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On some new inequalities for differentiable co-ordinated convex functions

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Abstract

Several new inequalities for differentiable co-ordinated convex and concave functions in two variables which are related to the left side of Hermite- Hadamard type inequality for co-ordinated convex functions in two variables are obtained.

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1. Introduction

The following definition is well known in literature:

A function f: $I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Many important inequalities have been established for the class of convex functions, but the most famous is the Hermite-Hadamard's inequality (see for instance [1]). This double inequality is stated as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where f:$I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ a convex function, $a, b \in I$ with $a < b$. The inequalities in (1.1) are in reversed order if f is a concave function.

The inequalities (1.1) have become an important cornerstone in mathematical analysis and optimization and many uses of these inequalities have been discovered in a variety of settings. Moreover, many inequalities of special means can be obtained for a particular choice of the function f. Due to the rich geometrical significance of Hermite-Hadamard's inequality (1.1), there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example [2-5] and the references therein.

Let us consider now a bidimensional interval $\Delta =: [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$, a mapping f: $\Delta \to \mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x, y) + (1-\lambda)f(z, w),$$

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holds for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$.

A modification for convex functions on Δ, which are also known as co-ordinated convex functions, was introduced by Dragomir [6,7] as follows:

A function $f: \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y: [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x: [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are convex where defined for all $x \in [a, b]$, $y \in [c, d]$.

A formal definition for co-ordinated convex functions may be stated as follows:

Definition 1. [8]*A function $f: \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the inequality*

$$\begin{aligned} & f(tx + (1-t)y, su + (1-s)w) \\ & \quad \le ts\, f(x, u) + t(1-s)f(x, w) + s(1-t)f(y, u) + (1-t)(1-s)f(y, w), \end{aligned}$$

holds for all $t, s \in [0, 1]$ and $(x, u), (y, w) \in \Delta$.

Clearly, every convex mapping $f: \Delta \to \mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex, (see for example [6,7]). For recent results on co-ordinated convex functions we refer the interested reader to [6,8-13].

The following Hermite-Hadamrd type inequality for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ was also proved in [6]:

Theorem 1. [6]*Suppose that $f: \Delta \to \mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ & \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x, y)dydx \\ & \le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x, c)dx + \frac{1}{b-a}\int_a^b f(x, d)dx \right. \\ & \left. + \frac{1}{d-c}\int_c^d f(a, y)dy + \frac{1}{d-c}\int_c^d f(b, y)dy\right] \\ & \le \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned} \tag{1.2}$$

The above inequalities are sharp.

In a recent article [13], Sarikaya et al. proved some new inequalities that give estimate of the difference between the middle and the rightmost terms in (1.2) for differentiable co-ordinated convex functions on rectangle from the plane $\mathbb{R}^2$. Motivated by notion given in [13], in the present article, we prove some new inequalities which give estimate between the middle and the leftmost terms in (1.2) for differentiable co-ordinated convex functions on rectangle from the plane $\mathbb{R}^2$.

2. Main results

The following lemma is necessary and plays an important role in establishing our main results:

Lemma 1. *Let f*: $\Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta := [a, b] \times [c, d]$ *with* $a < b$, $c < d$. *If* $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$, *then the following identity holds:*

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx + f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\&-\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx - \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \qquad (2.1)\\&= (b-a)(d-c)\int_0^1\int_0^1 K(t,s)\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)dsdt,\end{aligned}$$

where

$$K(t,s) = \begin{cases} ts, & (t,s) \in \left[0,\frac{1}{2}\right] \times \left[0,\frac{1}{2}\right] \\ t(s-1), & (t,s) \in \left[0,\frac{1}{2}\right] \times \left(\frac{1}{2},1\right] \\ s(t-1), & (t,s) \in \left(\frac{1}{2},1\right] \times \left[0,\frac{1}{2}\right] \\ (t-1)(s-1), & (t,s) \in \left(\frac{1}{2},1\right] \times \left(\frac{1}{2},1\right] \end{cases}$$

