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On some new double dynamic inequalities associated with Leibniz integral rule on time scales

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Abstract

In 2020, El-Deeb et al. proved several dynamic inequalities. It is our aim in this paper to give the retarded time scales case of these inequalities. We also give a new proof and formula of Leibniz integral rule on time scales. Beside that, we also apply our inequalities to discrete and continuous calculus to obtain some new inequalities as special cases. Furthermore, we study boundedness of some delay initial value problems by applying our results as application.

Keywords: Gronwall-type inequality; Boundedness; Time scales

1 Introduction

In 2020, El-Deeb et al. [1] have proved the following inequalities:

$$\begin{aligned}
&\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big)\\
&\quad\leq a(\hat{\varsigma},\hat{\varrho})+\int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}}\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)+p(\hat{\xi}_1,\hat{\xi}_2)\big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1\\
&\qquad+\int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}}b(\hat{\xi}_1,\hat{\xi}_2)\left[h(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)+\int_0^{\hat{\xi}_1}g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right]\Delta\hat{\xi}_2\Delta\hat{\xi}_1
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big)\\
&\quad\leq a(\hat{\varsigma},\hat{\varrho})+\int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}}\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)+p(\hat{\xi}_1,\hat{\xi}_2)\big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1\\
&\qquad+\int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}}f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\left(\int_0^{\hat{\xi}_1}g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1.
\end{aligned}$$

The objective of the theory of time scales, which was introduced by Stefan Hilger in his PhD thesis [2] in 1988, is to unify continuous and discrete calculus. Several foundational definitions and notations of basic calculus of time scales introduced in the excellent recent books [3, 4] by Bohner and Peterson will be employed in the sequel. For some Gronwall–Bellman-type integral, dynamic inequalities and other type of inequalities on time scales, see the papers [5–36].

We use the following notations:

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\begin{aligned} &\sigma(t) = t, \\ &\mu(t) = 0, \\ &f^{\Delta}(t) = f'(t), \\ &\int_a^b f(t)\Delta t = \int_a^b f(t)\,dt; \end{aligned} \tag{1.1}$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then

$$\begin{aligned} &\sigma(t) = t + 1, \\ &\mu(t) = 1, \\ &f^{\Delta}(t) = \Delta f(t), \\ &\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t), \end{aligned} \tag{1.2}$$

where Δ is the forward difference operator.

Theorem 1.1 (Chain rule on time scales [3]) *Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and Δ-differentiable on $\mathbb{T}^{\kappa}$, and let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Then there exists $c \in [t, \sigma(t)]$ with*

$$(f \circ g)^{\Delta}(t) = f'\big(g(c)\big)g^{\Delta}(t). \tag{1.3}$$

Theorem 1.2 (Chain rule on time scales [3]) *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable and the formula*

$$(f \circ g)^{\Delta}(t) = \left\{\int_0^1 \big[f'\big(hg^{\sigma}(t) + (1-h)g(t)\big)\big]\,dh\right\}(g)^{\Delta}(t), \tag{1.4}$$

holds.

Theorem 1.3 ([3]) *Let $t_0 \in \mathbb{T}^{\kappa}$ and $k : \mathbb{T} \times \mathbb{T}^{\kappa} \to \mathbb{R}$ be continuous at (t, t), where $t > t_0$ and $t \in \mathbb{T}^{\kappa}$. Assume that $k^{\Delta}(t, \cdot)$ is rd-continuous on $[t_0, \sigma(t)]$. Suppose that for any $\varepsilon > 0$, there exists a neighborhood U of t, independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$\big|\big[k\big(\sigma(t), \tau\big) - k(s, \tau)\big] - k^{\Delta}(t, \tau)\big[\sigma(t) - s\big]\big| \le \varepsilon\big|\sigma(t) - s\big|, \quad \forall s \in U.$$

If k^{Δ} denotes the derivative of k with respect to the first variable, then

$$f(t) = \int_{t_0}^{t} k(t, \tau)\Delta\tau$$

yields

$$f^{\Delta}(t) = \int_{t_0}^{t} k^{\Delta}(t,\tau)\Delta\tau + k\bigl(\sigma(t),t\bigr).$$

Other dynamic inequalities on time scales may be found in [37–40]. In this manuscript, we will discuss the retarded time scale case of the inequalities obtained in [1] using new techniques by replacing the upper limit $\hat{\varsigma}$ and $\hat{\varrho}$ of the integral by the delay function $\hat{\alpha}(\hat{\varsigma}) \leq \hat{\varsigma}$ and $\hat{\beta}(\hat{\varrho}) \leq \hat{\varrho}$. Furthermore, these inequalities that we obtained here extend some known results in the literature, and they also unify the continuous and discrete cases.

2 Main results

Throughout the paper, we suppose that $\mathbb{T}_1$ and $\mathbb{T}_2$ are two time scales.

First, we prove the following result.

Theorem 2.1 (Leibniz integral rule on time scales) *In the following by $f^{\Delta}(t,s)$ we mean the delta derivative of $f(t,s)$ with respect to t. Similarly, $f^{\nabla}(t,s)$ is understood. If f, f^{Δ} and f^{∇} are continuous, and $u,h : \mathbb{T} \to \mathbb{T}$ are delta differentiable functions, then the following formulas hold $\forall t \in \mathbb{T}^{\kappa}$:*

(i) $[\int_{u(t)}^{h(t)} f(t,s)\Delta s]^{\Delta} = \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\Delta s + h^{\Delta}(t)f(\sigma(t),h(t)) - u^{\Delta}(t)f(\sigma(t),u(t))$;

(ii) $[\int_{u(t)}^{h(t)} f(t,s)\Delta s]^{\nabla} = \int_{u(t)}^{h(t)} f^{\nabla}(t,s)\Delta s + h^{\nabla}(t)f(\rho(t),h(t)) - u^{\nabla}(t)f(\rho(t),u(t))$;

(iii) $[\int_{u(t)}^{h(t)} f(t,s)\nabla s]^{\Delta} = \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\nabla s + h^{\Delta}(t)f(\sigma(t),h(t)) - u^{\Delta}(t)f(\sigma(t),u(t))$;

(iv) $[\int_{u(t)}^{h(t)} f(t,s)\nabla s]^{\nabla} = \int_{u(t)}^{h(t)} f^{\nabla}(t,s)\nabla s + h^{\nabla}(t)f(\rho(t),h(t)) - u^{\nabla}(t)f(\rho(t),u(t))$.

Proof We will only prove part (i); the others may be proved similarly. Define a function g by

$$g(t) = \int_{u(t)}^{h(t)} f(t,s)\Delta s, \quad \text{for } t \in \mathbb{T}^{\kappa}. \tag{2.1}$$

We notice that g is a continuous function. Indeed, we have two cases for t. In the first case, if t is right-scattered, from (2.1), we get

$$\begin{aligned}
g^{\Delta}(t) &= \frac{g(\sigma(t)) - g(t)}{\sigma(t) - t} \\
&= \frac{1}{\sigma(t) - t}\left[\int_{u(\sigma(t))}^{h(\sigma(t))} f\bigl(\sigma(t),s\bigr)\Delta s - \int_{u(t)}^{h(t)} f(t,s)\Delta s\right] \\
&= \frac{1}{\sigma(t) - t}\left[-\int_{u(t)}^{u(\sigma(t))} f\bigl(\sigma(t),s\bigr)\Delta s + \int_{u(t)}^{h(t)} f\bigl(\sigma(t),s\bigr)\Delta s\right. \\
&\quad \left. + \int_{h(t)}^{h(\sigma(t))} f\bigl(\sigma(t),s\bigr)\Delta s - \int_{u(t)}^{h(t)} f(t,s)\Delta s\right] \\
&= \int_{u(t)}^{h(t)} \frac{f(\sigma(t),s) - f(t,s)}{\sigma(t) - t}\Delta s + \frac{1}{\sigma(t) - t}\int_{h(t)}^{h(\sigma(t))} f\bigl(\sigma(t),s\bigr)\Delta s \\
&\quad - \frac{1}{\sigma(t) - t}\int_{u(t)}^{u(\sigma(t))} f\bigl(\sigma(t),s\bigr)\Delta s \\
&= \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\Delta s + \frac{h(\sigma(t)) - h(t)}{\sigma(t) - t} f\bigl(\sigma(t),h(t)\bigr)
\end{aligned}$$

$$\begin{aligned}
&- \frac{u(\sigma(t)) - u(t)}{\sigma(t) - t} f\big(\sigma(t), u(t)\big) \\
&= \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\Delta s + h^{\Delta}(t) f\big(\sigma(t), h(t)\big) - u^{\Delta}(t) f\big(\sigma(t), u(t)\big). \qquad (2.2)
\end{aligned}$$

From (2.2), we get the required result.

