpha''(s) + t\beta''(s), \quad \frac{\partial^2 x}{\partial t^2} = 0, \quad \frac{\partial^2 x}{\partial t \partial s} = \beta'(s).$$

To calculate $L_1 x$, we need to know H_{11}, H_{22} and the shape operator of M. We obtain that

$$H_{11} = \frac{\partial 2}{\partial s^2} - \frac{1}{2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s} - \frac{q}{2}\frac{\partial q}{\partial t}\frac{\partial}{\partial t}, \quad H_{22} = \frac{\partial^2}{\partial t^2}.$$

We have

$$S(x_s) = \frac{l}{m}x_s + \frac{m}{G}x_t, \quad S(x_t) = \frac{m}{E}x_s + \frac{n}{G}x_t,$$
$$l = \langle S(x_s), x_s\rangle = \langle N, x_{ss}\rangle, \quad N = \frac{x_s \times x_t}{\|x_s \times x_t\|}, \quad S(x_t) = \frac{m}{E}x_s,$$
$$x_s = \frac{\partial x}{\partial s}, \quad x_t = \frac{\partial x}{\partial t},$$

l, m are polynomials in t of degree at most 2, with coefficients, functions in s, E is a polynomial in t of degree 2, with coefficients functions in s, G =constant, so the entries of S are rational functions in t with coefficients, functions in s. The nominators are of degree at most 2. Hence if P is a polynomial in t with coefficients, functions in s and $\deg P = d$, then

$$L_1(P(t)) = \frac{\widetilde{P}(t)}{q^3 Q}$$

where $\widetilde{P}$ is a polynomial in t with functions in s as coefficients, and $\deg \widetilde{P} \leq d + 6$, Q is a polynomial in t with functions as coefficients, $\deg Q = 2$.

If M is of L_1-k-type, $\exists c_1, \ldots, c_k \in \mathbb{R}$, s.t.,

$$L_1^{k+1}x + c_1 L_1^k x + \cdots + c_k L_1 x = 0 \tag{3.1}$$

We know that every component of x is a linear function in t with functions in s as coefficients. By the relation

$$L_1(P(t)) = \frac{\widetilde{P(t)}}{q^3.Q} \Rightarrow L_1^r x = \frac{P_r(t)}{q^{3r-1}.Q^r},$$

where P_r is a vector field whose components are polynomial in t with functions in s as coefficients, $\deg P_r \leq 1+6r$. Hence if r goes up by one, the degree of the numerator of any component of $L_1^r x$ goes up by at most 6, while the degree of the denominator goes up by 8. Hence the sum 3.1 can never be zero, unless of course $L_1 x = 0$, this means that H_2, the sectional (here the Gaussian) curvature of M is zero, (since $\dim M = 2$, $M \subseteq \mathbb{R}^{n+1}$). The converse is trivial. □

4. Revolution hypersurfaces

In the final section we consider some of the revolution hypersurfaces. We follow [5] and study such hypersurfaces in two different cases, polynomial and rational kind revolution hypersurfaces. We begin with the first case.

Definition 4.1. Let $M^2 \subseteq \mathbb{R}^3$ be a revolution hypersurface given by $x(u, v) = ((g(u), h(u)\cos v), h(u)\sin v), (u, v) \in \mathbb{R}^2$. Then M is called of polynomial (resp. rational) kind if g and h are polynomials in u (resp. if g is a rational function in h).

