Research Article

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On some new Hermite-Hadamard and Ostrowski type inequalities for *s*-convex functions in (*p*, *q*)-calculus with applications

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Abstract: In this study, we establish some new Hermite-Hadamard type inequalities for s-convex functions in the second sense using the post-quantum calculus. Moreover, we prove a new (p, q)-integral identity to prove some new Ostrowski type inequalities for (p, q)-differentiable functions. We also show that the newly discovered results are generalizations of comparable results in the literature. Finally, we give application to special means of real numbers using the newly proved inequalities.

Keywords: Hermite-Hadamard inequality, Ostrowski inequality, (p, q)-integral, post-quantum calculus, s-convex functions

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1 Introduction

The Hermite-Hadamard (HH) inequality, which was independently found by Hermite and Hadamard (see, also [1], and [2, p. 137]), is particularly important in convex functions theory:

$$\digamma\left(\frac{\pi_1+\pi_2}{2}\right) \le \frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\digamma(x)\mathrm{d}x \le \frac{\digamma(\pi_1)+\digamma(\pi_2)}{2}, \tag{1}$$

where $\digamma$ is a convex function on $[\pi_1, \pi_2]$ in this case. The aforementioned inequality is true in reverse order for concave mappings.

In [3], Hudzik and Maligranda defined s-convex functions in the second sense as follows: a mapping $\digamma : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+ = [0, \infty)$ is called s-convex in the second sense if

$$\digamma(tx + (1-t)y) \le t^s\digamma(x) + (1-t)^s\digamma(y)$$

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for all $x, y \in \mathbb{R}^+$ and $t \in [0,1]$ and $s \in (0,1]$. Dragomir and Fitzpatrick [4] then used this newly discovered class of functions to prove the HH inequality on $[\pi_1, \pi_2]$ as follows:

$$2^{s-1}F\left(\frac{\pi_1+\pi_2}{2}\right) \le \frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2} F(x)\mathrm{d}x \le \frac{F(\pi_1)+F(\pi_2)}{s+1}. \tag{2}$$

On the other hand, several works in the field of q-analysis are being carried out, beginning with Euler, to achieve mastery in the mathematics that underpins quantum computing. The link between physics and mathematics is referred to as q-calculus. It has a wide range of applications in different areas of pure and applied mathematics [5,6]. Euler is thought to be the inventor of this significant branch of mathematics. In Newton's work on infinite series, he used the q parameter. Later, Jackson [7,8] presented the q-calculus that knew without limits calculus in a logical approach. Al-Salam [9] presented the q-analogue of the q-fractional integral and the q-Riemann-Liouville fractional in 1966. Since then, the amount of study in this area has steadily expanded. In particular, in 2013, Tariboon and Ntouyas introduced ${}_{\pi_1}D_q$-difference operator and q_{π_1}-integral in [10]. In 2020, Bermudo et al. introduced the notion of ${}^{\pi_2}D_q$ derivative and q^{π_2}-integral in [11]. Sadjang generalized to quantum calculus and introduced the notions of post-quantum calculus or shortly (p,q)-calculus in [12]. Soontharanon and Sitthiwirattham [13] introduced the notions of fractional (p,q)-calculus later on. In [14], Tunç and Göv gave the post-quantum variant of ${}_{\pi_1}D_q$-difference operator and q_{π_1}-integral. Recently, in 2021, Vivas-Cortez et al. introduced the notions of ${}^{\pi_2}D_{p,q}$ derivative and $(p,q)^{\pi_2}$-integral in [15].

Many integral inequalities have been studied using quantum integrals for various types of functions. For example, in [16–19,11,20–23], the authors used ${}_{\pi_1}D_q$, ${}^{\pi_2}D_q$-derivatives and q_{π_1}, q^{π_2} -integrals to prove HH integral inequalities and their left-right estimates for convex and coordinated convex functions. In [24], Noor et al. presented a generalized version of quantum HH integral inequalities. For generalized quasi-convex functions, Nwaeze and Tameru proved certain parameterized quantum integral inequalities in [25]. Khan et al. proved quantum HH inequality using the green function in [26]. Budak et al. [27], Ali et al. [28,29] and Vivas-Cortez et al. [30] developed new quantum Simpson's and quantum Newton's type inequalities for convex and coordinated convex functions. For quantum Ostrowski's inequalities for convex and co-ordinated convex functions, readers refer to [31–33]. Kunt et al. [34] generalized the results of [18] and proved Hermite-Hadamard type inequalities and their left estimates using ${}_{\pi_1}D_{p,q}$-difference operator and $(p,q)_{\pi_1}$-integral. Recently, Latif et al. [35] found the right estimates of Hermite-Hadamard type inequalities proved by Kunt et al. [34].

Inspired by these ongoing studies, in the context of (p,q)-calculus, we prove several new Hermite-Hadamard and Ostrowski type inequalities for s-convex functions in the second sense.

The following is the structure of this article: Section 2 provides a brief overview of the fundamentals of q-calculus as well as other related studies in this field. In Section 3, we go over some basic (p,q)-calculus notions and inequalities. In Section 4, we show the relationship between the results presented here and related results in the literature by proving post-quantum HH inequalities for s-convex functions in the second sense. Post-quantum Ostrowski type inequalities for s-convex functions in the second are presented in Section 5. In Section 6, we present some applications to special means of real numbers for newly established inequalities. Section 7 concludes with some recommendations for future research.

2 Preliminaries of q-calculus and some inequalities

In this section, we revisit several previously regarded ideas. In addition, throughout the paper, $s \in (0,1]$, and we use the following notations (see, [6]):

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad q \in (0,1).$$

In [8], Jackson gave the q-Jackson integral from 0 to π_2 for $0 < q < 1$ as follows:

$$\int_0^{\pi_2} F(x) d_q x = (1 - q)\pi_2 \sum_{n=0}^{\infty} q^n F(\pi_2 q^n) \tag{3}$$

provided the sum converge absolutely.

Definition 1. [10] For a function $F : [\pi_1, \pi_2] \to \mathbb{R}$, the q_{π_1}-derivative of F at $x \in [\pi_1, \pi_2]$ is characterized by the expression:

$$_{\pi_1}D_q F(x) = \frac{F(x) - F(qx + (1 - q)\pi_1)}{(1 - q)(x - \pi_1)}, \quad x \neq \pi_1. \tag{4}$$

If $x = \pi_1$, we define $_{\pi_1}D_q F(\pi_1) = \lim_{x \to \pi_1 \pi_1} D_q F(x)$ if it exists, and it is finite.

Definition 2. [11] For a function $F : [\pi_1, \pi_2] \to \mathbb{R}$, the q^{π_2}-derivative of F at $x \in [\pi_1, \pi_2]$ is characterized by the expression:

$$^{\pi_2}D_q F(x) = \frac{F(qx + (1 - q)\pi_2) - F(x)}{(1 - q)(\pi_2 - x)}, \quad x \neq \pi_2. \tag{5}$$

If $x = \pi_2$, we define $^{\pi_2}D_q F(\pi_2) = \lim_{x \to \pi_2} {}^{\pi_2}D_q F(x)$ if it exists and it is finite.

Definition 3. [10] Let $F : [\pi_1, \pi_2] \to \mathbb{R}$ be a function. Then, the q_{π_1}-definite integral on $[\pi_1, \pi_2]$ is defined as follows:

$$\int_{\pi_1}^{\pi_2} F(x)_{\pi_1} d_q x = (1 - q)(\pi_2 - \pi_1) \sum_{n=0}^{\infty} q^n F(q^n \pi_2 + (1 - q^n)\pi_1) = (\pi_2 - \pi_1) \int_0^1 F((1 - t)\pi_1 + t\pi_2) d_q t. \tag{6}$$

Definition 4. [11] Let $F : [\pi_1, \pi_2] \to \mathbb{R}$ be a function. Then, the q^{π_2}-definite integral on $[\pi_1, \pi_2]$ is defined as follows:

$$\int_{\pi_1}^{\pi_2} F(x)^{\pi_2} d_q x = (1 - q)(\pi_2 - \pi_1) \sum_{n=0}^{\infty} q^n F(q^n \pi_1 + (1 - q^n)\pi_2) = (\pi_2 - \pi_1) \int_0^1 F(t\pi_1 + (1 - t)\pi_2) d_q t. \tag{7}$$

In [11], Bermudo et al. established the following quantum HH type inequality.