Now consider the second case when t is right-dense. Since f is continuous, it is rd-continuous, hence it has a delta partial anti-derivative with respect to the second variable s, say $F(t,s)$, that is, $f(t,s) = F^{\Delta_s}(t,s)$, and then we have

$$\begin{aligned}
\left[\int_{u(t)}^{h(t)} f(t,s)\Delta s\right]^{\Delta} &= g^{\Delta}(t) \\
&= \lim_{r\to t} \frac{g(t) - g(r)}{t - r} \\
&= \lim_{r\to t} \frac{1}{t - r}\left[\int_{u(t)}^{h(t)} f(t,s)\Delta s - \int_{u(r)}^{h(r)} f(r,s)\Delta s\right] \\
&= \lim_{r\to t} \frac{1}{t - r}\left[\int_{u(t)}^{h(t)} f(t,s)\Delta s - \int_{u(r)}^{u(t)} f(r,s)\Delta s\right. \\
&\quad \left. - \int_{u(t)}^{h(t)} f(r,s)\Delta s - \int_{h(t)}^{h(r)} f(r,s)\Delta s\right] \\
&= \lim_{r\to t} \int_{u(t)}^{h(t)} \frac{f(t,s) - f(r,s)}{t - r}\Delta s + \lim_{r\to t} \frac{1}{t - r}\int_{h(r)}^{h(t)} F^{\Delta_s}(r,s)\Delta s \\
&\quad - \lim_{r\to t} \frac{1}{t - r}\int_{u(r)}^{u(t)} F^{\Delta_s}(r,s)\Delta s. \qquad (2.3)
\end{aligned}$$

Thus, from (2.3), we get

$$\begin{aligned}
\left[\int_{u(t)}^{h(t)} f(t,s)\Delta s\right]^{\Delta} &= \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\Delta s + \lim_{r\to t} \frac{1}{t - r}\big[F\big(r, h(t)\big) - F\big(r, h(r)\big)\big] \\
&\quad - \lim_{r\to t} \frac{1}{t - r}\big[F\big(r, u(t)\big) - F\big(r, u(r)\big)\big] \\
&= \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\Delta s + \lim_{r\to t} \frac{h(t) - h(r)}{t - r} \frac{F(r,h(t)) - F(r,h(r))}{h(t) - h(r)} \\
&\quad - \lim_{r\to t} \frac{u(t) - u(r)}{t - r} \frac{F(r,u(t)) - F(r,u(r))}{u(t) - u(r)} \\
&= \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\Delta s + \lim_{r\to t} \frac{h(t) - h(r)}{t - r} \lim_{r\to t} \frac{F(r,h(t)) - F(r,h(r))}{h(t) - h(r)} \\
&\quad - \lim_{r\to t} \frac{u(t) - u(r)}{t - r} \lim_{r\to t} \frac{F(r,u(t)) - F(r,u(r))}{u(t) - u(r)} \\
&= \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\Delta s + h^{\Delta}(t) F^{\Delta_s}\big(t, h(t)\big) - u^{\Delta}(t) F^{\Delta_s}\big(t, u(t)\big) \\
&= \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\Delta s + h^{\Delta}(t) f\big(t, h(t)\big) - u^{\Delta}(t) f\big(t, u(t)\big).
\end{aligned}$$

This completes the proof. □

Remark 2.2 If we take $h(t) = t$ and $u(t) = a$ (where a is constant), then Theorem 2.1 reduces to [4, Theorem 5.37, p. 139].

Now, by using the result of Theorem 2.1, we state and prove the rest of our main results:

Theorem 2.3 *Suppose $a \in C_{\mathrm{rd}}(\Omega, \mathbb{R}_+)$ is nondecreasing with respect to $(\hat{\varsigma}, \hat{\varrho}) \in \Omega$, and $g, u, p, f \in C_{\mathrm{rd}}(\Omega, \mathbb{R}_+)$. Also let $\hat{\alpha} \in C^1_{\mathrm{rd}}(\mathbb{T}_1, \mathbb{T}_1)$ and $\hat{\beta} \in C^1_{\mathrm{rd}}(\mathbb{T}_2, \mathbb{T}_2)$ be nondecreasing functions with $\hat{\alpha}(\hat{\varsigma}) \le \hat{\varsigma}$ on $\mathbb{T}_1$, $\hat{\beta}(\hat{\varrho}) \le \hat{\varrho}$ on $\mathbb{T}_2$. Furthermore, suppose $\tilde{\Phi}$, $\tilde{\Psi} \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions with $\{\tilde{\Phi}, \tilde{\Psi}\}(u) > 0$ for $u > 0$, and $\lim_{u \to +\infty} \tilde{\Phi}(u) = +\infty$. If $u(\hat{\varsigma}, \hat{\varrho})$ satisfies*

$$\tilde{\Phi}\big(u(\hat{\varsigma}, \hat{\varrho})\big) \le a(\hat{\varsigma}, \hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})} \int_0^{\hat{\beta}(\hat{\varrho})} \big[f(\hat{\xi}_1, \hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1, \hat{\xi}_2)\big) + p(\hat{\xi}_1, \hat{\xi}_2)\big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1 + \int_0^{\hat{\alpha}(\hat{\varsigma})} \int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1, \hat{\xi}_2)\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta}, \hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta}, \hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \tag{2.4}$$

for $(\hat{\varsigma}, \hat{\varrho}) \in \Omega$, then

$$u(\hat{\varsigma}, \hat{\varrho}) \le \tilde{\Phi}^{-1}\left\{\tilde{\Lambda}^{-1}\left[\tilde{\Lambda}\big(q(\hat{\varsigma}, \hat{\varrho})\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})} \int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1, \hat{\xi}_2)\left(1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta}, \hat{\xi}_2)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right]\right\} \tag{2.5}$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \hat{\varrho}_1$, where

$$q(\hat{\varsigma}, \hat{\varrho}) = a(\hat{\varsigma}, \hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})} \int_0^{\hat{\beta}(\hat{\varrho})} p(\hat{\xi}_1, \hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \tag{2.6}$$

$$\tilde{\Lambda}(r) = \int_{r_0}^{r} \frac{\Delta\hat{\xi}_1}{\omega \circ \tilde{\Phi}^{-1}(\hat{\xi}_1)}, \quad r \ge r_0 > 0, \qquad \tilde{\Lambda}(+\infty) = \int_{r_0}^{+\infty} \frac{\Delta\hat{\xi}_1}{\omega \circ \tilde{\Phi}^{-1}(\hat{\xi}_1)} = +\infty, \tag{2.7}$$

and $(\hat{\varsigma}_1, \hat{\varrho}_1) \in \Omega$ is chosen so that

$$\left(\tilde{\Lambda}\big(q(\hat{\varsigma}, \hat{\varrho})\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})} \int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1, \hat{\xi}_2)\left(1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta}, \hat{\xi}_2)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right) \in \mathrm{Dom}\big(G^{-1}\big).$$

Proof Assume that $a(\hat{\varsigma}, \hat{\varrho}) > 0$. Since $q \ge 0$ and it is nondecreasing, fixing an arbitrary point $(\breve{\xi}, \breve{\zeta}) \in \Omega$ and defining $z(\hat{\varsigma}, \hat{\varrho})$ by

$$z(\hat{\varsigma}, \hat{\varrho}) = q(\breve{\xi}, \breve{\zeta}) + \int_0^{\hat{\alpha}(\hat{\varsigma})} \int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1, \hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1, \hat{\xi}_2)\big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1 + \int_0^{\hat{\alpha}(\hat{\varsigma})} \int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1, \hat{\xi}_2)\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta}, \hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta}, \hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1,$$

which is a positive and nondecreasing function for $0 \le \hat{\varsigma} \le \breve{\xi} \le \hat{\varsigma}_1$, $0 \le \hat{\varsigma} \le \breve{\zeta} \le \hat{\varrho}_1$, we then get $z(0, \hat{\varrho}) = z(\hat{\varsigma}, 0) = q(\breve{\xi}, \breve{\zeta})$ and

$$u(\hat{\varsigma}, \hat{\varrho}) \le \tilde{\Phi}^{-1}\big(z(\hat{\varsigma}, \hat{\varrho})\big). \tag{2.8}$$

By applying Theorem 2.1, differentiating $z(\hat{\varsigma},\hat{\varrho})$ with respect to $\hat{\varsigma}$, and using (2.8), we get

$$\begin{aligned}
&z^{\Delta}_{\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho})\\
&\quad= \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\Big[\tilde{\Psi}\big(u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\Big]\Delta\hat{\xi}_2\\
&\quad\le \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\Big[\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\\
&\qquad+ \int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\Big]\Delta\hat{\xi}_2.
\end{aligned}$$

Since $\tilde{\Psi}\circ\tilde{\Phi}^{-1}$ is nondecreasing with respect to $(\hat{\varsigma},\hat{\varrho})\in\mathbb{R}_+\times\mathbb{R}_+$, we then have

$$\begin{aligned}
&z^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho})\\
&\quad\le \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\Big[\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\\
&\qquad+ \tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\Big]\Delta\hat{\xi}_2\\
&\quad\le \tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big)\big)\hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\Big[1+\int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\Big]\Delta\hat{\xi}_2, \quad (2.9)
\end{aligned}$$

from which $\tilde{\Psi}\circ\tilde{\Phi}^{-1}(z(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})))\le\tilde{\Psi}\circ\tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho}))$, so from (2.9), we get

$$\frac{z^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho})}{\tilde{\Psi}\circ\tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho}))}\le\hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\Big(1+\int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\Big)\Delta\hat{\xi}_2. \tag{2.10}$$

Now from (2.10), we get

$$\tilde{\Lambda}\big(z(\hat{\varsigma},\hat{\varrho})\big)\le\tilde{\Lambda}\big(q(\breve{\xi},\breve{\zeta})\big)+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Big(1+\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\Big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1.$$

Since $(\breve{\xi},\breve{\zeta})\in\Omega$ is chosen arbitrarily,

$$z(\hat{\varsigma},\hat{\varrho})\le\tilde{\Lambda}^{-1}\Big[\tilde{\Lambda}\big(q(\hat{\varsigma},\hat{\varrho})\big)+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Big(1+\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\Big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\Big]. \tag{2.11}$$

So from (2.11) and (2.8), we get the desired inequality in (2.5). For $a(\hat{\varsigma},\hat{\varrho})=0$, we carry out the above procedure with $\epsilon>0$ instead of $a(\hat{\varsigma},\hat{\varrho})$ and subsequently let $\epsilon\to 0$. This completes the proof. □

Remark 2.4 If we take $\hat{\alpha}(\hat{\varsigma})=\hat{\varsigma}$ and $\hat{\alpha}(\hat{\varrho})=\hat{\varrho}$, then Theorem 2.3 reduces to [1, Theorem 2.1].