4.1. Polynomial kind revolution hypersurfaces

Let $M^2 \subseteq \mathbb{R}^3$ be parameterized by $x(u,v) = (g(u), h(u)\cos v, h(u)\sin v)$ where g, h are polynomials, $(u,v) \in \mathbb{R}^2$, we also assume that $g'^2 + h'^2 = A^2$ where A is a polynomial in u. Then $E = \langle x_u, x_u\rangle = g'^2+h'^2$, $G = \langle x_v, x_v\rangle = h^2$, $F = \langle x_u, x_v\rangle = 0$, so the matrix of the induced metric on M is $\left[\begin{smallmatrix} g'^2+h'^2 & 0 \\ 0 & h^2\end{smallmatrix}\right]$. If S is the shape operator of M, then $S = \left[\begin{smallmatrix} \frac{l}{E} & 0 \\ 0 & \frac{n}{G}\end{smallmatrix}\right]$ where

$$l = \frac{-g'h'' + g''h'}{\sqrt{g'^2 + h'^2}}, \quad n = \frac{-g'h}{\sqrt{g'^2 + h'^2}}.$$

In order to find $L_1 f = \operatorname{tr}(2H_1 I - S) \circ H^f, f \in C^\infty(M)$, it is enough to calculate H^f_{11} and H^f_{22}, we obtain that

$$H_{11} = \frac{\partial^2}{\partial u^2} - \frac{g'g'' + h'h''}{g'^2 + h'^2}\frac{\partial}{\partial u}, \quad H_{22} = \frac{\partial^2}{\partial v^2}.$$

Following [5], we prove the next lemma.

Lemma 4.2. *Let $F(u), G(u)$ be polynomial functions in u and M as introduced in the subsection* (4.1)*. Then $L_1(\frac{F}{G}) = \frac{F_1}{G_1}$ for some polynomial functions F_1, G_1 with*

$$d(F_1) - d(G_1) \leq d(F) - d(G) - d(h) - 2$$

Proof. By applying L_1 and straightforward computation, we obtain that $L_1(\frac{F(u)}{G(u)}) = \frac{F_1(u)}{G_1(u)}$, where

$$\begin{aligned} F_1 &= g'[(g'^2 + h'^2)(F'G - FG')'G - 2(g'^2 + h'^2)G'(F'G - FG') \\ &\quad - G(g'g'' + h'h'')(F'G - FG')], \\ G_1 &= G^3 h(g'^2 + h'^2)^{\frac{3}{2}} = G^3 hA^3. \end{aligned}$$

Then we get that

$$\begin{aligned} &d(G_1) = 3d(G) + d(h) + 3\max\{d(g), d(h)\} - 3 \\ &d(F_1) \leq 2d(G) + d(F) + 3\max\{d(g), d(h)\} - 5 \\ &\quad \Rightarrow d(F_1) - d(G_1) \leq d(F) - d(G) - d(h) - 2. \end{aligned}$$

□

Theorem 4.3. *Let M be a surface of revolution of the polynomial kind as introduced in the subsection* (4.1)*. Then M is a surface of L_1-finite type if and only if M is either an open piece of a plane or an open piece of a circular cylinder, or it is flat.*

Proof. Let

$$L_1^i g = \frac{F_i}{G_i}, i = 1, 2, 3, \ldots, \tag{4.1}$$

then similar to (3.5) of [5], we have

$$d(F_{i+1}) - d(G_{i+1}) < d(F_i) - d(G_i) < \cdots < d(F_1) - d(G_1) < d(g). \tag{4.2}$$

Assume that M of L_1-finite type, say of L_1-k-type. Let

$$p(T) = T^k + c_1 T^{k-1} + \cdots + c_{k-1}T + c_k \tag{4.3}$$

be the minimal polynomial of M, then p has k distinct real roots, and from the relations 4.1 and 4.3 we have that

$$\frac{F_k}{G_k} + c_1\frac{F_{k-1}}{G_{k-1}} + \cdots + c_{k-1}\frac{F_1}{G_1} + c_k(g-a) = 0$$

for some constant a. Let $K = G_1 \cdots G_k$, then

$$K\frac{F_k}{G_k} + c_1 K\frac{F_{k-1}}{G_{k-1}} + \cdots + c_{k-1}K\frac{F_1}{G_1} + c_k K(g-a) = 0.$$