Theorem 1. *For the convex mapping $F : [\pi_1, \pi_2] \to \mathbb{R}$, the following inequality holds*

$$F\left(\frac{\pi_1 + \pi_2}{2}\right) \leq \frac{1}{2(\pi_2 - \pi_1)} \left[\int_{\pi_1}^{\pi_2} F(x)_{\pi_1} d_q x + \int_{\pi_1}^{\pi_2} F(x)^{\pi_2} d_q x \right] \leq \frac{F(\pi_1) + F(\pi_2)}{2}. \tag{8}$$

In [33], Budak et al. proved the following Ostrowski inequality by using the concepts of quantum derivatives and integrals.

Theorem 2. *Let $F : [\pi_1, \pi_2] \subset \mathbb{R} \to \mathbb{R}$ be a function and $^{\pi_2}D_q F$ and $_{\pi_1}d_q F$ be two continuous and integrable functions on $[\pi_1, \pi_2]$. If $|^{\pi_2}D_q F(t)|, |_{\pi_1}D_q F(t)| \leq M$ for all $t \in [\pi_1, \pi_2]$, then we have the following quantum quantum Ostrowski inequality*:

$$\left| F(x) - \frac{1}{\pi_2 - \pi_1} \left[\int_{\pi_1}^{x} F(t)_{\pi_1} d_q t + \int_{x}^{\pi_2} F(t)^{\pi_2} d_q t \right] \right| \leq \frac{qM}{(\pi_2 - \pi_1)} \left[\frac{(x - \pi_1)^2 + (\pi_2 - x)^2}{[2]_q} \right]. \tag{9}$$

Recently, Asawasamrit et al. [36] gave the following generalizations of inequalities (8) and (9) using the s-convexity.

Theorem 3. *Assume that the mapping* $\digamma : [0, \infty) \to \mathbb{R}$ *is s-convex in the second sense and* $\pi_1, \pi_2 \in [0, \infty)$ *with* $\pi_1 < \pi_2$*, then the following inequality holds for* $s \in (0, 1]$:

$$2^{s-1}\digamma\left(\frac{\pi_1 + \pi_2}{2}\right) \leq \frac{1}{2(\pi_2 - \pi_1)}\left[\int_{\pi_1}^{\pi_2} \digamma(x)\,{}_{\pi_1}d_q x + \int_{\pi_1}^{\pi_2} \digamma(x)\,{}^{\pi_2}d_q x\right] \leq \frac{\digamma(\pi_1) + \digamma(\pi_2)}{[s + 1]_q}. \tag{10}$$

Theorem 4. *Let* $\digamma : [\pi_1, \pi_2] \subset \mathbb{R} \to \mathbb{R}$ *be function and* ${}^{\pi_2}D_q\digamma$ *and* ${}_{\pi_1}D_q\digamma$ *be two continuous and integrable functions on* $[\pi_1, \pi_2]$*. If* $|{}^{\pi_2}D_q\digamma(t)|, |{}_{\pi_1}D_q\digamma(t)| \leq M$ *for all* $t \in [\pi_1, \pi_2]$*, then we have the following quantum Ostrowski inequality for s-convex functions in the second sense*:

$$\left|\digamma(x) - \frac{1}{\pi_2 - \pi_1}\left[\int_{\pi_1}^{x} \digamma(t)\,{}_{\pi_1}d_q t + \int_{x}^{\pi_2} \digamma(t)\,{}^{\pi_2}d_q t\right]\right| \leq \frac{Mq}{\pi_2 - \pi_1}\left(\frac{1}{[s + 2]_q} + \Theta_{11}\right)[(x - \pi_1)^2 + (\pi_2 - x)^2], \tag{11}$$

where

$$\Theta_{11} = \int_0^1 t(1 - t)^s d_q t.$$

3 Post-quantum calculus and some inequalities

In this section, we review some fundamental notions and notations of (p, q)-calculus.

The $[n]_{p,q}$ is said to be (p, q)-integers and expressed as follows:

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}$$

with $0 < q < p \leq 1$. The $[n]_{p,q}!$ and $\begin{bmatrix} n \\ k \end{bmatrix}!$ are called (p, q)-factorial and (p, q)-binomial, respectively, and expressed as follows:

$$[n]_{p,q}! = \prod_{k=1}^{n}[k]_{p,q}, \quad n \geq 1, \quad [0]_{p,q}! = 1,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}! = \frac{[n]_{p,q}!}{[n - k]_{p,q}![k]_{p,q}!}.$$

Definition 5. [12] The (p, q)-derivative of mapping $\digamma : [\pi_1, \pi_2] \to \mathbb{R}$ is given as follows:

$$D_{p,q}\digamma(x) = \frac{\digamma(px) - \digamma(qx)}{(p - q)x}, \quad x \neq 0$$

with $0 < q < p \leq 1$.

Definition 6. [14] The $(p, q)_{\pi_1}$-derivative of mapping $\digamma : [\pi_1, \pi_2] \to \mathbb{R}$ is given as follows:

$${}_{\pi_1}D_{p,q}\digamma(x) = \frac{\digamma(px + (1 - p)\pi_1) - \digamma(qx + (1 - q)\pi_1)}{(p - q)(x - \pi_1)}, \quad x \neq \pi_1 \tag{12}$$

with $0 < q < p \leq 1$. For $x = \pi_1$, we state ${}_{\pi_1}D_{p,q}\digamma(\pi_1) = \lim_{x \to \pi_1 \pi_1} D_{p,q}\digamma(x)$ if it exists and it is finite.

Definition 7. [15] The $(p,q)^{\pi_2}$-derivative of mapping $\digamma:[\pi_1,\pi_2]\to\mathbb{R}$ is given as follows:

$$^{\pi_2}D_{p,q}\digamma(x)=\frac{\digamma(qx+(1-q)\pi_2)-\digamma(px+(1-p)\pi_2)}{(p-q)(\pi_2-x)},\quad x\neq\pi_2. \tag{13}$$

with $0<q<p\leq 1$. For $x=\pi_2$, we state $^{\pi_2}D_{p,q}\digamma(\pi_2)=\lim_{x\to\pi_2}{}^{\pi_2}D_{p,q}\digamma(x)$ if it exists and it is finite.

Remark 1. It is clear that if we use $p=1$ in (12) and (13), then the equalities (12) and (13) reduce to (4) and (5), respectively.

Definition 8. [14] The definite $(p,q)_{\pi_1}$-integral of mapping $\digamma:[\pi_1,\pi_2]\to\mathbb{R}$ on $[\pi_1,\pi_2]$ is stated as follows:

$$\int_{\pi_1}^{x}\digamma(\tau)\,_{\pi_1}d_{p,q}\tau=(p-q)(x-\pi_1)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\digamma\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)\pi_1\right) \tag{14}$$

with $0<q<p\leq 1$.

Definition 9. [15] The definite $(p,q)^{\pi_2}$-integral of mapping $\digamma:[\pi_1,\pi_2]\to\mathbb{R}$ on $[\pi_1,\pi_2]$ is stated as follows:

$$\int_{x}^{\pi_2}\digamma(\tau)\,^{\pi_2}d_{p,q}\tau=(p-q)(\pi_2-x)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\digamma\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)\pi_2\right) \tag{15}$$

with $0<q<p\leq 1$.

Remark 2. It is evident that if we pick $p=1$ in (14) and (15), then the equalities (14) and (15) change into (6) and (7), respectively.

Remark 3. If we take $\pi_1=0$ and $x=\pi_2=1$ in (14), then we have

$$\int_{0}^{1}\digamma(\tau)\,_{0}d_{p,q}\tau=(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\digamma\left(\frac{q^n}{p^{n+1}}\right).$$

Similarly, by taking $x=\pi_1=0$ and $\pi_2=1$ in (15), then we obtain that

$$\int_{0}^{1}\digamma(\tau)\,^{1}d_{p,q}\tau=(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\digamma\left(1-\frac{q^n}{p^{n+1}}\right).$$

Lemma 1. [15] *We have the following equalities*:

$$\int_{\pi_1}^{\pi_2}(\pi_2-x)^{\alpha}\,^{\pi_2}d_{p,q}x=\frac{(\pi_2-\pi_1)^{\alpha+1}}{[\alpha+1]_{p,q}}$$

$$\int_{\pi_1}^{\pi_2}(x-\pi_1)^{\alpha}\,_{\pi_1}d_{p,q}x=\frac{(\pi_2-\pi_1)^{\alpha+1}}{[\alpha+1]_{p,q}},$$

where $\alpha\in\mathbb{R}-\{-1\}$.