Corollary 2.5 *The discrete form can be obtained by letting* $\mathbb{T} = \mathbb{Z}$, *with the help of relations* (1.2), *and* $\hat{\alpha}(\hat{\varsigma}) = \hat{\varsigma}$, $\hat{\beta}(\hat{\varrho}) = \hat{\varrho}$ *in Theorem* 2.3. *If*

$$\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big) \le a(\hat{\varsigma},\hat{\varrho}) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1}\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) + p(\hat{\xi}_1,\hat{\xi}_2)\big]$$
$$+ \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\left(\sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\right)$$

holds for $(\hat{\varsigma},\hat{\varrho}) \in \Omega$, *then*

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\left\{\tilde{\Lambda}^{-1}\left[\tilde{\Lambda}\big(q(\hat{\varsigma},\hat{\varrho})\big) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\left(1 + \sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\right)\right]\right\}$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \hat{\varrho}_1$, *where*

$$q(\hat{\varsigma},\hat{\varrho}) = a(\hat{\varsigma},\hat{\varrho}) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} p(\hat{\xi}_1,\hat{\xi}_2),$$

$$\tilde{\Lambda}(r) = \sum_{\hat{\xi}_1=r_0}^{r-1} \frac{1}{\omega \circ \tilde{\Phi}^{-1}(\hat{\xi}_1)}, \quad r \ge r_0 > 0, \qquad \tilde{\Lambda}(+\infty) = \sum_{\hat{\xi}_1=r_0}^{+\infty} \frac{1}{\omega \circ \tilde{\Phi}^{-1}(\hat{\xi}_1)} = +\infty,$$

and $(\hat{\varsigma}_1,\hat{\varrho}_1) \in \Omega$ *is chosen so that*

$$\left(\tilde{\Lambda}\big(q(\hat{\varsigma},\hat{\varrho})\big) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\breve{st}=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\left(1 + \sum_{\hat{\zeta}0}^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\right)\right) \in \operatorname{Dom}\big(G^{-1}\big).$$

Theorem 2.6 *Assume that* $h, b \in C_{\mathrm{rd}}(\Omega,\mathbb{R}_+)$. *Let* $g, f, p, a, u, \tilde{\Phi}$, *and* $\tilde{\Psi}$ *be as in Theorem* 2.3. *If* $u(\hat{\varsigma},\hat{\varrho})$ *satisfies*

$$\begin{aligned}\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big) \le{}& a(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) + p(\hat{\xi}_1,\hat{\xi}_2)\big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \\ &+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} b(\hat{\xi}_1,\hat{\xi}_2)\bigg[h(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) \\ &+ \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\bigg]\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \end{aligned} \tag{2.12}$$

for $(\hat{\varsigma},\hat{\varrho}) \in \Omega$, *then*

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\left\{G^{-1}\left[G\big(q(\hat{\varsigma},\hat{\varrho})\big) + A(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right]\right\} \tag{2.13}$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \hat{\varrho}_1$, *where* $\tilde{\Lambda}$ *is defined by* (2.7),

$$\breve{A}(\hat{\varsigma},\hat{\varrho}) = \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} b(\hat{\xi}_1,\hat{\xi}_2)\left[h(\hat{\xi}_1,\hat{\xi}_2) + \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\right]\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \tag{2.14}$$

and $(\hat{\varsigma}_1, \hat{\varrho}_1) \in \Omega$ *is chosen so that*

$$\left(\tilde{\Lambda}\big(q(\hat{\varsigma},\hat{\varrho})\big) + \breve{A}(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right) \in \mathrm{Dom}\big(\tilde{\Lambda}^{-1}\big).$$

Proof Assume that $a(\hat{\varsigma},\hat{\varrho}) > 0$. Fixing an arbitrary $(\breve{\xi},\breve{\zeta}) \in \Omega$, we define a positive and nondecreasing function $z(\hat{\varsigma},\hat{\varrho})$ by

$$\begin{aligned} z(\hat{\varsigma},\hat{\varrho}) = q(\breve{\xi},\breve{\zeta}) &+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \\ &+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} b(\hat{\xi}_1,\hat{\xi}_2)\Big[h(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) \\ &+ \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\Big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \end{aligned}$$

for $0 \le \hat{\varsigma} \le \breve{\xi} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \breve{\zeta} \le y_1$, then $z(0,\hat{\varrho}) = z(\hat{\varsigma},0) = q(\breve{\xi},\breve{\zeta})$ and

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\big(z(\hat{\varsigma},\hat{\varrho})\big).$$

Now, by applying Theorem 2.1, we have

$$\begin{aligned} &z^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho}) \\ &\quad= \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\big(u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\Delta\hat{\xi}_2 + \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} b\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big) \\ &\qquad\times\left(h\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\big(u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big) + \int_0^{\hat{\varsigma}} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2 \\ &\quad\le \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\Delta\hat{\xi}_2 + \int_0^{\hat{\beta}(\hat{\varrho})} b\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big) \\ &\qquad\times\left(h\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big) + \int_0^{\hat{\varsigma}} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2 \\ &\quad\le \hat{\alpha}^{\Delta}(\hat{\varsigma})\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\varsigma},\hat{\varrho})\big)\left[\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\Delta\hat{\xi}_2\right. \\ &\qquad\left.+ \int_0^{\hat{\beta}(\hat{\varrho})} b\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\left(h\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big) + \int_0^{\hat{\varsigma}} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\right)\right]\Delta\hat{\xi}_2. \end{aligned}$$

Since $\tilde{\Psi}\circ\tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho})) \le \tilde{\Psi}\circ\tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho}))$, we then get

$$\begin{aligned} &\frac{z^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho})}{\tilde{\Psi}\circ\tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho}))} \\ &\quad\le \hat{\alpha}^{\Delta}(\hat{\varsigma})\left[\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\Delta\hat{\xi}_2\right. \\ &\qquad\left.+ \int_0^{\hat{\beta}(\hat{\varrho})} b\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\left(h\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\right)\right]\Delta\hat{\xi}_2. \end{aligned} \tag{2.15}$$

Integrating (2.15), we get

$$\tilde{\Lambda}\big(z(\hat{\varsigma},\hat{\varrho})\big) \le \tilde{\Lambda}\big(q(\breve{\xi},\breve{\zeta})\big) + \breve{A}(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1.$$

Since $(\breve{\xi},\breve{\zeta}) \in \Omega$ is chosen arbitrarily,

$$z(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Lambda}^{-1}\left[\tilde{\Lambda}\big(q(\hat{\varsigma},\hat{\varrho})\big) + \breve{A}(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right]. \tag{2.16}$$

Thus, from (2.16) and $u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho}))$, we get the required inequality in (2.13). For $a(\hat{\varsigma},\hat{\varrho}) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\hat{\varsigma},\hat{\varrho})$ and subsequently let $\epsilon \to 0$. This completes the proof. □

Remark 2.7 If we take $\hat{\alpha}(\hat{\varsigma}) = \hat{\varsigma}$ and $\hat{\alpha}(\hat{\varrho}) = \hat{\varrho}$, then Theorem 2.6 reduces to [1, Theorem 2.4].

Corollary 2.8 *If we take* $\mathbb{T} = \mathbb{R}$ *in Theorem* 2.6, *then, with the help of relations* (1.1), *we have the following inequality due to Boudeliou* [41]. *If*

$$\begin{aligned}
&\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big) \\
&\quad\le a(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} \big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) + p(\hat{\xi}_1,\hat{\xi}_2)\big]\,d\hat{\xi}_2\,d\hat{\xi}_1 \\
&\qquad + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} b(\hat{\xi}_1,\hat{\xi}_2)\left[h(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) + \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\,d\hat{\zeta}\right]d\hat{\xi}_2\,d\hat{\xi}_1
\end{aligned}$$

holds for $(\hat{\varsigma},\hat{\varrho}) \in \Omega$, *then*

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\left\{G^{-1}\left[G\big(q(\hat{\varsigma},\hat{\varrho})\big) + A(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\,d\hat{\xi}_2\,d\hat{\xi}_1\right]\right\}$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \hat{\varrho}_1$, *where* $\tilde{\Lambda}$ *is defined by* (2.7),

$$\breve{A}(\hat{\varsigma},\hat{\varrho}) = \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} b(\hat{\xi}_1,\hat{\xi}_2)\left[h(\hat{\xi}_1,\hat{\xi}_2) + \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\,d\hat{\zeta}\right]d\hat{\xi}_2\,d\hat{\xi}_1,$$

and $(\hat{\varsigma}_1,\hat{\varrho}_1) \in \Omega$ *is chosen so that*

$$\left(\tilde{\Lambda}\big(q(\hat{\varsigma},\hat{\varrho})\big) + \breve{A}(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\,d\hat{\xi}_2\,d\hat{\xi}_1\right) \in \mathrm{Dom}\big(\tilde{\Lambda}^{-1}\big).$$

Corollary 2.9 *The discrete form can be obtained by letting $\mathbb{T} = \mathbb{Z}$, with the help of relations (1.2) and $\hat{\alpha}(\hat{\varsigma}) = \hat{\varsigma}$, $\hat{\beta}(\hat{\varrho}) = \hat{\varrho}$ in Theorem 2.6. If*

$$\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big) \le a(\hat{\varsigma},\hat{\varrho}) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1}\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) + p(\hat{\xi}_1,\hat{\xi}_2)\big] + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} b(\hat{\xi}_1,\hat{\xi}_2)\Bigg[h(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) + \sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Bigg]$$

holds for $(\hat{\varsigma},\hat{\varrho}) \in \Omega$, then

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\Bigg\{G^{-1}\Bigg[G\big(q(\hat{\varsigma},\hat{\varrho})\big) + A(\hat{\varsigma},\hat{\varrho}) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\Bigg]\Bigg\}$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \hat{\varrho}_1$, where $\tilde{\Lambda}$ is defined by (2.7),

$$\breve{A}(\hat{\varsigma},\hat{\varrho}) = \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} b(\hat{\xi}_1,\hat{\xi}_2)\Bigg[h(\hat{\xi}_1,\hat{\xi}_2) + \sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\Bigg],$$

and $(\hat{\varsigma}_1,\hat{\varrho}_1) \in \Omega$ is chosen so that

$$\Bigg(\tilde{\Lambda}\big(q(\hat{\varsigma},\hat{\varrho})\big) + \breve{A}(\hat{\varsigma},\hat{\varrho}) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\Bigg) \in \operatorname{Dom}\big(\tilde{\Lambda}^{-1}\big).$$