If $d(g) = 0$, M is an open portion of a plane. If $d(g) > 0$, then by (4.2) we get the inequalities

$$d(K(g-a)) > d(K\frac{F_1}{G_1}) > \cdots > d(K\frac{F_{k-1}}{G_{k-1}}) > d(K\frac{F_k}{G_k})$$

which is impossible unless

$$p(T) = T^2 + c_1 T, c_1 \neq 0, L_1 g = 0,$$

since the minimal polynomial p has k distinct real roots. That is

$$L_1^2 x + c_1 L_1 x = 0,$$

so we get that M is either of L_1-1-type or of L_1-null 2-type. If M is of L_1-1-type, then $L_1 x = c_1 H_2 N = \lambda x$ for some $\lambda \in \mathbb{R}$, if $\lambda = 0 \Rightarrow H_2 = 0$, so M is flat. If $\lambda \neq 0$, then M has to be an open portion of an sphere, which is not possible, since M is a revolution hypersurface of polynomial kind in $\mathbb{R}^3$. If M is of L_1-null 2-type, we characterize it as follows. We have that $x = c + x_p + x_q$ where $c \in \mathbb{R}^3$, $x_p, x_q : M \to \mathbb{R}^3$ are smooth, $L_1 x_p = 0$,

$$\begin{aligned} L_1 x_q &= \lambda_q x_q, \lambda_q \in \mathbb{R}, \lambda_q \neq 0 \Rightarrow L_1 x_q = \lambda_q x_q = L_1 x = c_1 H_2 N \\ &\Rightarrow x_q = \frac{c_1 H_2 N}{\lambda_q} \\ &\Rightarrow L_1(\frac{c_1 H_2 N}{\lambda_q}) = \frac{c_1}{\lambda_q} L_1(H_2 N)\lambda_q x_q = c_1 H_2 N. \end{aligned} \tag{4.4}$$

Now we calculate $L_1(H_2 N)$ directly.

$$\begin{aligned} &X.\langle H_2 N, a\rangle = \langle (X.H_2)N, a\rangle - H_2\langle SX, a\rangle \\ &\Rightarrow \nabla\langle H_2 N, a\rangle = \langle N, a\rangle \nabla H_2 - S(a^T) \\ &\Rightarrow \nabla_X(\nabla\langle H_2 N, a\rangle) \end{aligned} \tag{4.5}$$

$$= X.\langle N,a\rangle\nabla H_2 + \langle N,a\rangle\nabla_X\nabla H_2 - \nabla_X(S(a^T))$$
$$= -\langle SX,a\rangle\nabla H_2 + \langle N,a\rangle\nabla_X\nabla H_2 - \nabla S(a^T,X) - S(\nabla_X a^T)$$
$$= -\langle S(a^T),X\rangle\nabla H_2 + \langle N,a\rangle\nabla_X\nabla H_2 - (\nabla_{a^T}S)X - \langle N,a\rangle S^2X$$
$$\Rightarrow L_1\langle H_2N,a\rangle = tr(p_1\circ\nabla_{a^T}S) - \langle N,a\rangle tr\, S^2\circ P_1 - \sum_i \langle\langle Sa^T,e_i\rangle\nabla H_2, P_1e_i\rangle$$
$$+ \langle N,a\rangle\sum_i\langle P_1(\nabla_{e_i}\nabla H_2),e_i\rangle$$
$$\Rightarrow L_1\langle H_2N,a\rangle = L_1\langle N,a\rangle + \langle N,a\rangle\sum_i H_{H_2}(e_i,P_1e_i) - \langle S\circ P_1(\nabla H_2),a\rangle$$
$$\Rightarrow L_1(H_2N) = L_1N + (\sum_i H_{H_2}(e_i,P_1e_i))N - S\circ P_1(\nabla H_2)$$

using the formula of L_1N and comparing (4.5), (4.4) one gets that $S\circ P_1(\nabla H_2) = -\nabla H_2$. By using the relation $P_1 = 2H_1I - S$ and that $S = [\begin{smallmatrix}\frac{l}{E} & 0\\ 0 & \frac{n}{G}\end{smallmatrix}]$ with respect to the ordered basis $\{x_u,x_v\}$ one obtains that the relation $S\circ P_1(\nabla H2) = -\nabla H_2$ (by counting the degrees of the polynomials involved in this equation) holds only when h equals some constant λ, $\lambda\in\mathbb{R},\lambda\neq 0$, i.e., $x(u,v) = (g(u),\lambda\cos v,\lambda\sin v)$, $\lambda\in\mathbb{R},\lambda\neq 0$, so M is a circular cylinder. We leave the proof of the converse to the reader as an easy exercise. □