Recently, Vivas-Cortez et al. [15] proved the following HH type inequalities for convex functions using the $(p,q)^{\pi_2}$-integral:

Theorem 5. [15] *For a convex mapping* $\digamma:[\pi_1,\pi_2]\to\mathbb{R}$, *which is differentiable on* $[\pi_1,\pi_2]$, *the following inequalities hold for* $(p,q)^{\pi_2}$*-integral*:

$$F\left(\frac{p\pi_1+q\pi_2}{[2]_{p,q}}\right)\leq\frac{1}{p(\pi_2-\pi_1)}\int_{p\pi_1+(1-p)\pi_2}^{\pi_2}F(x)^{\pi_2}d_{p,q}x\leq\frac{pF(\pi_1)+qF(\pi_2)}{[2]_{p,q}},\tag{16}$$

where $0<q<p\leq1$.

Theorem 6. [15] *For a convex function* $F:[\pi_1,\pi_2]\to\mathbb{R}$, *the following inequality holds*:

$$F\left(\frac{\pi_1+\pi_2}{2}\right)\leq\frac{1}{2p(\pi_2-\pi_1)}\left[\int_{\pi_1}^{p\pi_2+(1-p)\pi_1}F(x)_{\pi_1}d_{p,q}x+\int_{p\pi_1+(1-p)\pi_2}^{\pi_2}F(x)^{\pi_2}d_{p,q}x\right]\leq\frac{F(\pi_1)+F(\pi_2)}{2},\tag{17}$$

where $0<q<p\leq1$.

4 Hermite-Hadamard inequalities

In this section, we prove HH inequalities for s-convex functions in the second kind using the post-quantum integrals.

Theorem 7. *Assume that the mapping* $F:[0,\infty)\to\mathbb{R}$ *is s-convex in the second sense and* $\pi_1,\pi_2\in[0,\infty)$ *with* $\pi_1<\pi_2$, *then the following inequality holds for* $s\in(0,1]$:

$$2^{s-1}F\left(\frac{\pi_1+\pi_2}{2}\right)\leq\frac{1}{2p(\pi_2-\pi_1)}\left[\int_{\pi_1}^{\pi_2}F(x)_{\pi_1}d_{p,q}x+\int_{\pi_1}^{\pi_2}F(x)^{\pi_2}d_{p,q}x\right]\leq\frac{F(\pi_1)+F(\pi_2)}{[s+1]_{p,q}}.\tag{18}$$

Proof. We have s-convexity, as we know from s-convexity

$$2^sF\left(\frac{x+y}{2}\right)\leq F(x)+F(y).\tag{19}$$

We obtain the following by putting $x=t\pi_2+(1-t)\pi_1$ and $y=t\pi_1+(1-t)\pi_2$ in (19)

$$2^sF\left(\frac{\pi_1+\pi_2}{2}\right)\leq F(t\pi_2+(1-t)\pi_1)+F(t\pi_1+(1-t)\pi_2).$$

From Definitions 8 and 9, we have

$$2^{s-1}F\left(\frac{\pi_1+\pi_2}{2}\right)\leq\frac{1}{2p(\pi_2-\pi_1)}\left[\int_{\pi_1}^{p\pi_2+(1-p)\pi_1}F(x)_{\pi_1}d_{p,q}x+\int_{p\pi_1+(1-p)\pi_2}^{\pi_2}F(x)^{\pi_2}d_{p,q}x\right],$$

and the first inequality in (18) is proved.

To prove the second inequality, we use the s-convexity, and we have

$$F(t\pi_2+(1-t)\pi_1)\leq t^sF(\pi_2)+(1-t)^sF(\pi_1)\tag{20}$$

and

$$F(t\pi_1+(1-t)\pi_2)\leq t^sF(\pi_1)+(1-t)^sF(\pi_2).\tag{21}$$

By adding (20) and (21), from Definitions 8 and 9, we have

$$\frac{1}{2p(\pi_2-\pi_1)}\left[\int_{\pi_1}^{p\pi_2+(1-p)\pi_1}F(x)_{\pi_1}d_{p,q}x+\int_{p\pi_1+(1-p)\pi_2}^{\pi_2}F(x)^{\pi_2}d_{p,q}x\right]\leq\frac{F(\pi_1)+F(\pi_2)}{[s+1]_{p,q}},$$

and the proof is completed. □

Example 1. For s-convex function $f(x) = x^s$, from inequality (18) with $a = s = 1$, $b = 2$, $p = \frac{1}{2}$, and $q = \frac{1}{4}$, we have

$$2^{s-1}\mathrm{F}\left(\frac{\pi_1 + \pi_2}{2}\right) = \frac{3}{2},$$

$$\begin{aligned}\frac{1}{2p(\pi_2 - \pi_1)}\left[\int_{\pi_1}^{\pi_2} \mathrm{F}(x)_{\pi_1} d_{p,q}x + \int_{\pi_1}^{\pi_2} \mathrm{F}(x)^{\pi_2} d_{p,q}x\right] &= \left(\frac{1}{2} - \frac{1}{4}\right)\sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)^n}{\left(\frac{1}{2}\right)^{n+1}}\left(\frac{\left(\frac{1}{4}\right)^n}{\left(\frac{1}{2}\right)^{n+1}}2 + \left(1 - \frac{\left(\frac{1}{4}\right)^n}{\left(\frac{1}{2}\right)^{n+1}}\right)\right) \\ &\quad + \left(\frac{1}{2} - \frac{1}{4}\right)\sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)^n}{\left(\frac{1}{2}\right)^{n+1}}\left(\frac{\left(\frac{1}{4}\right)^n}{\left(\frac{1}{2}\right)^{n+1}} + \left(1 - \frac{\left(\frac{1}{4}\right)^n}{\left(\frac{1}{2}\right)^{n+1}}\right)2\right) = 3\end{aligned}$$

and

$$\frac{\mathrm{F}(\pi_1) + \mathrm{F}(\pi_2)}{[1 + s]_{p,q}} = \frac{1 + 2}{\frac{1}{4} + \frac{1}{2}} = 4.$$

Thus,

$$\frac{3}{2} < 3 < 4,$$

which shows that the inequality proved in Theorem 7 is true.

Remark 4. If we set $s = 1$ in Theorem 7, then we recapture the inequality (17).

Remark 5. In Theorem 7, if we take the limit as $p = 1$, then inequality (18) becomes the inequality (10).

Remark 6. In Theorem 7, if we take $p = 1$ and later take the limit as $q \to 1^-$, then inequality (18) becomes the inequality (2).

5 Ostrowski's inequalities

In this section, we prove post-quantum Ostrowski type inequalities for s-convex functions in the second sense.

We begin with the following identity.

Lemma 2. *Let* $\mathrm{F} : [\pi_1, \pi_2] \subset \mathbb{R} \to \mathbb{R}$ *be a function. If* ${}^{\pi_2}D_{p,q}\mathrm{F}$ *and* ${}_{\pi_1}D_{p,q}\mathrm{F}$ *are two continuous and integrable functions on* $[\pi_1, \pi_2]$*, then for all* $x \in [\pi_1, \pi_2]$*, we have*

$$\begin{aligned}&\mathrm{F}(x) - \frac{1}{p(\pi_2 - \pi_1)}\left[\int_{\pi_1}^{px+(1-p)\pi_1} \mathrm{F}(t)_{\pi_1} d_{p,q}t + \int_{px+(1-p)\pi_2}^{\pi_2} \mathrm{F}(t)^{\pi_2} d_{p,q}t\right] \\ &= \frac{q(x - \pi_1)^2}{\pi_2 - \pi_1}\int_0^1 t_{\pi_1}D_{p,q}\mathrm{F}(tx + (1 - t)\pi_1)_0 d_{p,q}t - \frac{q(\pi_2 - x)^2}{\pi_2 - \pi_1}\int_0^1 t^{\pi_2}D_{p,q}\mathrm{F}(tx + (1 - t)\pi_2)^1 d_{p,q}t.\end{aligned} \tag{22}$$

Proof. From Definitions 6 and 7, we have

$${}_{\pi_1}D_{p,q}\mathrm{F}(tx + (1 - t)\pi_1) = \frac{\mathrm{F}(ptx + (1 - pt)\pi_1) - \mathrm{F}(qtx + (1 - qt)\pi_1)}{t(x - \pi_1)(p - q)}$$

and

$$ {}^{\pi_2}D_{p,q}F(tx+(1-t)\pi_2) = \frac{F(qtx+(1-qt)\pi_2) - F(ptx+(1-pt)\pi_2)}{t(\pi_2 - x)(p-q)}. $$