Theorem 2.10 *Assume that g, a, u, f, p, $\tilde{\Phi}$, and $\tilde{\Psi}$ are as in Theorem 2.3. If $u(\hat{\varsigma},\hat{\varrho})$ satisfies*

$$\begin{aligned} &\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big) \\ &\quad \le a(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} \tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) + p(\hat{\xi}_1,\hat{\xi}_2)\big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \\ &\qquad + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\bigg(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \end{aligned} \tag{2.17}$$

for $(\hat{\varsigma},\hat{\varrho}) \in \Omega$, then

$$\begin{aligned} &u(\hat{\varsigma},\hat{\varrho}) \\ &\quad \le \tilde{\Phi}^{-1}\Bigg\{\tilde{\Lambda}^{-1}\Bigg(\tilde{\Theta}^{-1}\Bigg[\tilde{\Theta}\big(q_1(\hat{\varsigma},\hat{\varrho})\big) \\ &\qquad + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\bigg(1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\Bigg]\Bigg)\Bigg\}, \end{aligned} \tag{2.18}$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \hat{\varrho}_1$, where

$$q_1(\hat{\varsigma},\hat{\varrho}) = \tilde{\Lambda}\big(a(\hat{\varsigma},\hat{\varrho})\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} p(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \tag{2.19}$$

$$\begin{aligned}
&\tilde{\Theta}(r) = \int_{r_0}^{r} \frac{\Delta\hat{\xi}_1}{((\tilde{\Psi}\circ\tilde{\Phi}^{-1})\circ\tilde{\Lambda}^{-1})(\hat{\xi}_1)}, \quad r \geq r_0 > 0, \\
&\tilde{\Theta}(+\infty) = \int_{r_0}^{+\infty} \frac{\Delta\hat{\xi}_1}{(\omega\circ\tilde{\Phi}^{-1})\circ\tilde{\Lambda}^{-1}(\hat{\xi}_1)} = +\infty,
\end{aligned} \tag{2.20}$$

and $(\hat{\varsigma}_1, \hat{\varrho}_1) \in \Omega$ *is chosen so that*

$$\left(\tilde{\Theta}\big(q_1(\hat{\varsigma},\hat{\varrho})\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\left(1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right) \in \operatorname{Dom}\big(\tilde{\Theta}^{-1}\big).$$

Proof Suppose that $a(\breve{\xi},\breve{\zeta}) > 0$. Fixing an arbitrary $(\breve{\xi},\breve{\zeta}) \in \Omega$, we define a positive and nondecreasing function $z(\hat{\varsigma},\hat{\varrho})$ by

$$\begin{aligned}
z(\hat{\varsigma},\hat{\varrho}) = a(\breve{\xi},\breve{\zeta}) &+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} \tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) + p(\hat{\xi}_1,\hat{\xi}_2)\big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \\
&+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1,
\end{aligned}$$

for $0 \leq \hat{\varsigma} \leq \breve{\xi} \leq \hat{\varsigma}_1$, $0 \leq \hat{\varrho} \leq \breve{\zeta} \leq \hat{\varrho}_1$, then $z(0,\hat{\varrho}) = z(\hat{\varsigma},0) = a(\breve{\xi},\breve{\zeta})$ and

$$u(\hat{\varsigma},\hat{\varrho}) \leq \tilde{\Phi}^{-1}\big(z(\hat{\varsigma},\hat{\varrho})\big).$$

Now, by applying Theorem 2.1, we have

$$\begin{aligned}
&z^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho}) \\
&= \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} \tilde{\Psi}\big(u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\big[f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\big(u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big) + p(\hat{\varsigma},\hat{\xi}_2)\big]\Delta\hat{\xi}_2 \\
&\quad + \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\big(u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\left(\int_0^{\hat{\varsigma}} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2 \\
&\leq \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} \tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\big[f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big) + p(\hat{\varsigma},\hat{\xi}_2)\big]\Delta\hat{\xi}_2 \\
&\quad + \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big) \\
&\quad \times \left(\int_0^{\hat{\varsigma}} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2 \\
&\leq \tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big)\big)\hat{\alpha}^{\Delta}(\hat{\varsigma}) \\
&\quad \times \int_0^{\hat{\beta}(\hat{\varrho})} \big[f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big) + p(\hat{\varsigma},\hat{\xi}_2)\big]\Delta\hat{\xi}_2 \\
&\quad + \tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big)\big)\hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big) \\
&\quad \times \left(\int_0^{\hat{\varsigma}} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2,
\end{aligned}$$

or

$$\begin{aligned}
&\frac{z^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho})}{\tilde{\Psi}\circ\tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho}))}\\
&\quad\le \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})}\big[f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)+p(\hat{\varsigma},\hat{\xi}_2)\big]\Delta\hat{\xi}_2\\
&\qquad+\hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})}f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\biggl(\int_0^{\hat{\varsigma}}g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\biggr)\Delta\hat{\xi}_2.
\end{aligned}\tag{2.21}$$

Integrating (2.21), we get

$$\begin{aligned}
\tilde{\Lambda}\big(z(\hat{\varsigma},\hat{\varrho})\big)&\le\tilde{\Lambda}\big(a(\breve{\xi},\breve{\zeta})\big)+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\xi}_1,\hat{\xi}_2)\big)+p(\hat{\xi}_1,\hat{\xi}_2)\big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1\\
&\quad+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}f(\hat{\xi}_1,\hat{\xi}_2)\biggl(\int_0^{\hat{\xi}_1}g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\biggr)\Delta\hat{\xi}_2\Delta\hat{\xi}_1.
\end{aligned}$$

If $(\breve{\xi},\breve{\zeta})\in\Omega$ is chosen arbitrarily, then

$$\begin{aligned}
\tilde{\Lambda}\big(z(\hat{\varsigma},\hat{\varrho})\big)&\le q_1(\hat{\varsigma},\hat{\varrho})+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\xi}_1,\hat{\xi}_2)\big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\\
&\quad+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}f(\hat{\xi}_1,\hat{\xi}_2)\biggl(\int_0^{\hat{\xi}_1}g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\biggr)\Delta\hat{\xi}_2\Delta\hat{\xi}_1.
\end{aligned}$$

Since $q_1(\hat{\varsigma},\hat{\varrho})>0$ is a nondecreasing function, fixing an arbitrary point $(\breve{\xi},\breve{\zeta})\in\Omega$ and defining $\nu(\hat{\varsigma},\hat{\varrho})>0$ to be a nondecreasing function given by

$$\begin{aligned}
\nu(\hat{\varsigma},\hat{\varrho})&=q_1(\breve{\xi},\breve{\zeta})+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\xi}_1,\hat{\xi}_2)\big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\\
&\quad+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}f(\hat{\xi}_1,\hat{\xi}_2)\biggl(\int_0^{\hat{\xi}_1}g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\biggr)\Delta\hat{\xi}_2\Delta\hat{\xi}_1,
\end{aligned}$$

for $0\le\hat{\varsigma}\le\breve{\xi}\le\hat{\varsigma}_1$, $0\le\hat{\varrho}\le\breve{\zeta}\le y_1$, we obtain $\nu(0,\hat{\varrho})=\nu(\hat{\varsigma},0)=q_1(\breve{\xi},\breve{\zeta})$ and

$$z(\hat{\varsigma},\hat{\varrho})\le\tilde{\Lambda}^{-1}\big(\nu(\hat{\varsigma},\hat{\varrho})\big).\tag{2.22}$$

Now, by applying Theorem 2.1, we have

$$\begin{aligned}
\nu^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho})&=\hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})}f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\Delta\hat{\xi}_2\\
&\quad+\hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})}f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\biggl(\int_0^{\hat{\varsigma}}g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\biggr)\Delta\hat{\xi}_2\\
&\le\hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})}f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(G^{-1}\big(\nu\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\big)\Delta\hat{\xi}_2\\
&\quad+\hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})}f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\biggl(\int_0^{\hat{\varsigma}}g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(G^{-1}\big(\nu(\hat{\zeta},\hat{\xi}_2)\big)\big)\Delta\hat{\zeta}\biggr)\Delta\hat{\xi}_2
\end{aligned}$$

$$\leq \left(\tilde{\Psi} \circ \tilde{\Phi}^{-1}\right) \circ \tilde{\Lambda}^{-1}\left(v\left(\hat{\alpha}(\hat{\varsigma}), \hat{\beta}(\hat{\varrho})\right)\right)\hat{\alpha}^{\Delta}(\hat{\varsigma}) \times \left[\int_0^{\hat{\beta}(\hat{\varrho})} f\left(\hat{\alpha}(\hat{\varsigma}), \hat{\xi}_2\right)\Delta\hat{\xi}_2 + \int_0^{\hat{\beta}(\hat{\varrho})} f\left(\hat{\alpha}(\hat{\varsigma}), \hat{\xi}_2\right)\left(\int_0^{\hat{\varsigma}} g(\hat{\zeta}, \hat{\xi}_2)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\right],$$

or

$$\frac{v^{\Delta\hat{\varsigma}}(\hat{\varsigma}, \hat{\varrho})}{(\tilde{\Psi} \circ \tilde{\Phi}^{-1}) \circ \tilde{\Lambda}^{-1}(v(\hat{\varsigma}, \hat{\varrho}))} \leq \hat{\alpha}^{\Delta}(\hat{\varsigma})\left[\int_0^{\hat{\beta}(\hat{\varrho})} f\left(\hat{\alpha}(\hat{\varsigma}), \hat{\xi}_2\right)\Delta\hat{\xi}_2 + \int_0^{\hat{\beta}(\hat{\varrho})} f\left(\hat{\alpha}(\hat{\varsigma}), \hat{\xi}_2\right)\left(\int_0^{\hat{\varsigma}} g(\hat{\zeta}, \hat{\xi}_2)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\right]. \quad (2.23)$$

Integrating (2.23), we get

$$\tilde{\Theta}\left(v(\hat{\varsigma}, \hat{\varrho})\right) \leq \tilde{\Theta}\left(q_1(\breve{\xi}, \breve{\zeta})\right) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1, \hat{\xi}_2)\left[1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta}, \hat{\xi}_2)\Delta\hat{\zeta}\right]\Delta\hat{\xi}_2\Delta\hat{\xi}_1.$$

Since we chose $(\breve{\xi}, \breve{\zeta}) \in \Omega$ arbitrarily,

$$v(\hat{\varsigma}, \hat{\varrho}) \leq \tilde{\Theta}^{-1}\left[\tilde{\Theta}\left(q_1(\hat{\varsigma}, \hat{\varrho})\right) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1, \hat{\xi}_2)\left[1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta}, \hat{\xi}_2)\Delta\hat{\zeta}\right]\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right]. \quad (2.24)$$

From (2.24), (2.22), and $u(\hat{\varsigma}, \hat{\varrho}) \leq \tilde{\Phi}^{-1}(z(\hat{\varsigma}, \hat{\varrho}))$, we get the desired inequality in (2.18). For $a(\hat{\varsigma}, \hat{\varrho}) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\hat{\varsigma}, \hat{\varrho})$ and subsequently let $\epsilon \to 0$. This completes the proof. □

Remark 2.11 If we take $\hat{\alpha}(\hat{\varsigma}) = \hat{\varsigma}$ and $\hat{\alpha}(\hat{\varrho}) = \hat{\varrho}$, then Theorem 2.10 reduces to [1, Theorem 2.7].