4.2. Revolution hypersurfaces of rational kind

Let M be parameterized by $x(u,v) = (g(u),h(u)\cos v,h(u)\sin v)$, we recall that M is said to be of rational kind if g is a rational function in h, i.e., g is the quotient of two polynomial functions in h.
Following [5], we get the final result.

Theorem 4.4. *Let M be a surface of revolution of the rational kind such that if $g' = \frac{Q}{R}$, $(Q,R) = 1$, then $Q^2+R^2 = A^2$, where A is a polynomial. Then M is a surface of L_1-finite type if and only if M is flat.*

In fact Theorem 4.4 follows from the following more general result.

Theorem 4.5. *Let M be an L_1-finite type surface of revolution parameterized by $x(t,v) = (g(t),t\cos v,t\sin v)$, where g is a rational function in t and if $g'(t) = \frac{Q}{R}$, $(Q,R) = 1$, then $Q^2+R^2 = A^2$, where A is a polynomial in t. Then M is flat or* $\deg Q = \deg R = 2 + \deg(Q+R)$.

Proof of Theorem 4.4. Without loss of generality, we may assume that M is parameterized by $x(t,v) = (g(t),t\cos v,t\sin v)$, where $g(t) = \frac{G(t)}{H(t)}$ for some polynomial functions G,H, $(G,H) = 1$. Assume that M is of L_1-finite type, then by Theorem 4.5, M is flat. The converse is trivial. □

Proof of Theorem 4.5. Let M be a surface of revolution parameterized by $x(t,v) = (g(t),t\cos v,t\sin v)$ as in Theorem 4.5. In order to find L_1 we need the metric, the shape operator S and H_{11}, H_{22}. The induced metric on x is $\begin{bmatrix} g'^2+1 & 0 \\ 0 & t^2 \end{bmatrix}$, the shape operator S is

$$\begin{bmatrix} \frac{g''}{(g'^2+1)^{\frac{3}{2}}} & 0 \\ 0 & \frac{g'}{t(g'^2+1)^{\frac{1}{2}}} \end{bmatrix}.$$

We obtain that

$$H_{11} = \frac{\partial^2}{\partial t^2} - \frac{g'g''}{g'^2+1}\frac{\partial}{\partial t}, \quad H_{22} = \frac{\partial}{\partial v^2}.$$

So,

$$L_1 = \mathrm{tr}(s_1 H_1 I - S) \circ H, \quad H = \begin{bmatrix} H_{11} & * \\ * & H_{22} \end{bmatrix}$$

$$\Rightarrow L_1 = \mathrm{tr} \begin{bmatrix} \frac{g'}{t(g'^2+1)^{\frac{1}{2}}} & 0 \\ 0 & \frac{g''}{(g'^2+1)^{\frac{3}{2}}} \end{bmatrix} \circ \begin{bmatrix} \frac{\partial^2}{\partial t^2} - \frac{g'g''}{g'^2+1}\frac{\partial}{\partial t} & * \\ * & \frac{\partial}{\partial v^2} \end{bmatrix}$$

$$\Rightarrow L_1 = \frac{g'}{t(g'^2+1)^{\frac{1}{2}}}\left(\frac{\partial^2}{\partial t^2} - \frac{g'g''}{g'^2+1}\frac{\partial}{\partial t}\right) + \frac{g''}{(g'^2+1)^{\frac{3}{2}}}\frac{\partial^2}{\partial v^2}$$