By using Definition 9, we have

$$ \begin{aligned} I_1 &= \int_0^1 t\, {}^{\pi_2}D_{p,q}F(tx+(1-t)\pi_2)\, {}^1d_{p,q}t \\ &= \frac{1}{(\pi_2 - x)(p-q)} \int_0^1 [F(qtx+(1-qt)\pi_2) - F(ptx+(1-pt)\pi_2)]\, {}^1d_{p,q}t \\ &= \frac{1}{\pi_2 - x}\left[\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}F\left(\frac{q^{n+1}}{p^{n+1}}x + \left(1-\frac{q^{n+1}}{p^{n+1}}\right)\pi_2\right) - \sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}F\left(\frac{q^n}{p^n}x + \left(1-\frac{q^n}{p^n}\right)\pi_2\right)\right] \\ &= \frac{1}{\pi_2 - x}\left[\frac{1}{q}\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+1}}F\left(\frac{q^{n+1}}{p^{n+1}}x + \left(1-\frac{q^{n+1}}{p^{n+1}}\right)\pi_2\right) - \frac{1}{p}\sum_{n=0}^{\infty}\frac{q^n}{p^n}F\left(\frac{q^n}{p^n}x + \left(1-\frac{q^n}{p^n}\right)\pi_2\right)\right] \\ &= \frac{1}{\pi_2 - x}\left[\left(\frac{1}{q}-\frac{1}{p}\right)\sum_{n=0}^{\infty}\frac{q^n}{p^n}F\left(\frac{q^n}{p^n}x + \left(1-\frac{q^n}{p^n}\right)\pi_2\right) - \frac{1}{q}F(x)\right] \\ &= \frac{1}{\pi_2 - x}\left[\frac{p-q}{pq}\sum_{n=0}^{\infty}\frac{q^n}{p^n}F\left(\frac{q^n}{p^n}x + \left(1-\frac{q^n}{p^n}\right)\pi_2\right) - \frac{1}{q}F(x)\right] \\ &= \frac{1}{\pi_2 - x}\left[\frac{1}{pq(\pi_2 - x)}\int_{px+(1-p)\pi_2}^{\pi_2} F(x)\, {}^{\pi_2}d_{p,q}x - \frac{1}{q}F(x)\right]. \end{aligned} \tag{23} $$

Similarly, from Definition 8, we have

$$ I_2 = \int_0^1 t\, {}_{\pi_1}D_{p,q}F(tx+(1-t)\pi_1)\, {}_0d_{p,q}t = \frac{1}{x-\pi_1}\left[\frac{1}{q}F(x) - \frac{1}{pq(x-\pi_1)}\int_{\pi_1}^{px+(1-p)\pi_1} F(x)\, {}_{\pi_1}d_{p,q}x\right]. \tag{24} $$

Thus, we obtain the resultant equality (22) by subtracting (23) from (24). □

Remark 7. In Lemma 2, if we set $p = 1$, then we obtain the equality:

$$ \begin{aligned} F(x) - \frac{1}{(\pi_2 - \pi_1)}\left[\int_{\pi_1}^{x} F(t)\, {}_{\pi_1}d_qt + \int_x^{\pi_2} F(t)\, {}^{\pi_2}d_qt\right] &= \frac{q(x-\pi_1)^2}{\pi_2 - \pi_1}\int_0^1 t\, {}_{\pi_1}D_qF(tx+(1-t)\pi_1)\, {}_0d_qt \\ &\quad - \frac{q(\pi_2 - x)^2}{\pi_2 - \pi_1}\int_0^1 t\, {}^{\pi_2}D_qF(tx+(1-t)\pi_2)\, {}^1d_qt, \end{aligned} $$

which is proved by Budak et al. in [33].

Remark 8. In Lemma 2, if we set $p = 1$ and later taking the limit as $q \to 1^-$, then we obtain [37, Lemma 1].

Theorem 8. *Assume that the mapping* $F : I \subset [0,\infty) \to \mathbb{R}$ *is differentiable and* $\pi_1, \pi_2 \in I$ *with* $\pi_1 < \pi_2$. *If* $|{}_{\pi_1}D_{p,q}F|$ *and* $|{}^{\pi_2}D_{p,q}F|$ *are s-convex mappings in the second sense, then the following inequality holds*:

$$ \left| F(x) - \frac{1}{p(\pi_2 - \pi_1)}\left[\int_{\pi_1}^{px+(1-p)\pi_1} F(t)\, {}_{\pi_1}d_{p,q}t + \int_{px+(1-p)\pi_2}^{\pi_2} F(t)\, {}^{\pi_2}d_{p,q}t\right]\right| \tag{25} $$

$$\leq \frac{q(x-\pi_1)^2}{\pi_2-\pi_1}\left[\frac{1}{[s+2]_{p,q}}|_{\pi_1}D_{p,q}F(x)| + \Theta_1|_{\pi_1}D_{p,q}F(\pi_1)|\right] + \frac{q(\pi_2-x)^2}{\pi_2-\pi_1}\left[\frac{1}{[s+2]_{p,q}}|^{\pi_2}D_{p,q}F(x)| + \Theta_2|^{\pi_2}D_{p,q}F(\pi_2)|\right],$$

where

$$\Theta_1 = \int_0^1 t(1-t)^s{}_0 d_{p,q}t$$

and

$$\Theta_2 = \int_0^1 t(1-t)^{s1} d_{p,q}t.$$

Proof. From Lemma 2 and properties of the modulus, we have

$$\left|F(x) - \frac{1}{p(\pi_2-\pi_1)}\left[\int_{\pi_1}^{px+(1-p)\pi_1} F(t)_{\pi_1}d_{p,q}t + \int_{px+(1-p)\pi_2}^{\pi_2} F(t)^{\pi_2}d_{p,q}t\right]\right| \leq \frac{q(x-\pi_1)^2}{\pi_2-\pi_1}\int_0^1 t|_{\pi_1}D_{p,q}F(tx+(1-t)\pi_1)|_0 d_{p,q}t + \frac{q(\pi_2-x)^2}{\pi_2-\pi_1}\int_0^1 t|^{\pi_2}D_{p,q}F(tx+(1-t)\pi_2)|^1 d_{p,q}t. \tag{26}$$

Since the mapping $|_{\pi_1}D_{p,q}F|$ and $|^{\pi_2}D_{p,q}F|$ are s-convexities in the second sense, therefore

$$\int_0^1 t|_{\pi_1}D_{p,q}F(tx+(1-t)\pi_1)|_0 d_{p,q}t \leq \int_0^1 t^{s+1}|_{\pi_1}D_{p,q}F(x)|_0 d_{p,q}t + \int_0^1 t(1-t)^s|_{\pi_1}D_{p,q}F(\pi_1)|_0 d_{p,q}t = \frac{1}{[s+2]_{p,q}}|_{\pi_1}D_{p,q}F(x)| + \Theta_1|_{\pi_1}D_{p,q}F(\pi_1)| \tag{27}$$

and

$$\int_0^1 t|^{\pi_2}D_{p,q}F(tx+(1-t)\pi_2)|^1 d_{p,q}t \leq \int_0^1 t^{s+1}|^{\pi_2}D_{p,q}F(x)|^1 d_{p,q}t + \int_0^1 t(1-t)^s|^{\pi_2}D_{p,q}F(\pi_2)|^1 d_{p,q}t = \frac{1}{[s+2]_{p,q}}|^{\pi_2}D_{p,q}F(x)| + \Theta_2|^{\pi_2}D_{p,q}F(\pi_2)|. \tag{28}$$

We obtain the resultant inequality (25) by putting (27) and (28) in (26). □

Corollary 1. *If we set* $s = 1$ *in* Theorem 8, *then we obtain the following new Ostrowski type inequality for convex functions*:

$$\left|F(x) - \frac{1}{p(\pi_2-\pi_1)}\left[\int_{\pi_1}^{px+(1-p)\pi_1} F(t)_{\pi_1}d_{p,q}t + \int_{px+(1-p)\pi_2}^{\pi_2} F(t)^{\pi_2}d_{p,q}t\right]\right| \leq \frac{q(x-\pi_1)^2}{\pi_2-\pi_1}\left[\frac{1}{[3]_{p,q}}|_{\pi_1}D_{p,q}F(x)| + \frac{[3]_{p,q}-[2]_{p,q}}{[3]_{p,q}[2]_{p,q}}|_{\pi_1}D_{p,q}F(\pi_1)|\right] + \frac{q(\pi_2-x)^2}{\pi_2-\pi_1}\left[\frac{1}{[3]_{p,q}}|^{\pi_2}D_{p,q}F(x)| + \frac{[3]_{p,q}-[2]_{p,q}}{[3]_{p,q}[2]_{p,q}}|^{\pi_2}D_{p,q}F(\pi_2)|\right].$$

Remark 9. In Theorem 8, if we set $p = 1$, then Theorem 8 reduces to [36, Theorem 4.1].