Corollary 2.12 *If we take* $\mathbb{T} = \mathbb{R}$ *in Theorem* 2.10, *then, with the help of relations* (1.1), *we get the following inequality due to Boudeliou* [41]. *If*

$$\tilde{\Phi}\left(u(\hat{\varsigma}, \hat{\varrho})\right) \leq a(\hat{\varsigma}, \hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} \tilde{\Psi}\left(u(\hat{\xi}_1, \hat{\xi}_2)\right)\left[f(\hat{\xi}_1, \hat{\xi}_2)\tilde{\Psi}\left(u(\hat{\xi}_1, \hat{\xi}_2)\right) + p(\hat{\xi}_1, \hat{\xi}_2)\right] d\hat{\xi}_2\, d\hat{\xi}_1 + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1, \hat{\xi}_2)\tilde{\Psi}\left(u(\hat{\xi}_1, \hat{\xi}_2)\right)\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta}, \hat{\xi}_2)\tilde{\Psi}\left(u(\hat{\zeta}, \hat{\xi}_2)\right)\Delta\hat{\zeta}\right) d\hat{\xi}_2\, d\hat{\xi}_1$$

holds for $(\hat{\varsigma}, \hat{\varrho}) \in \Omega$, *then*

$$u(\hat{\varsigma}, \hat{\varrho}) \leq \tilde{\Phi}^{-1}\left\{\tilde{\Lambda}^{-1}\left(\tilde{\Theta}^{-1}\left[\tilde{\Theta}\left(q_2(\hat{\varsigma}, \hat{\varrho})\right) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1, \hat{\xi}_2)\left(1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta}, \hat{\xi}_2)\, d\hat{\zeta}\right) d\hat{\xi}_2\, d\hat{\xi}_1\right]\right)\right\},$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \hat{\varrho}_1$, *where*

$$q_2(\hat{\varsigma},\hat{\varrho}) = \tilde{\Lambda}\big(a(\hat{\varsigma},\hat{\varrho})\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} p(\hat{\xi}_1,\hat{\xi}_2)\,d\hat{\xi}_2\,d\hat{\xi}_1,$$

$$\tilde{\Theta}(r) = \int_{r_0}^{r} \frac{d\hat{\xi}_1}{((\tilde{\Psi}\circ\tilde{\Phi}^{-1})\circ\tilde{\Lambda}^{-1})(\hat{\xi}_1)}, \quad r \ge r_0 > 0,$$

$$\tilde{\Theta}(+\infty) = \int_{r_0}^{+\infty} \frac{d\hat{\xi}_1}{(\omega\circ\tilde{\Phi}^{-1})\circ\tilde{\Lambda}^{-1}(\hat{\xi}_1)} = +\infty,$$

and $(\hat{\varsigma}_1,\hat{\varrho}_1) \in \Omega$ *is chosen so that*

$$\left(\tilde{\Theta}\big(q_2(\hat{\varsigma},\hat{\varrho})\big) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\left(1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\,d\hat{\zeta}\right)d\hat{\xi}_2\,d\hat{\xi}_1\right) \in \operatorname{Dom}\big(\tilde{\Theta}^{-1}\big).$$

Corollary 2.13 *The discrete form, due to El-Deeb et al. [1], can be obtained by letting* $\mathbb{T} = \mathbb{Z}$ *in Theorem 2.10, with the help of relations (1.2) and* $\hat{\alpha}(\hat{\varsigma}) = \hat{\varsigma}$, $\hat{\beta}(\hat{\varrho}) = \hat{\varrho}$ *as follows. If*

$$\begin{aligned}\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big) &\le a(\hat{\varsigma},\hat{\varrho}) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} \tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big) + p(\hat{\xi}_1,\hat{\xi}_2)\big] \\ &\quad + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\left(\sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\right),\end{aligned}$$

holds for $(\hat{\varsigma},\hat{\varrho}) \in \Omega$, *then*

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\left\{\bar{G}^{-1}\left(\bar{F}^{-1}\left[\bar{F}\big(\bar{q}_2(\hat{\varsigma},\hat{\varrho})\big) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\left(1 + \sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\right)\right]\right)\right\},$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \hat{\varrho}_1$, *where*

$$\bar{q}_2(\hat{\varsigma},\hat{\varrho}) = \tilde{\Lambda}\big(a(\hat{\varsigma},\hat{\varrho})\big) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} p(\hat{\xi}_1,\hat{\xi}_2),$$

$$\bar{F}(r) = \sum_{\hat{\xi}_1=r_0}^{r-1} \frac{1}{((\tilde{\Psi}\circ\tilde{\Phi}^{-1})\circ\bar{G}^{-1})(\hat{\xi}_1)}, \quad r \ge r_0 > 0,$$

$$\bar{F}(+\infty) = \sum_{\hat{\xi}_1=r_0}^{+\infty} \frac{1}{(\omega\circ\tilde{\Phi}^{-1})\circ\bar{G}^{-1}(\hat{\xi}_1)} = +\infty,$$

and $(\hat{\varsigma}_1,\hat{\varrho}_1) \in \Omega$ *is chosen so that*

$$\left(\bar{F}\big(\bar{q}_2(\hat{\varsigma},\hat{\varrho})\big) + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\left(1 + \sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\right)\right) \in \operatorname{Dom}\big(\bar{F}^{-1}\big).$$

Theorem 2.14 *Assume that $g, a, f, u, \tilde{\Phi}$, and $\tilde{\Psi}$ are as in Theorem 2.3. If $u(\hat{\varsigma},\hat{\varrho})$ satisfies*

$$\begin{aligned}
&\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big)\\
&\quad\le a(\hat{\varsigma},\hat{\varrho}) + \left(\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right)^2\\
&\qquad+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \qquad (2.25)
\end{aligned}$$

for $(\hat{\varsigma},\hat{\varrho}) \in \Omega$, then

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\left\{\breve{H}^{-1}\left[\breve{H}\big(a(\hat{\varsigma},\hat{\varrho})\big) + \breve{B}(\hat{\varsigma},\hat{\varrho}) + \left(\int_0^{\hat{\beta}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right)^2\right]\right\}, \qquad (2.26)$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \hat{\varrho}_1$, where

$$\breve{B}(\hat{\varsigma},\hat{\varrho}) = \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \qquad (2.27)$$

$$\begin{aligned}
&\breve{H}(r) = \int_{r_0}^{r}\frac{\Delta\hat{\xi}_1}{(\tilde{\Psi}\circ\tilde{\Phi}^{-1})^2(\hat{\xi}_1)}, \quad r \ge r_0 > 0,\\
&\tilde{\Theta}(+\infty) = \int_{r_0}^{+\infty}\frac{\Delta\hat{\xi}_1}{(\tilde{\Psi}\circ\tilde{\Phi}^{-1})^2(\hat{\xi}_1)} = +\infty,
\end{aligned} \qquad (2.28)$$

and $(\hat{\varsigma}_1,\hat{\varrho}_1) \in \Omega$ is chosen so that

$$\left(\breve{H}\big(a(\hat{\varsigma},\hat{\varrho})\big) + B(\hat{\varsigma},\hat{\varrho}) + 2\left(\int_0^{\sigma(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right)^2\right) \in \operatorname{Dom}\big(\breve{H}^{-1}\big).$$

Proof Assume that $a(\hat{\varsigma},\hat{\varrho}) > 0$. Taking $(\breve{\xi},\breve{\zeta}) \in \Omega$ as a fixed arbitrary point, we define $z(\hat{\varsigma},\hat{\varrho}) > 0$ to be a nondecreasing function by

$$\begin{aligned}
&z(\hat{\varsigma},\hat{\varrho})\\
&\quad= a(\breve{\xi},\breve{\zeta}) + \left(\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right)^2 \qquad (2.29)\\
&\qquad+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \qquad (2.30)
\end{aligned}$$

for $0 \le \hat{\varsigma} \le \breve{\xi} \le \hat{\varsigma}_1$, $0 \le \hat{\varrho} \le \breve{\zeta} \le \hat{\varrho}_1$, hence $z(0,\hat{\varrho}) = z(\hat{\varsigma},0) = a(\breve{\xi},\breve{\zeta})$ and

$$u(\hat{\varsigma},\hat{\varrho}) \le \tilde{\Phi}^{-1}\big(z(\hat{\varsigma},\hat{\varrho})\big).$$