Proof. Since

$$\begin{aligned}&(b-a)(d-c)\int_0^1\int_0^1 K(t,s)\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)dsdt\\&= (b-a)(d-c)\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} ts\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)dsdt\\&+(b-a)(d-c)\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 t(s-1)\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)dsdt \qquad (2.2)\\&+(b-a)(d-c)\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} s(t-1)\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)dsdt\\&+(b-a)(d-c)\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 (t-1)(s-1)\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)dsdt\\&= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Now by integration by parts, we have

$$
\begin{aligned}
I_1 &= (b-a)(d-c)\int_0^{\frac{1}{2}} t\left[\int_0^{\frac{1}{2}} s\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)ds\right]dt \\
&= \frac{1}{4}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{2}\int_0^{\frac{1}{2}} f\left(ta+(1-t)b,\frac{c+d}{2}\right)dt \qquad (2.3)\\
&\quad -\frac{1}{2}\int_0^{\frac{1}{2}} f\left(\frac{a+b}{2}, sc+(1-s)d\right)ds+\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} f(ta+(1-t)b, sc+(1-s)d)dsdt.
\end{aligned}
$$

If we make use of the substitutions $x = ta + (1 - t)b$ and $y = sc + (1 - s)d$, $(t, s) \in [0, 1]^2$, in (2.3), we observe that

$$
\begin{aligned}
I_1 &= \frac{1}{4}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{2(b-a)}\int_{\frac{a+b}{2}}^{b} f\left(x,\frac{c+d}{2}\right)dx \\
&\quad -\frac{1}{2(d-c)}\int_{\frac{c+d}{2}}^{d} f\left(\frac{a+b}{2},y\right)dy+\frac{1}{(b-a)(d-c)}\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)dydx.
\end{aligned}
$$

Similarly, by integration by parts, we also have that

$$
\begin{aligned}
I_2 &= \frac{1}{4}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{2(b-a)}\int_{\frac{a+b}{2}}^{b} f\left(x,\frac{c+d}{2}\right)dx \\
&\quad -\frac{1}{2(d-c)}\int_{c}^{\frac{c+d}{2}} f\left(\frac{a+b}{2},y\right)dy+\frac{1}{(b-a)(d-c)}\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}} f(x,y)dydx,
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \frac{1}{4}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{2(b-a)}\int_{a}^{\frac{a+b}{2}} f\left(x,\frac{c+d}{2}\right)dx \\
&\quad -\frac{1}{2(d-c)}\int_{\frac{c+d}{2}}^{d} f\left(\frac{a+b}{2},y\right)dy+\frac{1}{(b-a)(d-c)}\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y)dydx
\end{aligned}
$$

and

$$I_4 = \frac{1}{4}f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{2(b-a)} \int_a^{\frac{a+b}{2}} f\left(x, \frac{c+d}{2}\right)dx$$
$$- \frac{1}{2(d-c)} \int_c^{\frac{c+d}{2}} f\left(\frac{a+b}{2}, y\right) dy + \frac{1}{(b-a)(d-c)} \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(x,y)dydx.$$

Substitution of the I_1, I_2, I_3, and I_4 in (2.2) gives the desired identity (2.1).

Theorem 2. *Let f*: $\Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta := [a, b] \times [c, d]$ *with* $a < b$, $c < d$. *If* $\left|\frac{\partial^2 f}{\partial s \partial t}\right|$ *is convex on the co-ordinates on* Δ, *then the following inequality holds:*

$$\left|\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A\right|$$
$$\leq \frac{(b-a)(d-c)}{16}\left[\frac{\left|\frac{\partial^2}{\partial s \partial t}f(a,c)\right| + \left|\frac{\partial^2}{\partial s \partial t}t(a,d)\right| + \left|\frac{\partial^2}{\partial s \partial t}f(b,c)\right| + \left|\frac{\partial^2}{\partial s \partial t}f(b,d)\right|}{4}\right], \tag{2.4}$$

where

$$A = \frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) dy.$$

Proof. From Lemma 1, we have

$$\left|\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A\right|$$
$$\leq (b-a)(d-c)\int_0^1 \int_0^1 \left|K(t,s)\right| \left|\frac{\partial^2}{\partial s \partial t}f(ta + (1-t)b, sc + (1-s)d)\right| dsdt \tag{2.5}$$