Remark 10. In Corollary 1, if we set $p = 1$, then we obtain the following inequality:

$$\left|F(x) - \frac{1}{\pi_2 - \pi_1}\left[\int_{\pi_1}^{x} F(t)_{\pi_1}d_q t + \int_{x}^{\pi_2} F(t)^{\pi_2}d_q t\right]\right| \le \frac{q}{(\pi_2 - \pi_1)[2]_q[3]_q}[(x - \pi_1)^2([2]_q|_{\pi_1}D_qF(x)| + q^2|_{\pi_1}D_qF(\pi_1)|) + (\pi_2 - x)^2([2]_q|^{\pi_2}D_qF(x)| + q^2|^{\pi_2}D_qF(\pi_2)|)],$$

which is given by Budak et al. in [33].

Corollary 2. *If we assume* $|_{\pi_1}D_{p,q}F(x)|, |_{\pi_1}D_{p,q}F(x)| \le M$ *in Theorem* 8, *then we have following post-quantum Ostrowski type inequality for s-convex functions in the second sense*:

$$\left|F(x) - \frac{1}{p(\pi_2 - \pi_1)}\left[\int_{\pi_1}^{px+(1-p)\pi_1} F(t)_{\pi_1}d_{p,q}t + \int_{px+(1-p)\pi_2}^{\pi_2} F(t)^{\pi_2}d_{p,q}t\right]\right| \le \frac{Mq(x - \pi_1)^2}{\pi_2 - \pi_1}\left[\frac{1}{[s + 2]_{p,q}} + \Theta_1\right] + \frac{Mq(\pi_2 - x)^2}{\pi_2 - \pi_1}\left[\frac{1}{[s + 2]_{p,q}} + \Theta_2\right]. \tag{29}$$

Remark 11. In Corollary 2, if we set $p = 1$, then Corollary 2 reduces to [36, Corollary 4.1].

Remark 12. If we set $s = p = 1$ in Corollary 2, then we recapture inequality (9).

Remark 13. In Corollary 2, if we set $p = 1$ and later take the limit as $q \to 1^-$, then Corollary 2 reduces to [38, Theorem 2].

Theorem 9. *Assume that the mapping* $F : I \subset [0, \infty) \to \mathbb{R}$ *is differentiable and* $\pi_1, \pi_2 \in I$ *with* $\pi_1 < \pi_2$. If $|_{\pi_1}D_{p,q}F|^{p_1}$ *and* $|^{\pi_2}D_{p,q}F|^{p_1}$, $p_1 \ge 1$ *are s-convex mappings in the second sense, then the following inequality holds*:

$$\begin{aligned}&\left|F(x) - \frac{1}{p(\pi_2 - \pi_1)}\left[\int_{\pi_1}^{px+(1-p)\pi_1} F(t)_{\pi_1}d_{p,q}t + \int_{px+(1-p)\pi_2}^{\pi_2} F(t)^{\pi_2}d_{p,q}t\right]\right| \\ &\le \frac{q}{\pi_2 - \pi_1}\left(\frac{1}{[2]_{p,q}}\right)^{1-\frac{1}{p_1}}\left[(x - \pi_1)^2\left(\frac{1}{[s + 2]_{p,q}}|_{\pi_1}D_{p,q}F(x)|^{p_1} + \Theta_1|_{\pi_1}D_{p,q}F(\pi_1)|^{p_1}\right)^{\frac{1}{p_1}}\right. \\ &\left.+ (\pi_2 - x)^2\left(\frac{1}{[s + 2]_{p,q}}|^{\pi_2}D_{p,q}F(x)|^{p_1} + \Theta_2|^{\pi_2}D_{p,q}F(\pi_2)|^{p_1}\right)^{\frac{1}{p_1}}\right].\end{aligned} \tag{30}$$

Proof. From Lemma 2, by using properties of the modulus and power mean inequality, we have

$$\left|F(x) - \frac{1}{p(\pi_2 - \pi_1)}\left[\int_{\pi_1}^{px+(1-p)\pi_1} F(t)_{\pi_1}d_{p,q}t + \int_{px+(1-p)\pi_2}^{\pi_2} F(t)^{\pi_2}d_{p,q}t\right]\right| \le \frac{q(x - \pi_1)^2}{\pi_2 - \pi_1}\int_0^1 t|_{\pi_1}D_{p,q}F(tx + (1 - t)\pi_1)|_0 d_{p,q}t + \frac{q(\pi_2 - x)^2}{\pi_2 - \pi_1}\int_0^1 t|^{\pi_2}D_{p,q}F(tx + (1 - t)\pi_2)|^1 d_{p,q}t \tag{31}$$

$$\le \frac{q(x-\pi_1)^2}{\pi_2-\pi_1}\left(\int_0^1 t_0 d_{p,q}t\right)^{1-\frac{1}{p_1}}\left(\int_0^1 t\,|_{\pi_1}D_{p,q}F(tx+(1-t)\pi_1)|^{p_1}{}_0d_{p,q}t\right)^{\frac{1}{p_1}}$$
$$+\frac{q(\pi_2-x)^2}{\pi_2-\pi_1}\left(\int_0^1 t^1 d_{p,q}t\right)^{1-\frac{1}{p_1}}\left(\int_0^1 t\,|^{\pi_2}D_{p,q}F(tx+(1-t)\pi_2)|^{p_1}{}^1d_{p,q}t\right)^{\frac{1}{p_1}}. \tag{31}$$

Since the mapping $|_{\pi_1}D_{p,q}F|^{p_1}$ and $|^{\pi_2}D_{p,q}F|^{p_1}$ are s-convexities in the second sense, therefore

$$\left(\int_0^1 t_0 d_{p,q}t\right)^{1-\frac{1}{p_1}}\left(\int_0^1 t\,|_{\pi_1}D_{p,q}F(tx+(1-t)\pi_1)|^{p_1}{}_0d_{p,q}t\right)^{\frac{1}{p_1}}$$
$$\le\left(\frac{1}{[2]_{p,q}}\right)^{1-\frac{1}{p_1}}\left(\frac{1}{[s+2]_{p,q}}\,|_{\pi_1}D_{p,q}F(x)|^{p_1}+\Theta_1|_{\pi_1}D_{p,q}F(\pi_1)|^{p_1}\right)^{\frac{1}{p_1}} \tag{32}$$

and

$$\left(\int_0^1 t^1 d_{p,q}t\right)^{1-\frac{1}{p_1}}\left(\int_0^1 t\,|^{\pi_2}D_{p,q}F(tx+(1-t)\pi_2)|^{p_1}{}^1d_{p,q}t\right)^{\frac{1}{p_1}}$$
$$\le\left(\frac{1}{[2]_{p,q}}\right)^{1-\frac{1}{p_1}}\left(\frac{1}{[s+2]_{p,q}}\,|^{\pi_2}D_{p,q}F(x)|^{p_1}+\Theta_2|^{\pi_2}D_{p,q}F(\pi_2)|^{p_1}\right)^{\frac{1}{p_1}}. \tag{33}$$

We obtain the resultant inequality (30) by putting (32) and (33) in (31). □

Corollary 3. *If we set* $s = 1$ *in* Theorem 9, *then we obtain the following new Ostrowski type inequality for convex functions*:

$$\left|F(x)-\frac{1}{p(\pi_2-\pi_1)}\left[\int_{\pi_1}^{px+(1-p)\pi_1}F(t)\,{}_{\pi_1}d_{p,q}t+\int_{px+(1-p)\pi_2}^{\pi_2}F(t)\,{}^{\pi_2}d_{p,q}t\right]\right|$$
$$\le\frac{q}{\pi_2-\pi_1}\left(\frac{1}{[2]_{p,q}}\right)^{1-\frac{1}{p_1}}\left[(x-\pi_1)^2\left(\frac{1}{[3]_{p,q}}\,|_{\pi_1}D_{p,q}F(x)|^{p_1}+\frac{[3]_{p,q}-[2]_{p,q}}{[3]_{p,q}[2]_{p,q}}\,|_{\pi_1}D_{p,q}F(\pi_1)|^{p_1}\right)^{\frac{1}{p_1}}\right.$$
$$\left.+(\pi_2-x)^2\left(\frac{1}{[3]_{p,q}}\,|^{\pi_2}D_{p,q}F(x)|^{p_1}+\frac{[3]_{p,q}-[2]_{p,q}}{[3]_{p,q}[2]_{p,q}}\,|^{\pi_2}D_{p,q}F(\pi_2)|^{p_1}\right)^{\frac{1}{p_1}}\right].$$

Remark 14. In Theorem 9, if we set $p = 1$, then Theorem 9 reduces to [36, Theorem 4.2].