From (2.29), and applying the chain rule on time scales (1.2), we get

$$\begin{aligned}
&z^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho})\\
&\quad= 2\left(\int_0^{\hat{\alpha}(c)}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \hat{\alpha}^{\Delta}(\hat{\varsigma}) \int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\big(u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\Delta\hat{\xi}_2 \\
&+ \hat{\alpha}^{\Delta}(\hat{\varsigma}) \int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\big(u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\bigg(\int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2 \\
\le{}& 2\bigg(\int_0^{\hat{\alpha}(c)}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\xi}_1,\hat{\xi}_2)\big)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\bigg) \\
&\times \hat{\alpha}^{\Delta}(\hat{\varsigma}) \int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big)\Delta\hat{\xi}_2 \\
&+ \hat{\alpha}^{\Delta}(\hat{\varsigma}) \int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\big) \\
&\times \bigg(\int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z(\hat{\zeta},\hat{\xi}_2)\big)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2 \\
\le{}& 2\big(\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big)\big)\big)^2\hat{\alpha}^{\Delta}(\hat{\varsigma})\bigg(\int_0^{\hat{\alpha}(c)}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\bigg) \\
&\times \int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\Delta\hat{\xi}_2 \\
&+ \big(\tilde{\Psi}\circ\tilde{\Phi}^{-1}\big(z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big)\big)\big)^2\hat{\alpha}^{\Delta}(\hat{\varsigma}) \int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\bigg(\int_0^{\hat{\varsigma}} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2,
\end{aligned}
$$

thus we have

$$
\begin{aligned}
\frac{z^{\Delta\hat{\varsigma}}(\hat{\varsigma},\hat{\varrho})}{(\tilde{\Psi}\circ\tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho})))^2} \le{}& 2\bigg(\int_0^{\hat{\alpha}(c)}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\bigg)\hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\Delta\hat{\xi}_2 \\
&+ \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\bigg(\int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2, \\
={}& \bigg[\bigg(\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\bigg)^2\bigg]^{\Delta_{\hat{\varsigma}}} \\
&+ \hat{\alpha}^{\Delta}(\hat{\varsigma})\int_0^{\hat{\beta}(\hat{\varrho})} f\big(\hat{\alpha}(\hat{\varsigma}),\hat{\xi}_2\big)\bigg(\int_0^{\hat{\alpha}(\hat{\varsigma})} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2. \qquad (2.31)
\end{aligned}
$$

Integrating (2.31), we get

$$
\begin{aligned}
\breve{H}\big(z(\hat{\varsigma},\hat{\varrho})\big) \le{}& \breve{H}\big(a(\breve{\xi},\breve{\zeta})\big) + \bigg(\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\bigg)^2 \\
&+ \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\bigg(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\bigg)\Delta\hat{\xi}_2\Delta\hat{\xi}_1.
\end{aligned}
$$

Since $(\breve{\xi},\breve{\zeta}) \in \Omega$ is chosen arbitrarily,

$$
z(\hat{\varsigma},\hat{\varrho}) \le \breve{H}^{-1}\bigg[\breve{H}\big(a(\hat{\varsigma},\hat{\varrho})\big) + \breve{B}(\hat{\varsigma},\hat{\varrho}) + \bigg(\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\bigg)^2\bigg]. \qquad (2.32)
$$

From (2.32) and $u(\hat{\varsigma},\hat{\varrho}) \leq \tilde{\Phi}^{-1}(z(\hat{\varsigma},\hat{\varrho}))$, we get the desired inequality (2.26). For $a(\hat{\varsigma},\hat{\varrho}) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\hat{\varsigma},\hat{\varrho})$ and subsequently let $\epsilon \to 0$. This completes the proof. □

Remark 2.15 If we take $\hat{\alpha}(\hat{\varsigma}) = \hat{\varsigma}$ and $\hat{\alpha}(\hat{\varrho}) = \hat{\varrho}$, then Theorem 2.14 reduces to [1, Theorem 10].

Theorem 2.16 *If we take $\mathbb{T} = \mathbb{R}$ in Theorem 2.14, with the help of relations (1.1), we have the following inequality due to Boudeliou. If*

$$\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big) \leq a(\hat{\varsigma},\hat{\varrho}) + \left(\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\,d\hat{\xi}_2\,d\hat{\xi}_1\right)^2 + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\,d\hat{\zeta}\right)d\hat{\xi}_2\,d\hat{\xi}_1,$$

for $(\hat{\varsigma},\hat{\varrho}) \in \Omega$, then

$$u(\hat{\varsigma},\hat{\varrho}) \leq \tilde{\Phi}^{-1}\left\{\breve{H}^{-1}\left[\breve{H}\big(a(\hat{\varsigma},\hat{\varrho})\big) + \breve{B}(\hat{\varsigma},\hat{\varrho}) + \left(\int_0^{\hat{\beta}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\,d\hat{\xi}_2\,d\hat{\xi}_1\right)^2\right]\right\},$$

for $0 \leq \hat{\varsigma} \leq \hat{\varsigma}_1$, $0 \leq \hat{\varrho} \leq \hat{\varrho}_1$, where

$$\breve{B}(\hat{\varsigma},\hat{\varrho}) = \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\left(\int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\,d\hat{\zeta}\right)d\hat{\xi}_2\,d\hat{\xi}_1,$$

$$\breve{H}(r) = \int_{r_0}^{r} \frac{d\hat{\xi}_1}{(\tilde{\Psi}\circ\tilde{\Phi}^{-1})^2(\hat{\xi}_1)}, \quad r \geq r_0 > 0, \qquad \tilde{\Theta}(+\infty) = \int_{r_0}^{+\infty} \frac{d\hat{\xi}_1}{(\tilde{\Psi}\circ\tilde{\Phi}^{-1})^2(\hat{\xi}_1)} = +\infty,$$

and $(\hat{\varsigma}_1,\hat{\varrho}_1) \in \Omega$ is chosen so that

$$\left(\breve{H}\big(a(\hat{\varsigma},\hat{\varrho})\big) + B(\hat{\varsigma},\hat{\varrho}) + 2\left(\int_0^{\sigma(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})} f(\hat{\xi}_1,\hat{\xi}_2)\,d\hat{\xi}_2\,d\hat{\xi}_1\right)^2\right) \in \operatorname{Dom}\big(\breve{H}^{-1}\big).$$

Corollary 2.17 *The discrete form, due to El-Deeb et al. [1], can be obtained by letting $\mathbb{T} = \mathbb{Z}$ and $\hat{\alpha}(\hat{\varsigma}) = \hat{\varsigma}$, $\hat{\beta}(\hat{\varrho}) = \hat{\varrho}$ in Theorem 2.14 as follows. If*

$$\tilde{\Phi}\big(u(\hat{\varsigma},\hat{\varrho})\big) \leq a(\hat{\varsigma},\hat{\varrho}) + \left(\sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\right)^2 + \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\xi}_1,\hat{\xi}_2)\big)\left(\sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(u(\hat{\zeta},\hat{\xi}_2)\big)\right)$$

holds for $(\hat{\varsigma},\hat{\varrho}) \in \Omega$, then

$$u(\hat{\varsigma},\hat{\varrho}) \leq \tilde{\Phi}^{-1}\left\{\breve{H}^{-1}\left[\breve{H}\big(a(\hat{\varsigma},\hat{\varrho})\big) + \breve{B}(\hat{\varsigma},\hat{\varrho}) + \left(\sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\right)^2\right]\right\},$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1, 0 \le \hat{\varrho} \le \hat{\varrho}_1$, *where*

$$\breve{B}(\hat{\varsigma},\hat{\varrho}) = \sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}} f(\hat{\xi}_1,\hat{\xi}_2)\left(\sum_{\hat{\zeta}=0}^{\hat{\xi}_1-1} g(\hat{\zeta},\hat{\xi}_2)\right),$$

$$\breve{H}(r) = \sum_{\hat{\xi}_1=r_0}^{r-1} \frac{1}{(\tilde{\Psi}\circ\tilde{\Phi}^{-1})^2(\hat{\xi}_1)}, \quad r \ge r_0 > 0, \qquad \tilde{\Theta}(+\infty) = \sum_{\hat{\xi}_1=r_0}^{+\infty} \frac{1}{(\tilde{\Psi}\circ\tilde{\Phi}^{-1})^2(\hat{\xi}_1)} = +\infty,$$

and $(\hat{\varsigma}_1,\hat{\varrho}_1) \in \Omega$ *is chosen so that*

$$\left(\breve{H}\big(a(\hat{\varsigma},\hat{\varrho})\big) + B(\hat{\varsigma},\hat{\varrho}) + \left(\sum_{\hat{\xi}_1=0}^{\hat{\varsigma}-1}\sum_{\hat{\xi}_2=0}^{\hat{\varrho}-1} f(\hat{\xi}_1,\hat{\xi}_2)\right)^2\right) \in \operatorname{Dom}\big(\breve{H}^{-1}\big).$$

3 Applications

In this section we would like to show the beauty behind our results by applying Theorems 2.10 and 2.3 to study the boundedness of the solutions of some delay initial boundary value problems.

Consider the problem

$$u^{\Delta\hat{\varsigma}\Delta\hat{\varrho}}(\hat{\varsigma},\hat{\varrho}) = \tilde{\Theta}\left(\hat{\varsigma},\hat{\varrho},u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big),\int_0^{\hat{\alpha}(\hat{\varsigma})} \breve{k}\big(\hat{\xi}_1,\hat{\varrho},u(s,\hat{\varrho})\big)\Delta\hat{\xi}_1\right), \tag{3.1}$$

$$u(\hat{\varsigma},0) = a_1(\hat{\varsigma}), \qquad u(0,\hat{\varrho}) = a_2(\hat{\varrho}), \qquad a_1(0) = a_2(0) = 0, \tag{3.2}$$

for any $(\hat{\varsigma},\hat{\varrho}) \in \Omega$, where $\breve{k} \in C_{\mathrm{rd}}(\Omega\times\mathbb{R},\mathbb{R})$, $\tilde{\Theta} \in C_{\mathrm{rd}}(\Omega\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $a_1 \in C_{\mathrm{rd}}(\mathbb{T}_1,\mathbb{R})$, and $a_2 \in C_{\mathrm{rd}}(\mathbb{T}_2,\mathbb{R})$.