Since $\left|\frac{\partial^2 f}{\partial s \partial t}\right|$ is convex on the co-ordinates on Δ, we have

$$\left|\frac{\partial^2}{\partial s \partial t}f(ta + (1-t)b, sc + (1-s)d)\right| \leq ts\left|\frac{\partial^2}{\partial s \partial t}f(a,c)\right| + t(1-s)\left|\frac{\partial^2}{\partial s \partial t}f(a,d)\right|$$
$$+ s(1-t)\left|\frac{\partial^2}{\partial s \partial t}f(b,c)\right| + (1-t)(1-s)\left|\frac{\partial^2}{\partial s \partial t}f(b,d)\right|. \tag{2.6}$$

Substitution of (2.6) in (2.5) gives the following inequality:

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx + f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - A\right|$$
$$\le (b-a)(d-c)\int_0^1\int_0^1 |K(t,s)|\left[\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|ts + \left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|t(1-s)\right.$$
$$\left.+\left|\frac{\partial^2}{\partial s\partial t}(b,c)\right|s(1-t) + \left|\frac{\partial^2}{\partial s\partial t}(b,d)\right|(1-t)(1-s)\right]dsdt = (b-a)(d-c)$$
$$\times\left\{\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} ts\left[\left|\frac{\partial^2}{\partial s\partial t}(a,c)\right|ts + \left|\frac{\partial^2}{\partial s\partial t}(a,d)\right|t(1-s) + \left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|s(1-t)\right.\right.$$
$$\left.+\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|(1-t)(1-s)\right]dsdt + \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 t(1-s)\left[\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|ts\right. \tag{2.7}$$
$$\left.+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|t(1-s) + \left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|s(1-t) + \left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|(1-t)(1-s)\right]dsdt$$
$$+\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} s(1-t)\left[\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|ts + \left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|t(1-s) + \left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|s(1-t)\right.$$
$$\left.+\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|(1-t)(1-s)\right]dsdt + \int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 (1-t)(1-s)\left[\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|ts+\right.$$
$$\left.\left.\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|t(1-s) + \left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|s(1-t) + \left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|(1-t)(1-s)\right]dsdt\right\}$$

Evaluating each integral in (2.7) and simplifying, we get (2.4). Hence the proof of the theorem is complete.

Theorem 3. *Let* $\boldsymbol{f}$: $\Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a partial differentiable mapping on* Δ: = $[a, b] \times [c, d]$ *with* $a < b$, $c < d$. *If* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ *is convex on the co-ordinates on* Δ *and* $p, q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, *then the following inequality holds:*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_a^d f(x,y)dydx + f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - A\right|$$
$$\le \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left[\frac{\left|\frac{\partial^2}{\partial s\partial t}(a,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t}(a,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t}(b,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t}(b,d)\right|^q}{4}\right]^{\frac{1}{q}}, \tag{2.8}$$

where A is as given in Theorem 2.

Proof. From Lemma 1, we have

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx + f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - A\right| \\ \leq (b-a)(d-c)\int_0^1\int_0^1 |K(t,s)|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|dsdt. \tag{2.9}$$

Now using the well-known Hölder inequality for double integrals, we obtain

$$\int_0^1\int_0^1 |K(t,s)|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|dsdt \\ \leq \left(\int_0^1\int_0^1 |K(t,s)|^p dsdt\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}. \tag{2.10}$$

Since $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is convex on the co-ordinates on Δ, we have

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|^q dsdt \\ \leq \int_0^1\int_0^1\left[ts\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q + t(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q\right. \\ \left.+s(1-t)\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q + (1-t)(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q\right]dsdt \\ = \frac{\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q}{4}. \tag{2.11}$$

Also, we notice that

$$\int_0^1\int_0^1 |K(t,s)|^p dsdt = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^p s^p dsdt + \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 t^p(1-s)^p dsdt \\ + \int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} s^p(1-t)^p dsdt + \int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 (1-t)^p(1-s)^p dsdt \\ = \frac{4}{(p+1)^2}\left(\frac{1}{2}\right)^{2(p+1)}. \tag{2.12}$$

Using (2.11) and (2.12) in (2.10), we obtain

$$\int_0^1\int_0^1 |K(t,s)|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|dsdt$$

$$\leq \frac{1}{4(p+1)^{\frac{2}{p}}}\left[\frac{\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q}{4}\right]^{\frac{1}{q}}.$$

Utilizing the last inequality in (2.9) gives us (2.8). This completes the proof of the theorem.