Remark 15. In Corollary 3, if we set $p = 1$, then we obtain the following inequality:

$$\left|F(x)-\frac{1}{\pi_2-\pi_1}\left[\int_{\pi_1}^{x}F(t)\,{}_{\pi_1}d_qt+\int_{x}^{\pi_2}F(t)\,{}^{\pi_2}d_qt\right]\right|$$
$$\le\frac{q}{(\pi_2-\pi_1)[2]_q}\left[(x-\pi_1)^2\left(\frac{[2]_q|_{\pi_1}D_qF(x)|^p+q^2\,|_{\pi_1}D_qF(\pi_1)|^p}{[3]_q}\right)^{\frac{1}{p}}\right.$$
$$\left.+(\pi_2-x)^2\left(\frac{[2]_q|^{\pi_2}D_qF(x)|^p+q^2\,|^{\pi_2}D_qF(\pi_2)|^p}{[3]_q}\right)^{\frac{1}{p}}\right],$$

which is proved by Budak et al. in [33].

Corollary 4. *If we assume* $|{}_{\pi_1}D_{p,q}\mathrm{F}(x)|, |{}_{\pi_1}D_{p,q}\mathrm{F}(x)| \leq M$ *in* Theorem 9, *then we have following post-quantum Ostrowski type inequality for s-convex functions in the second sense*:

$$\left|\mathrm{F}(x) - \frac{1}{p(\pi_2 - \pi_1)}\left[\int\limits_{\pi_1}^{px+(1-p)\pi_1} \mathrm{F}(t)\,{}_{\pi_1}d_{p,q}t + \int\limits_{px+(1-p)\pi_2}^{\pi_2} \mathrm{F}(t)\,{}^{\pi_2}d_{p,q}t\right]\right|$$
$$\leq \frac{Mq}{\pi_2 - \pi_1}\left(\frac{1}{[2]_{p,q}}\right)^{1-\frac{1}{p_1}}\left[(x-\pi_1)^2\left(\frac{1}{[s+2]_{p,q}} + \Theta_1\right)^{\frac{1}{p_1}} + (\pi_2 - x)^2\left(\frac{1}{[s+2]_{p,q}} + \Theta_2\right)^{\frac{1}{p_1}}\right].$$

Remark 16. In Corollary 4, if we set $p = 1$, then Corollary 4 reduces to [36, Corollary 4.2].

Remark 17. In Corollary 4, if we set $p = 1$ and later take the limit as $q \to 1^-$, then Corollary 4 reduces to [38, Theorem 4].

Theorem 10. *Assume that the mapping* $\mathrm{F} : I \subset [0, \infty) \to \mathbb{R}$ *is differentiable and* $\pi_1, \pi_2 \in I$ *with* $\pi_1 < \pi_2$. *If* $|{}_{\pi_1}D_{p,q}\mathrm{F}|^{p_1}$ *and* $|{}^{\pi_2}D_{p,q}\mathrm{F}|^{p_1}$, $p_1 > 1$ *are s-convex mappings in the second sense, then the following inequality holds*:

$$\left|\mathrm{F}(x) - \frac{1}{p(\pi_2 - \pi_1)}\left[\int\limits_{\pi_1}^{px+(1-p)\pi_1} \mathrm{F}(t)\,{}_{\pi_1}d_{p,q}t + \int\limits_{px+(1-p)\pi_2}^{\pi_2} \mathrm{F}(t)\,{}^{\pi_2}d_{p,q}t\right]\right|$$
$$\leq \frac{q}{\pi_2 - \pi_1}\left(\frac{1}{[r_1+1]_{p,q}}\right)^{\frac{1}{r_1}}\left[(x-\pi_1)^2\left(\frac{1}{[s+1]_{p,q}}(|{}_{\pi_1}D_{p,q}\mathrm{F}(x)|^{p_1} + |{}_{\pi_1}D_{p,q}\mathrm{F}(\pi_1)|^{p_1})\right)^{\frac{1}{p_1}}\right. \tag{34}$$
$$\left. + (\pi_2 - x)^2\left(\frac{1}{[s+1]_{p,q}}(|{}^{\pi_2}D_{p,q}\mathrm{F}(x)|^{p_1} + |{}^{\pi_2}D_{p,q}\mathrm{F}(\pi_2)|^{p_1})\right)^{\frac{1}{p_1}}\right],$$

where $r_1^{-1} + p_1^{-1} = 1$.

Proof. From Lemma 2, by using properties of the modulus and Hölder's inequality, we have

$$\left|\mathrm{F}(x) - \frac{1}{p(\pi_2 - \pi_1)}\left[\int\limits_{\pi_1}^{px+(1-p)\pi_1} \mathrm{F}(t)\,{}_{\pi_1}d_{p,q}t + \int\limits_{px+(1-p)\pi_2}^{\pi_2} \mathrm{F}(t)\,{}^{\pi_2}d_{p,q}t\right]\right|$$
$$\leq \frac{q(x-\pi_1)^2}{\pi_2 - \pi_1}\int\limits_0^1 t|{}_{\pi_1}D_{p,q}\mathrm{F}(tx + (1-t)\pi_1)|\,{}_0d_{p,q}t + \frac{q(\pi_2 - x)^2}{\pi_2 - \pi_1}\int\limits_0^1 t|{}^{\pi_2}D_{p,q}\mathrm{F}(tx + (1-t)\pi_2)|\,{}^1d_{p,q}t$$
$$\leq \frac{q(x-\pi_1)^2}{\pi_2 - \pi_1}\left(\int\limits_0^1 t^{r_1}\,{}_0d_{p,q}t\right)^{\frac{1}{r_1}}\left(\int\limits_0^1 |{}_{\pi_1}D_{p,q}\mathrm{F}(tx + (1-t)\pi_1)|^{p_1}\,{}_0d_{p,q}t\right)^{\frac{1}{p_1}} \tag{35}$$
$$+ \frac{q(\pi_2 - x)^2}{\pi_2 - \pi_1}\left(\int\limits_0^1 t^{r_1}\,{}^1d_{p,q}t\right)^{\frac{1}{r_1}}\left(\int\limits_0^1 |{}^{\pi_2}D_{p,q}\mathrm{F}(tx + (1-t)\pi_2)|^{p_1}\,{}^1d_{p,q}t\right)^{\frac{1}{p_1}}.$$

Since the mapping $|{}_{\pi_1}D_{p,q}\mathrm{F}|^{p_1}$ and $|{}^{\pi_2}D_{p,q}\mathrm{F}|^{p_1}$ are s-convexities in the second sense, therefore

$$\left(\int\limits_0^1 t^{r_1}\,{}_0d_{p,q}t\right)^{\frac{1}{r_1}}\left(\int\limits_0^1 |{}_{\pi_1}D_{p,q}\mathrm{F}(tx + (1-t)\pi_1)|^{p_1}\,{}_0d_{p,q}t\right)^{\frac{1}{p_1}}$$
$$\leq \left(\frac{1}{[r_1+1]_{p,q}}\right)^{\frac{1}{r_1}}\left(\frac{1}{[s+1]_{p,q}}(|{}_{\pi_1}D_{p,q}\mathrm{F}(x)|^{p_1} + |{}_{\pi_1}D_{p,q}\mathrm{F}(\pi_1)|^{p_1})\right) \tag{36}$$

and

$$\left(\int_0^1 t^{r_1 1} d_{p,q}t\right)^{\frac{1}{r_1}}\left(\int_0^1 |^{\pi_2}D_{p,q}\mathsf{F}(tx+(1-t)\pi_2)|^{p_1 1} d_{p,q}t\right)^{\frac{1}{p_1}} \le \left(\frac{1}{[r_1+1]_{p,q}}\right)^{\frac{1}{r_1}}\left(\frac{1}{[s+1]_{p,q}}(|^{\pi_2}D_{p,q}\mathsf{F}(x)|^{p_1}+|^{\pi_2}D_{p,q}\mathsf{F}(\pi_2)|^{p_1})\right). \tag{37}$$

We obtain the resultant inequality (34) by putting (36) and (37) in (35). □

Corollary 5. *If we set* $s = 1$ *in* Theorem 10, *then we obtain the following new Ostrowski type inequality for convex functions*:

$$\left|\mathsf{F}(x)-\frac{1}{p(\pi_2-\pi_1)}\left[\int_{\pi_1}^{px+(1-p)\pi_1}\mathsf{F}(t)_{\pi_1}d_{p,q}t+\int_{px+(1-p)\pi_2}^{\pi_2}\mathsf{F}(t)^{\pi_2}d_{p,q}t\right]\right| \le \frac{q}{\pi_2-\pi_1}\left(\frac{1}{[r_1+1]_{p,q}}\right)^{\frac{1}{r_1}}\left[(x-\pi_1)^2\left(\frac{1}{[3]_{p,q}}(|_{\pi_1}D_{p,q}\mathsf{F}(x)|^{p_1}+|_{\pi_1}D_{p,q}\mathsf{F}(\pi_1)|^{p_1})\right)^{\frac{1}{p_1}} + (\pi_2-x)^2\left(\frac{1}{[3]_{p,q}}(|^{\pi_2}D_{p,q}\mathsf{F}(x)|^{p_1}+|^{\pi_2}D_{p,q}\mathsf{F}(\pi_2)|^{p_1})\right)^{\frac{1}{p_1}}\right].$$

Remark 18. In Theorem 9, if we set $p = 1$, then Theorem 9 reduces to [36, Theorem 4.3].