Theorem 3.1 *Suppose that the functions* $\breve{k}, \tilde{\Theta}, a_2, a_1$ *in* (3.1) *and* (3.2) *satisfy the conditions*

$$\begin{aligned} &\big|\tilde{\Theta}\big(\hat{\varsigma},\hat{\varrho},u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho}),v\big)\big| \\ &\quad\le \tilde{\Psi}\big(\big|u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big|\big)\big[f(\hat{\varsigma},\hat{\varrho})\tilde{\Psi}\big(\big|u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big|\big) + p(\hat{\varsigma},\hat{\varrho})\big] \\ &\qquad + f(\hat{\varsigma},\hat{\varrho})\tilde{\Psi}\big(\big|u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big|\big)v, \end{aligned} \tag{3.3}$$

$$\big|\breve{k}\big(\hat{\varsigma},\hat{\varrho},u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big)\big| \le g(\hat{\varsigma},\hat{\varrho})\tilde{\Psi}\big(\big|u\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big|\big), \tag{3.4}$$

$$\big|a_1(\hat{\varsigma}) + a_2(\hat{\varrho})\big| \le a(\hat{\varsigma},\hat{\varrho}), \tag{3.5}$$

where the functions $p, g, a, f, \hat{\alpha}, \hat{\beta}$, *and* $\tilde{\Psi}$ *are defined as in Theorem* 2.10 *with* $a(\hat{\varsigma},\hat{\varrho}) > 0$, *for all* $(\hat{\varsigma},\hat{\varrho}) \in \Omega$. *Then*

$$\begin{aligned} \big|u(\hat{\varsigma},\hat{\varrho})\big| \le \tilde{\Lambda}^{-1}\Bigg(\tilde{\Theta}^{-1}\Bigg[&\tilde{\Theta}\big(q_2(\hat{\varsigma},\hat{\varrho})\big) + \int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}} \frac{f(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2))}{\hat{\alpha}'(\hat{\alpha}^{-1}(\hat{\xi}_1))\hat{\beta}'(\hat{\beta}^{-1}(\hat{\xi}_2))} \\ &\times\left[1 + \int_0^{\hat{\xi}_1} g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\right]\Delta\hat{\xi}_2\Delta\hat{\xi}_1\Bigg]\Bigg), \end{aligned} \tag{3.6}$$

for $0 \le \hat{\varsigma} \le \hat{\varsigma}_1, 0 \le \hat{\varrho} \le \hat{\varrho}_1$, *where* F *and* G *are defined as in Theorem* 2.10,

$$q_2\big(\hat{\varsigma},\hat{\varrho} = G\big(a(\hat{\varsigma},\hat{\varrho})\big)\big) + \int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}} \frac{p(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2))}{\hat{\alpha}'(\hat{\alpha}^{-1}(\hat{\xi}_1))\hat{\beta}'(\hat{\beta}^{-1}(\hat{\xi}_2))}\Delta t\Delta s, \tag{3.7}$$

and $(\hat{\varsigma},\hat{\varrho})\in\Omega$ *is chosen so that*

$$\tilde{\Theta}\big(q_2(\hat{\varsigma},\hat{\varrho})\big)+\int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}}\frac{f(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2))}{\hat{\alpha}'(\hat{\alpha}^{-1}(\hat{\xi}_1))\hat{\beta}'(\hat{\beta}^{-1}(\hat{\xi}_2))}\left[1+\int_0^{\hat{\xi}_1}g(\hat{\zeta},\hat{\xi}_2)\Delta\hat{\zeta}\right]\Delta\hat{\xi}_2\Delta\hat{\xi}_1 \in \mathrm{Dom}\big(F^{-1}\big).$$

Proof If the problem (3.1) and (3.2) has a solution $u(\hat{\varsigma},\hat{\varrho})$, it can be written as

$$u(\hat{\varsigma},\hat{\varrho})=a_1(\hat{\varsigma})+a_2(\hat{\varrho})+\int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}}\tilde{\Theta}\left(\hat{\xi}_1,\hat{\xi}_2,u\big(\hat{\alpha}(\hat{\xi}_1),\hat{\beta}(\hat{\xi}_2)\big),\int_0^{\hat{\xi}_1}\breve{k}\big(\hat{\zeta},\hat{\xi}_2,u(\hat{\zeta},t)\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \tag{3.8}$$

for any $(\hat{\varsigma},\hat{\varrho})\in\Omega$. Using the conditions (3.3), (3.4), and (3.5) in (3.8), we get

$$\begin{aligned}\big|u(\hat{\varsigma},\hat{\varrho})\big| &\le a(\hat{\varsigma},\hat{\varrho})+\int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}}\tilde{\Psi}\big(\big|u\big(\hat{\alpha}(\hat{\xi}_1),\hat{\beta}(\hat{\xi}_2)\big)\big|\big)\\ &\quad\times\big[f(\hat{\xi}_1,\hat{\xi}_2)\tilde{\Psi}\big(\big|u\big(\hat{\alpha}(\hat{\xi}_1),\hat{\beta}(\hat{\xi}_2)\big)\big|\big)+p(\hat{\xi}_1,\hat{\xi}_2)\big]\Delta\hat{\xi}_2\Delta\hat{\xi}_1\\ &\quad+\int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}}f(s,t)\tilde{\Psi}\big(\big|u\big(\hat{\alpha}(\hat{\xi}_1),\hat{\beta}(\hat{\xi}_2)\big)\big|\big)\\ &\quad\times\left(\int_0^{\hat{\xi}_1}g(\hat{\zeta},\hat{\xi}_2)\tilde{\Psi}\big(\big|u(\hat{\zeta},\hat{\xi}_2)\big|\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1.\end{aligned} \tag{3.9}$$

Now, from (3.9), we get

$$\begin{aligned}\big|u(\hat{\varsigma},\hat{\varrho})\big| &\le a(\hat{\varsigma},\hat{\varrho})+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}\frac{\varphi(|u(\hat{\xi}_1,\hat{\xi}_2)|)}{\hat{\alpha}'(\hat{\alpha}^{-1}(\hat{\xi}_1))\hat{\beta}'(\hat{\beta}^{-1}(\hat{\xi}_2))}\\ &\quad\times\big[f\big(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2)\big)\varphi\big(\big|u(\hat{\xi}_1,\hat{\xi}_2)\big|\big)\\ &\quad+p\big(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2)\big)\big]\Delta t\Delta s\\ &\quad+\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}\frac{f(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2))}{\hat{\alpha}'(\hat{\alpha}^{-1}(\hat{\xi}_1))\hat{\beta}'(\hat{\beta}^{-1}(\hat{\xi}_2))}\varphi\big(\big|u(\hat{\xi}_1,\hat{\xi}_2)\big|\big)\\ &\quad\times\left(\int_0^{\hat{\xi}_1}g(\hat{\zeta},\hat{\xi}_2)\varphi\big(\big|u(\hat{\zeta},\hat{\xi}_2)\big|\big)\Delta\hat{\zeta}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1,\end{aligned} \tag{3.10}$$

for any $(\hat{\varsigma},\hat{\varrho})\in\Omega$. Now, an application of Theorem 2.10 to (3.10) yields the required inequality in (3.6). □

Consider the initial boundary value problem of the form

$$\big(z^q\big)^{\Delta\hat{\varsigma}\Delta\hat{\varrho}}(\hat{\varsigma},\hat{\varrho})=\breve{A}\left(\hat{\varsigma},\hat{\varrho},z\big(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho})\big),\int_0^{\hat{\alpha}(\hat{\varsigma})}h\big(\hat{\xi}_1,\hat{\varrho},z(\hat{\xi}_1,\hat{\varrho})\big)\Delta\hat{\xi}_1\right), \tag{3.11}$$

$$z(\hat{\varsigma},0)=a_1(\hat{\varsigma}),\qquad z(0,\hat{\varrho})=a_2(\hat{\varrho}),\qquad a_1(0)=a_2(0)=0, \tag{3.12}$$

for any $(\hat{\varsigma},\hat{\varrho})\in\Omega$.

Theorem 3.2 *Assume that the functions $h, \breve{A}, a_2, a_1$ in (3.11) and (3.12) satisfy the conditions*

$$\left|\breve{A}(\hat{\varsigma},\hat{\varrho},z(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho}),v)\right| \leq f(\hat{\varsigma},\hat{\varrho})\left|z^r(\hat{\alpha}(\hat{\varsigma}),\hat{\beta}(\hat{\varrho}))\right| + f(\hat{\varsigma},\hat{\varrho})v, \tag{3.13}$$

$$\left|h(\hat{\varsigma},\hat{\varrho},z(\hat{\varsigma},\hat{\varrho}))\right| \leq g(\hat{\varsigma},\hat{\varrho})\left|z^r(\hat{\varsigma},\hat{\varrho})\right|, \tag{3.14}$$

$$\left|a_1(\hat{\varsigma}) + a_2(\hat{\varrho})\right| \leq a(\hat{\varsigma},\hat{\varrho}), \tag{3.15}$$

where $r \geq q > 0$. Then

$$\left|z(\hat{\varsigma},\hat{\varrho})\right| \leq \left[\left(a(\hat{\varsigma},\hat{\varrho})\right)^{\frac{q-r}{q}} + \frac{q-r}{q}\int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}\frac{f(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2))}{\hat{\alpha}'(\hat{\alpha}^{-1}(\hat{\xi}_1))\hat{\beta}'(\hat{\beta}^{-1}(\hat{\xi}_2))} \times\left(1+\int_0^{\hat{\xi}_1} g(\hat{\varsigma},\hat{\xi}_2)\Delta\hat{\varsigma}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1\right]^{\frac{1}{q-r}}, \tag{3.16}$$

for $0 \leq \hat{\varsigma} \leq \hat{\varsigma}_1, 0 \leq \hat{\varrho} \leq \hat{\varrho}_1$.