Now we state our next result in:

Theorem 4. *Let f*: $\Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta := [a, b] \times [c, d]$ *with* $a < b$, $c < d$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ *is convex on the co-ordinates on* Δ *and* $q \geq 1$, *then the following inequality holds:*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|$$

$$\leq \frac{(b-a)(d-c)}{16}\left[\frac{\left|\frac{\partial^2}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}(b,d)\right|^q}{4}\right]^{\frac{1}{q}}, \tag{2.13}$$

where A is as given in Theorem 2.

Proof. By using Lemma 1, we have that the following inequality:

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_a^d f(x,y)dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|$$

$$\leq (b-a)(d-c)\int_0^1\int_0^1 |K(t,s)|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|dsdt. \tag{2.14}$$

By the power mean inequality, we have

$$\int_0^1\int_0^1 |K(t,s)|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|dsdt$$

$$\leq \left(\int_0^1\int_0^1 |K(t,s)|\,dsdt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\int_0^1 |K(t,s)|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}} \tag{2.15}$$

$$=\left(\frac{1}{16}\right)^{1-\frac{1}{q}}\left(\int_0^1\int_0^1 |K(t,s)|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}.$$

Using the fact that $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is convex on the co-ordinates on Δ, we get

$$\begin{aligned}&\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|^q\\&= ts\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q + t(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q + s(1-t)\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q\\&+(1-t)(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q\end{aligned}$$

and hence, we obtain

$$\begin{aligned}&\int_0^1\int_0^1 |K(t,s)|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|^q dsdt\\&\le \int_0^1\int_0^1 |K(t,s)|\left[ts\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q + t(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q\right.\\&\left.+s(1-t)\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q + (1-t)(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q\right] dsdt\\&= \frac{1}{64}\left[\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q\right].\end{aligned}$$

Therefore (2.15) becomes

$$\begin{aligned}&\int_0^1\int_0^1 |K(t,s)|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right| dsdt\\&\le \frac{1}{16}\left[\frac{\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q}{4}\right]^{\frac{1}{q}}\end{aligned} \tag{2.16}$$

Substitution of (2.16) in (2.14), we obtain (2.13). Hence the proof is complete.

Remark 1. *Since* $2^p > p + 1$ *if* $p > 1$ *and accordingly*

$$\frac{1}{4} < \frac{1}{2(p+1)^{\frac{1}{p}}}$$

and hence we have that the following inequality:

$$\frac{1}{16} < \frac{1}{4}\cdot\frac{1}{4} < \frac{1}{2(p+1)^{\frac{1}{p}}}\cdot\frac{1}{2(p+1)^{\frac{1}{p}}} = \frac{1}{4(p+1)^{\frac{2}{p}}},$$

and as a consequence we get an improvement of the constant in Theorem 3.

Following theorem is about concave functions on the co-ordinates on Δ:

Theorem 5. *Let f: $\Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on Δ: = $[a, b] \times [c, d]$ with $a < b$, $c < d$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ is concave on the co-ordinates on Δ and $q \geq 1$, then we have the inequality:*

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A\right| \\
&\leq \frac{(b-a)(d-c)}{64}\left[\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+2b}{3}, \frac{c+2d}{3}\right)\right| + \left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+2b}{3}, \frac{2c+d}{3}\right)\right|\right. \qquad (2.17)\\
&\left.\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{2a+b}{3}, \frac{c+2d}{3}\right)\right| + \left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{2a+b}{3}, \frac{2c+d}{3}\right)\right|\right],
\end{aligned}$$

where A is as defined in Theorem 2.