Remark 19. In Corollary 5, if we set $p = 1$, then we obtain the following inequality:

$$\left|\mathsf{F}(x)-\frac{1}{\pi_2-\pi_1}\left[\int_{\pi_1}^{x}\mathsf{F}(t)_{\pi_1}d_qt+\int_{x}^{\pi_2}\mathsf{F}(t)^{\pi_2}d_qt\right]\right| \le \frac{q}{\pi_2-\pi_1}\left(\frac{1}{[r_1+1]_q}\right)^{\frac{1}{r_1}}\left[(x-\pi_1)^2\left(\frac{|_{\pi_1}D_q\mathsf{F}(x)|^p+q\,|_{\pi_1}D_q\mathsf{F}(\pi_1)|^p}{[2]_q}\right)^{\frac{1}{p}}+(\pi_2-x)^2\left(\frac{|^{\pi_2}D_q\mathsf{F}(x)|^p+q\,|^{\pi_2}D_q\mathsf{F}(\pi_2)|^p}{[2]_q}\right)\right]^{\frac{1}{p}},$$

which is proved by Budak et al. in [33].

Corollary 6. *If we assume* $|_{\pi_1}D_{p,q}\mathsf{F}(x)|, |_{\pi_1}D_{p,q}\mathsf{F}(x)| \le M$ *in* Theorem 10, *then we have following post-quantum Ostrowski type inequality for s-convex functions in the second sense*:

$$\left|\mathsf{F}(x)-\frac{1}{p(\pi_2-\pi_1)}\left[\int_{\pi_1}^{px+(1-p)\pi_1}\mathsf{F}(t)_{\pi_1}d_{p,q}t+\int_{px+(1-p)\pi_2}^{\pi_2}\mathsf{F}(t)^{\pi_2}d_{p,q}t\right]\right| \le \frac{Mq}{\pi_2-\pi_1}\left(\frac{1}{[r_1+1]_{p,q}}\right)^{\frac{1}{r_1}}\left(\frac{2}{[s+1]_{p,q}}\right)^{\frac{1}{p_1}}[(x-\pi_1)^2+(\pi_2-x)^2]. \tag{38}$$

Remark 20. In Corollary 6, if we set $p = 1$, then Corollary 6 reduces to [36, Corollary 4.3].

Remark 21. In Corollary 6, if we set $p = 1$ and later take the limit as $q \to 1^-$, then Corollary 6 reduces to [38, Theorem 3].

6 Applications to special means

For arbitrary positive numbers $\pi_1, \pi_2 (\pi_1 \neq \pi_2)$, we consider the means as follows:

1. The arithmetic mean

$$\mathcal{A} = \mathcal{A}(\pi_1, \pi_2) = \frac{\pi_1 + \pi_2}{2}.$$

2. The logarithmic mean

$$\mathcal{L}_\sigma^\sigma = \mathcal{L}_\sigma^\sigma(\pi_1, \pi_2) = \frac{\pi_2^{\sigma+1} - \pi_1^{\sigma+1}}{(\sigma + 1)(\pi_2 - \pi_1)}.$$

Proposition 1. *For* $0 < \pi_1 < \pi_2$ *and* $0 < q < p \leq 1$, *the following inequality is true*:

$$\left|\frac{1}{s+1}[\mathcal{A}^{s+1}(\pi_1, \pi_2) - \mathcal{A}(\Bbbk_1, \Bbbk_2)]\right| \leq \frac{q(\pi_2 - \pi_1)}{2}\left[\frac{1}{[s+2]_{p,q}}\left\{\mathcal{L}_s^s\left(q\left(\frac{\pi_2 - \pi_1}{2}\right) + \pi_1, p\left(\frac{\pi_2 - \pi_1}{2}\right) + \pi_1\right)\right.\right.$$
$$\left.\left. + \mathcal{L}_s^s\left(\pi_2 - q\left(\frac{\pi_2 - \pi_1}{2}\right), \pi_2 - p\left(\frac{\pi_2 - \pi_1}{2}\right)\right)\right\} + \Theta_1\mathcal{A}(\pi_1^s, \pi_2^s) + \Theta_2\mathcal{A}(\pi_1^s, \pi_2^s)\right],$$

where

$$\Bbbk_1 = (p - q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^n}\left(\frac{\pi_2 - \pi_1}{2}\right) + \pi_1\right)^{s+1},$$
$$\Bbbk_2 = (p - q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\pi_2 - \frac{q^n}{p^n}\left(\frac{\pi_2 - \pi_1}{2}\right)\right)^{s+1}.$$

Proof. The inequality (25) in Theorem 8 with $x = \frac{\pi_1 + \pi_2}{2}$ for $\digamma(x) = \frac{x^{s+1}}{s+1}$, where $x > 0$ leads to this conclusion. □

Proposition 2. *For* $0 < \pi_1 < \pi_2$ *and* $0 < q < p \leq 1$, *the following inequality is true*:

$$\left|\frac{1}{s+1}[\mathcal{A}^{s+1}(\pi_1, \pi_2) - \mathcal{A}(\Bbbk_1, \Bbbk_2)]\right| \leq \frac{Mq(\pi_2 - \pi_1)}{4}\left[\frac{2}{[s+2]_{p,q}} + \Theta_1 + \Theta_2\right].$$

Proof. The inequality (29) in Corollary 2 with $x = \frac{\pi_1 + \pi_2}{2}$ for $\digamma(x) = \frac{x^{s+1}}{s+1}$, where $x > 0$ leads to this conclusion. □

Proposition 3. *For* $0 < \pi_1 < \pi_2$ *and* $0 < q < p \leq 1$, *the following inequality is true*:

$$\left|\frac{1}{s+1}[\mathcal{A}^{s+1}(\pi_1, \pi_2) - \mathcal{A}(\Bbbk_1, \Bbbk_2)]\right|$$
$$\leq \frac{q(\pi_2 - \pi_1)}{2}\left(\frac{1}{[2]_{p,q}}\right)^{1-\frac{1}{p_1}}\left[\left(\frac{1}{[s+2]_{p,q}}\left|\mathcal{L}_s^s\left(q\left(\frac{\pi_2 - \pi_1}{2}\right) + \pi_1, p\left(\frac{\pi_2 - \pi_1}{2}\right) + \pi_1\right)\right|^{p_1} + \Theta_1 |\pi_1^s|^{p_1}\right)^{\frac{1}{p_1}}\right.$$
$$\left. + \left(\frac{1}{[s+2]_{p,q}}\left|\mathcal{L}_s^s\left(\pi_2 - q\left(\frac{\pi_2 - \pi_1}{2}\right), \pi_2 - p\left(\frac{\pi_2 - \pi_1}{2}\right)\right)\right|^{p_1} + \Theta_2 |\pi_2^s|^{p_1}\right)^{\frac{1}{p_1}}\right].$$

Proof. The inequality (30) in Theorem 9 with $x = \frac{\pi_1 + \pi_2}{2}$ for $\digamma(x) = \frac{x^{s+1}}{s+1}$, where $x > 0$ leads to this conclusion. □

Proposition 4. *For* $0 < \pi_1 < \pi_2$ *and* $0 < q < p \le 1$, *the following inequality is true*:

$$\left|\frac{1}{s+1}[\mathcal{A}^{s+1}(\pi_1,\pi_2) - \mathcal{A}(\Bbbk_1,\Bbbk_2)]\right|$$
$$\le \frac{q(\pi_2-\pi_1)}{2}\left(\frac{1}{[r_1+1]_{p,q}}\right)^{\frac{1}{r_1}}\left[\left(\frac{1}{[s+1]_{p,q}}\left(\left|\mathcal{L}_s^s\left(q\left(\frac{\pi_2-\pi_1}{2}\right)+\pi_1, p\left(\frac{\pi_2-\pi_1}{2}\right)+\pi_1\right)\right|^{p_1} + |\pi_1^s|^{p_1}\right)\right)^{\frac{1}{p_1}}\right.$$
$$\left. + \left(\frac{1}{[s+1]_{p,q}}\left(\left|\mathcal{L}_s^s\left(\pi_2 - q\left(\frac{\pi_2-\pi_1}{2}\right), \pi_2 - p\left(\frac{\pi_2-\pi_1}{2}\right)\right)\right|^{p_1} + |\pi_2^s|^{p_1}\right)\right)^{\frac{1}{p_1}}\right].$$