Proof If the problem (3.11) and (3.12) has a solution $z(\hat{\varsigma},\hat{\varrho})$, it can be written as

$$z^q(\hat{\varsigma},\hat{\varrho}) = a_1(x) + a_2(y) + \int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}}\tilde{\Theta}\left(\hat{\xi}_1,\hat{\xi}_1,u(\hat{\alpha}(\hat{\xi}_1),\hat{\beta}(\hat{\xi}_2)), \int_0^{\hat{\alpha}(\hat{\xi}_1)}\breve{k}(\hat{\varsigma},\hat{\xi}_2,u(\hat{\varsigma},\hat{\xi}_2))\Delta\hat{\varsigma}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \tag{3.17}$$

for any $(\hat{\varsigma},\hat{\varrho}) \in \Omega$. Using the conditions (3.13), (3.14), and (3.15) in (3.17), we get

$$\left|z^q(\hat{\varsigma},\hat{\varrho})\right| \leq a(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}} f(\hat{\xi}_1,\hat{\xi}_2)\left|z^r(\hat{\alpha}(s),\hat{\beta}(t))\right|\Delta\hat{\xi}_2\Delta\hat{\xi}_1 + \int_0^{\hat{\varsigma}}\int_0^{\hat{\varrho}} f(\hat{\xi}_1,\hat{\xi}_2)\left(\int_0^{\hat{\xi}_1} g(\hat{\varsigma},\hat{\xi}_2)\left|z^r(\hat{\varsigma},\hat{\xi}_2)\right|\Delta\hat{\varsigma}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1. \tag{3.18}$$

From (3.18), we get

$$\left|z^q(\hat{\varsigma},\hat{\varrho})\right| \leq a(\hat{\varsigma},\hat{\varrho}) + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\alpha}(\hat{\varrho})}\frac{f(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2))}{\hat{\alpha}'(\hat{\alpha}^{-1}(\hat{\xi}_1))\hat{\beta}'(\hat{\beta}^{-1}(\hat{\xi}_2))}\left|z^r(\hat{\xi}_1,\hat{\xi}_2)\right|\Delta\hat{\xi}_2\Delta\hat{\xi}_1 + \int_0^{\hat{\alpha}(\hat{\varsigma})}\int_0^{\hat{\beta}(\hat{\varrho})}\frac{f(\hat{\alpha}^{-1}(\hat{\xi}_1),\hat{\beta}^{-1}(\hat{\xi}_2))}{\hat{\alpha}'(\hat{\alpha}^{-1}(\hat{\xi}_1))\hat{\beta}'(\hat{\beta}^{-1}(\hat{\xi}_2))} \times\left(\int_0^{\hat{\xi}_1} g(\hat{\varsigma},\hat{\xi}_2)\left|z^r(\hat{\varsigma},\hat{\xi}_2)\right|\Delta\hat{\varsigma}\right)\Delta\hat{\xi}_2\Delta\hat{\xi}_1, \tag{3.19}$$

for any $(\hat{\varsigma},\hat{\varrho}) \in \Omega$. A suitable application of Theorem 2.3 to (3.19) with $\tilde{\Phi}(u) = u^q$, $\tilde{\Psi}(u) = u^r$ and $p(\hat{\varsigma},\hat{\varrho}) = 0$ gives the required inequality in (3.16). □

4 Conclusion

In this work, by using a new technique, we proved several nonlinear retarded dynamic inequalities in two independent variables of Gronwall type on time scales. We also gave a new proof and formula of Leibniz integral rule on time scales. Further, we also applied our inequalities to discrete and continuous calculus to obtain some new inequalities as special cases. Furthermore, we studied the boundedness of some delay initial value problems by applying our results.

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References
1. El-Deeb, A.A., Khan, Z.A.: Certain new dynamic nonlinear inequalities in two independent variables and applications. Bound. Value Probl. **2020**(1), 31 (2020)
2. Hilger, S.: Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten. Ph.D. thesis, Universität Würzburg (1988)
3. Bohner, M., Peterson, A.: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston (2001)
4. Bohner, M., Peterson, A.: Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston (2003)
5. Abdeldaim, A., El-Deeb, A.A., Agarwal, P., El-Sennary, H.A.: On some dynamic inequalities of Steffensen type on time scales. Math. Methods Appl. Sci. **41**(12), 4737–4753 (2018)
6. Agarwal, R., O'Regan, D., Saker, S.: Dynamic Inequalities on Time Scales. Springer, Cham (2014)
7. Akin-Bohner, E., Bohner, M., Akin, F.: Pachpatte inequalities on time scales. JIPAM. J. Inequal. Pure Appl. Math. **6**(1), Article ID 23 (2005)
8. Bohner, M., Matthews, T.: The Grüss inequality on time scales. Commun. Math. Anal. **3**(1), 1–8 (2007)
9. Bohner, M., Matthews, T.: Ostrowski inequalities on time scales. JIPAM. J. Inequal. Pure Appl. Math. **9**(1), Article ID 8 (2008)
10. Dinu, C.: Hermite–Hadamard inequality on time scales. J. Inequal. Appl. **2008**, Article ID 287947 (2008)
11. El-Deeb, A.A.: On some generalizations of nonlinear dynamic inequalities on time scales and their applications. Appl. Anal. Discrete Math. **13**(2), 440–462 (2019)
12. El-Deeb, A.A., Cheung, W.-S.: A variety of dynamic inequalities on time scales with retardation. J. Nonlinear Sci. Appl. **11**(10), 1185–1206 (2018)
13. El-Deeb, A.A., El-Sennary, H.A., Khan, Z.A.: Some Steffensen-type dynamic inequalities on time scales. Adv. Differ. Equ. **2019**, 246 (2019)
14. El-Deeb, A.A., Elsennary, H.A., Cheung, W.-S.: Some reverse Hölder inequalities with Specht's ratio on time scales. J. Nonlinear Sci. Appl. **11**(4), 444–455 (2018)
15. El-Deeb, A.A., Elsennary, H.A., Nwaeze, E.R.: Generalized weighted Ostrowski, trapezoid and Grüss type inequalities on time scales. Fasc. Math. **60**, 123–144 (2018)
16. El-Deeb, A.A., Xu, H., Abdeldaim, A., Wang, G.: Some dynamic inequalities on time scales and their applications. Adv. Differ. Equ. **2019**, 130 (2019)
17. El-Deeb, A.A.: Some Gronwall–Bellman type inequalities on time scales for Volterra–Fredholm dynamic integral equations. J. Egypt. Math. Soc. **26**(1), 1–17 (2018)
18. Hilscher, R.: A time scales version of a Wirtinger-type inequality and applications. J. Comput. Appl. Math. **141**(1–2), 219–226 (2002)
19. Li, W.N.: Some delay integral inequalities on time scales. Comput. Math. Appl. **59**(6), 1929–1936 (2010)
20. Řehák, P.: Hardy inequality on time scales and its application to half-linear dynamic equations. J. Inequal. Appl. **2005**, 942973 (2005)
21. Saker, S.H., El-Deeb, A.A., Rezk, H.M., Agarwal, R.P.: On Hilbert's inequality on time scales. Appl. Anal. Discrete Math. **11**(2), 399–423 (2017)

22. Tian, Y., El-Deeb, A.A., Meng, F.: Some nonlinear delay Volterra–Fredholm type dynamic integral inequalities on time scales. Discrete Dyn. Nat. Soc. **2018**, Article ID 5841985 (2018)
23. El-Deeb, A.A., Kh, F.M., Ismail, G.A.F., Khan, Z.A.: Weighted dynamic inequalities of Opial-type on time scales. Adv. Differ. Equ. **2019**(1), 393 (2019)
24. Kh, F.M., El-Deeb, A.A., Abdeldaim, A., Khan, Z.A.: On some generalizations of dynamic Opial-type inequalities on time scales. Adv. Differ. Equ. **2019**(1), 323 (2019)
25. Abdeldaim, A., El-Deeb, A.A.: Some new retarded nonlinear integral inequalities with iterated integrals and their applications in retarded differential equations and integral equations. J. Fract. Calc. Appl. **5**(suppl. 3S), Paper no. 9 (2014)
26. Abdeldaim, A., El-Deeb, A.A.: On generalized of certain retarded nonlinear integral inequalities and its applications in retarded integro-differential equations. Appl. Math. Comput. **256**, 375–380 (2015)
27. Abdeldaim, A., El-Deeb, A.A.: On some generalizations of certain retarded nonlinear integral inequalities with iterated integrals and an application in retarded differential equation. J. Egypt. Math. Soc. **23**(3), 470–475 (2015)
28. Abdeldaim, A., El-Deeb, A.A.: On some new nonlinear retarded integral inequalities with iterated integrals and their applications in integro-differential equations. Br. J. Math. Comput. Sci. **5**(4), 479–491 (2015)
29. Agarwal, R.P., Lakshmikantham, V.: Uniqueness and Nonuniqueness Criteria for Ordinary Differential Equations. Series in Real Analysis, vol. 6. World Scientific, Singapore (1993)
30. El-Deeb, A.A.: On Integral Inequalities and Their Applications. LAP Lambert Academic Publishing, Saarbrücken (2017)
31. El-Deeb, A.A.: A variety of nonlinear retarded integral inequalities of Gronwall type and their applications. In: Advances in Mathematical Inequalities and Applications, pp. 143–164. Springer, Berlin (2018)
32. El-Deeb, A.A., Ahmed, R.G.: On some explicit bounds on certain retarded nonlinear integral inequalities with applications. Adv. Inequal. Appl. **2016**, Article ID 15 (2016)
33. El-Deeb, A.A., Ahmed, R.G.: On some generalizations of certain nonlinear retarded integral inequalities for Volterra–Fredholm integral equations and their applications in delay differential equations. J. Egypt. Math. Soc. **25**(3), 279–285 (2017)
34. El-Owaidy, H., Abdeldaim, A., El-Deeb, A.A.: On some new retarded nonlinear integral inequalities and their applications. Math. Sci. Lett. **3**(3), 157–164 (2014)
35. El-Owaidy, H.M., Ragab, A.A., Eldeeb, A.A., Abuelela, W.M.K.: On some new nonlinear integral inequalities of Gronwall–Bellman type. Kyungpook Math. J. **54**(4), 555–575 (2014)
36. Li, J.D.: Opial-type integral inequalities involving several higher order derivatives. J. Math. Anal. Appl. **167**(1), 98–110 (1992)
37. El-Deeb, A.A., Makharesh, S.D., Baleanu, D.: Dynamic Hilbert-type inequalities with Fenchel–Legendre transform. Symmetry **12**(4), 582 (2020)
38. El-Deeb, A.A., Baleanu, D.: New weighted Opial-type inequalities on time scales for convex functions. Symmetry **12**(5), 842 (2020)
39. El-Deeb, A.A., El-Sennary, H.A., Khan, Z.A.: Some reverse inequalities of Hardy type on time scales. Adv. Differ. Equ. **2020**(1), 402 (2020)
40. El-Deeb, A.A., Elsennary, H.A., Baleanu, D.: Some new Hardy-type inequalities on time scales. Adv. Differ. Equ. **2020**(1), 441 (2020)
41. Ammar, B.: On certain new nonlinear retarded integral inequalities in two independent variables and applications. Appl. Math. Comput. **2019**(335), 103–111 (2018)