Proof. By the concavity of $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ on the co-ordinates on Δ and power mean inequality, we note that the following inequality holds:

$$\begin{aligned}
\left|\frac{\partial^2}{\partial s\partial t}f(\lambda x + (1-\lambda(y, v)\right|^q &\geq \lambda\left|\frac{\partial^2}{\partial s\partial t}f(x,v)\right|^q + (1-\lambda)\left|\frac{\partial^2}{\partial s\partial t}f(y,v)\right|^q \\
&\geq \left(\lambda\left|\frac{\partial^2}{\partial s\partial t}f(x,v)\right| + (1-\lambda)\left|\frac{\partial^2}{\partial s\partial t}f(y,v)\right|\right)^q,
\end{aligned}$$

for all x, $y \in [a, b]$, $\lambda \in [0, 1]$ and for fixed $v \in [c, d]$.

Hence,

$$\left|\frac{\partial^2}{\partial s\partial t}f(\lambda x + (1-\lambda)y, v)\right| \geq \lambda\left|\frac{\partial^2}{\partial s\partial t}f(x,v)\right| + (1-\lambda)\left|\frac{\partial^2}{\partial s\partial t}f(y,v)\right|,$$

for all x, $y \in [a, b]$, $\lambda \in [0, 1]$ and for fixed $v \in [c, d]$.

Similarly, we can show that

$$\left|\frac{\partial^2}{\partial s\partial t}f(u, \lambda z + (1-\lambda)w)\right| \geq \lambda\left|\frac{\partial^2}{\partial s\partial t}f(u,z)\right| + (1-\lambda)\left|\frac{\partial^2}{\partial s\partial t}f(u,w)\right|,$$

for all z, $w \in [c, d]$, $\lambda \in [0, 1]$ and for fixed $u \in [a, d]$, thus $\left|\frac{\partial^2 f}{\partial s \partial t}\right|$ is concave on the co-ordinates on Δ.

It is clear from Lemma 1 that

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx + f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - A\right| \\
&\le (b-a)(d-c)\int_0^1\int_0^1 |K(t,s)|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|dsdt \\
&= (b-a)(d-c)\left[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} st\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|dsdt\right. \\
&+\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 t(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|dsdt \\
&+\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} s(1-t)\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|dsdt \\
&\left.+\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 (1-t)(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|dsdt\right].
\end{aligned} \tag{2.18}$$

Since $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is concave on the co-ordinates, we have, by Jensen's inequality for integrals, that:

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} st\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|dsdt \\
&= \int_0^{\frac{1}{2}} t\left[\int_0^{\frac{1}{2}} s\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|ds\right]dt \\
&\le \int_0^{\frac{1}{2}} r\left(\int_0^{\frac{1}{2}} sds\right)\left|\frac{\partial^2}{\partial s\partial t}f\left(ta+(1-t)b, \frac{\int_0^{\frac{1}{2}} s(sc+(1-s)d)ds}{\int_0^{\frac{1}{2}} sds}\right)\right|dt \\
&= \frac{1}{8}\int_0^{\frac{1}{2}} t\left|\frac{\partial^2}{\partial s\partial t}f\left(ta+(1-t)b, \frac{c+2d}{3}\right)\right|dt \\
&\le \frac{1}{8}\left(\int_0^{\frac{1}{2}} tdt\right)\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{\int_0^{\frac{1}{2}} t(ta+(1-t)b)dt}{\int_0^{\frac{1}{2}} tdt}, \frac{c+2d}{3}\right)\right| \\
&= \frac{1}{64}\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right|.
\end{aligned} \tag{2.19}$$

In a similar way, we also have that

$$\begin{aligned}&\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right| dsdt\\&\leq \frac{1}{64}\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)\right|,\end{aligned} \tag{2.20}$$

$$\begin{aligned}&\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} s(1-t)\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right| dsdt\\&\leq \frac{1}{64}\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)\right|\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}&\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} (1-t)(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right| dsdt\\&\leq \frac{1}{64}\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)\right|.\end{aligned} \tag{2.22}$$

By making use of (2.19)-(2.22) in (2.18), we get the desired result. This completes the proof.

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Authors' contributions
MAL and SSD carried out the design of the study and performed the analysis. Both of the authors read and approved the final version of the manuscript.

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