Proof. The inequality (34) in Theorem 10 with $x = \frac{\pi_1+\pi_2}{2}$ for $\digamma(x) = \frac{x^{s+1}}{s+1}$, where $x > 0$ leads to this conclusion. □

Proposition 5. *For* $0 < \pi_1 < \pi_2$ *and* $0 < q < p \le 1$, *the following inequality is true*:

$$\left|\frac{1}{s+1}[\mathcal{A}^{s+1}(\pi_1,\pi_2) - \mathcal{A}(\Bbbk_1,\Bbbk_2)]\right| \le \frac{Mq(\pi_2-\pi_1)}{2}\left(\frac{1}{[r_1+1]_{p,q}}\right)^{\frac{1}{r_1}}\left(\frac{2}{[s+1]_{p,q}}\right)^{\frac{1}{p_1}}.$$

Proof. The inequality (38) in Corollary 6 with $x = \frac{\pi_1+\pi_2}{2}$ for $\digamma(x) = \frac{x^{s+1}}{s+1}$, where $x > 0$ leads to this conclusion. □

7 Conclusion

In this work, we proved some new variants of post-quantum Hermite-Hadamard and Ostrowski type inequalities using the (p, q)-differentiable s-convex functions in the second sense. We also proved that the newly established results are strong generalizations of the related existing results. Finally, we presented various applications based on the newly established inequalities to demonstrate the utility of our findings. It is a new and interesting problem that upcoming researchers can obtain similar inequalities for different kinds of convexity in their future work.

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References

[1] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[2] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.

[3] H. Hudzik and L. Maligranda, *Some remarks on* s*-convex functions*, Aequationes Math. **48** (1994), 100–111

[4] S. S. Dragomir and S. Fitzpatrick, *The Hadamard inequalities for* s*-convex functions in the second sense*, Demonstr. Math. **32** (1999), 68–696.

[5] T. Ernst, *A Comprehensive Treatment of* q*-Calculus*, *Springer*, Basel, 2012.

[6] V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, 2002.

[7] T. Ernst, *The History of* q*-Calculus and New Method*, Department of Mathematics, Uppsala University, Sweden, 2000.

[8] F. H. Jackson, *On a q-definite integrals*, Quarterly J. Pure Appl. Math. **41** (1910), 193–203.

[9] W. Al-Salam, *Some fractional q-integrals and q-derivatives*, Proc. Edinburgh Math. Soc. **15** (1966), 135–140.

[10] J. Tariboon and S. K. Ntouyas, *Quantum calculus on finite intervals and applications to impulsive difference equations*, Adv. Differ. Equ. **2013** (2013), 282.

[11] S. Bermudo, P. Kórus, and J. N. Valdés, *On q-Hermite-Hadamard inequalities for general convex functions*, Acta Math. Hungar. **162** (2020), 364–374.

[12] P. N. Sadjang, *On the fundamental theorem of* (p, q)*-calculus and some* (p, q)*-Taylor formulas*, Results Math. **73** (2018), 1–21.

[13] J. Soontharanon and T. Sitthiwirattham, *On Fractional* (p, q)*-Calculus*, Adv. Differ. Equ. **2020** (2020), 1–18.

[14] M. Tunç and E. Göv, *Some integral inequalities via* (p, q)*-calculus on finite intervals*, RGMIA Res. Rep. Coll. **19** (2016), 1–12.

[15] M. Vivas-Cortez, M. A. Ali, H. Budak, H. Kalsoom, and P. Agarwal, *Some new Hermite-Hadamard and related inequalities for convex functions via* (p, q)*-integral*, Entropy **23** (2021), 828.

[16] M. A. Ali, H. Budak, H. M. Abbas, and Y.-M. Chu, *Quantum Hermite–Hadamard-type inequalities for functions with convex absolute values of second* q^{π_2}*-derivatives*, Adv. Differ. Equ. **2021** (2021), 1–12.

[17] M. A. Ali, N. Alp, H. Budak, Y-M. Chu, and Z. Zhang, *On some new quantum midpoint type inequalities for twice quantum differentiable convex functions*, Open Math. **19** (2021), 427–439.

[18] N. Alp, M. Z. Sarikaya, M. Kunt, and İ. İşcan, *q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions*, J. King Saud Univ. Sci. **30** (2018), 193–203.

[19] N. Alp and M. Z. Sarikaya, *Hermite Hadamardas type inequalities for co-ordinated convex functions on quantum integral*, Appl. Math. E-Notes **20** (2020), 341–356.

[20] H. Budak, *Some trapezoid and midpoint type inequalities for newly defined quantum integrals*, Proyecciones **40** (2021), 199–215.

[21] H. Budak, M. A. Ali, and M. Tarhanaci, *Some new quantum Hermite-Hadamard-like inequalities for coordinated convex functions*, J. Optim. Theory Appl. **186** (2020), 899–910.

[22] S. Jhanthanam, J. Tariboon, S. K. Ntouyas, and K. Nonlaopon, *On q-Hermite-Hadamard inequalities for differentiable convex functions*, Mathematics **7** (2019), 632.

[23] M. A. Noor, K. I. Noor, and M. U. Awan, *Some quantum estimatesfor Hermite-Hadamard inequalities*, Appl. Math. Comput. **251** (2015), 675–679.

[24] M. A. Noor, K. I. Noor, and M. U. Awan, *Some quantum integral inequalities via preinvex functions*, Appl. Math. Comput. **269** (2015), 242–251.

[25] E. R. Nwaeze and A. M. Tameru, *New parameterized quantumintegral inequalities via* η*-quasiconvexity*, Adv. Differ. Equ. **2019** (2019), 1–12.

[26] M. A. Khan, M. Noor, E. R. Nwaeze, and Y-M. Chu, *Quantum Hermite-Hadamard inequality by means of a Green function*, Adv. Differ. Equ. **2020** (2020), 1–20.

[27] H. Budak, S. Erden, and M. A. Ali, *Simpson and Newton type in equalities for convex functions via newly defined quantum integrals*, Math. Methods Appl. Sci. **44** (2020), 378–390.

[28] M. A. Ali, H. Budak, Z. Zhang, and H. Yildrim, *Some new Simpson's type inequalities for co-ordinated convex functions in quantum calculus*, Math. Methods Appl. Sci. **44** (2021), 4515–4540.

[29] M. A. Ali, M. Abbas, H. Budak, P. Agarwal, G. Murtaza, and Y.-M. Chu, *New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions*, Adv. Differ. Equ. **2021** (2021), 1–21.

[30] M. Vivas-Cortez, M. A. Ali, A. Kashuri, I. B. Sial, and Z. Zhang, *Some new Newton's type integral inequalities for co-ordinated convex functions in quantum calculus*, Symmetry **12** (2020), 1476.

[31] M. A. Ali, Y.-M. Chu, H. Budak, A. Akkurt, and H. Yildrim, *Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables*, Adv. Differ. Equ. **2021** (2021), 1–26.

[32] M. A. Ali, H. Budak, A. Akkurt, and Y-M. Chu, *Quantum Ostrowskitype inequalities for twice quantum differentiable functions in quantum calculus*, Open Math. **19** (2021), 440–449.

[33] H. Budak, M. A. Ali, N. Alp, and Y.-M. Chu, *Quantum Ostrowski type integral inequalities*, J. Math. Inequal. **2021** (2021).

[34] M. Kunt, İ. İşcan, N. Alp, and M. Z. Sarikaya, (p, q)-*Hermite-Hadamard inequalities and* (p, q)-*estimates for midpoint inequalities via convex quasi-convex functions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM. **112** (2018), 969–992.

[35] M. A. Latif, M. Kunt, S. S. Dragomir, and İ. İşcan, *Postquantum trapezoid type inequalities*, AIMS Math. **5** (2020), 4011–4026.

[36] S. Asawasamrit, M. A. Ali, H. Budak, S. K. Ntouyas, and J. Tariboon, *Quantum Hermite-Hadamard and quantum Ostrowski type inequalities for* s-*convex functions in the second sense with applications*, AIMS Math. **6** (2021), 13327–13346.

[37] P. Cerone and S. S. Dragomir, *Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions*, Demonstratio Math. **37** (2004), 299–308.

[38] M. Alomari, M. Daus, and S. S. Dragomir, *Ostrowski type inequalities for functions whose derivatives are* s-*convex in the second sense*, Appl. Math. Lett. **23** (2010), 1071–1076.