then if $L_1(t\cos v) = \frac{Q_1}{R_1}\cos v \Rightarrow \frac{Q_1}{R_1} = \frac{-g'^2 g'' - t^2 g''}{t(g'^2+1)^{\frac{3}{2}}} = -\frac{g''(g'^2+t^2)}{t(g'^2+1)^{\frac{3}{2}}}$ if $g' = \frac{Q}{R} \Rightarrow \frac{Q_1}{R_1} = \frac{(QR'-Q'R)(Q^+t^2R^2)}{tR(Q^2+R^2)^{\frac{3}{2}}}$, $Q^2+R^2 = A^2$. If inductively $L_1^{i+1}(t\cos v) = \frac{Q_{i+1}}{R_{i+1}}\cos v$, $i = 1,2,3,\ldots$ for some polynomial functions Q_i, R_i, put $\overline{Q}_i = \overline{Q}_i R_i - Q_i R_i'$, $\overline{\overline{Q}}_i = (Q_i'R_i - Q_iR_i')'R_i^2 - 2R_iR_i'(Q_i'R_i - Q_iR_i')$, then have

$$\begin{aligned} \frac{Q_{i+1}}{R_{i+1}} &= \frac{g'}{t(g'^2+1)^{\frac{1}{2}}}\frac{\overline{\overline{Q}}_i}{R_i^4} - \frac{g'^2 g''}{t(g'^2+1)^{\frac{3}{2}}}\frac{\overline{Q}_i}{R_i^2} - \frac{g''}{(g'^2+1)^{\frac{3}{2}}}\frac{Q_i}{R_i} \\ &= \frac{Q}{t(Q^2+R^2)^{\frac{1}{2}}} \times \frac{\overline{\overline{Q}}_i}{R_i^4} - \frac{Q^2(Q'R-R'Q)}{tR(Q^2+R^2)^{\frac{3}{2}}} \times \frac{\overline{Q}_i}{R_i^2} - \frac{R(Q'R-R'Q)}{(Q^2+R^2)^{\frac{3}{2}}}\frac{Q_i}{R_i} \\ &= \frac{QR(Q^2+R^2)\overline{\overline{Q}}_i - Q^2R_i^2(Q'R-R'Q)\overline{Q}_i - tR^2R_i^3(Q'R-R'Q)Q_i}{tRR_i^4(Q^2+R^2)^{\frac{3}{2}}} \end{aligned}$$

$$\begin{aligned} \Rightarrow Q_{i+1} &= QR(Q^2+R^2)\overline{\overline{Q}}_i - Q_i^2R_i^2(Q'R-R'Q)\overline{Q}_i - tR^2R_i^3(Q'R-R'Q)Q_i, \\ R_{i+1} &= tRR_i^4(Q^2+R^2)^{\frac{3}{2}} = tRR_i^4A^3. \end{aligned}$$

Assume that M is of L_1-finite type, say of k-type, let $p(T) = T^k + c_1T^{k-1} + \cdots + c_{k-1}T + c_k$ be the minimal polynomial of M, then p has k distinct real roots. We have that

$$\frac{Q_k}{R_k} + c_1\frac{Q_{k-1}}{R_{k-1}} + \cdots + c_{k-1}\frac{Q_1}{R_1} + c_k t = 0.$$

Let $K = R_1 \cdots R_k$, then we get

$$K\frac{Q_k}{R_k} + c_1 K\frac{Q_{k-1}}{R_{k-1}} + \cdots + c_{k-1}K\frac{Q_1}{R_1} + c_k tK = 0.$$

Following ([5], cases i-iii of Theorem 3) by similar discussion about $\deg(Q)$, $\deg(R)$ according to the cases $\deg(Q) > \deg(R)$ or $\deg(Q) < \deg(R)$ or $\deg(Q) = \deg(R)$, we see that the only possibility is that either $p(T) = T$ or $\deg(Q) = \deg(R) = 2 + \deg(Q + R)$. When $p(T) = T \Rightarrow H_2 = 0$, in this case, M is flat. □

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