On Some New Inequalities of Hermite-Hadamard-Féjer Type Involving Convex Functions

ON SOME NEW INEQUALITIES OF HERMITE-HADAMARD-FEJÉR TYPE INVOLVING CONVEX FUNCTIONS

KUEI-LIN TSENG, SHIOW RU HWANG, AND S.S. DRAGOMIR

Abstract. In this paper, we establish some inequalities of Hermite-Hadamard-Fejér type for m-convex functions and s-convex functions.

1. Introduction

If $f : [a, b] \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as Hermite-Hadamard inequality.

Fejér [14] gave a generalization of the inequalities (1.1) as the following:

If $f : [a, b] \to \mathbb{R}$ is a convex function, and $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$, then

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \tag{1.2}$$

For some results which generalize, improve, and extend the inequalities (1.1) and (1.2) see [1] – [12], [14] – [16], [19] – [23].

Definition 1 (see [6, 13, 18]). *A function $f : [0, b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have:*

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y). \tag{1.3}$$

We denote the set of all m-convex functions on $[0, b]$ by $K_m(b)$.

Dragomir and Toader [13] (see also [6]) proved the following two theorems:

Theorem 1. *Let $f : [0, \infty) \to \mathbb{R}$ be an m-convex function with $m \in (0, 1]$. If $0 \le a < b < \infty$ and $f \in L_1[0, b]$, then*

$$\int_a^b f(x)dx \le (b-a)\min\left\{\frac{f(a)+mf(\frac{b}{m})}{2}, \frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}. \tag{1.4}$$

1991 *Mathematics Subject Classification.* 26D15, 26D10.

Key words and phrases. Hermite-Hadamard inequality, m-convex functions, s-convex functions in the first and second senses.

Theorem 2. *Let f, m, a and b be defined as in Theorem 1. If f is also differentiable on $(0,\infty)$, then*

$$\left[\frac{f(mb)}{m} - \frac{b-a}{2}f'(mb)\right](b-a) \le \int_a^b f(x)dx \le \frac{1}{2}[(b-ma)f(b) - (a-mb)f(a)]. \tag{1.5}$$

The following two theorems are due to Dragomir [6]:

Theorem 3. *Let f be defined as in Theorem 1. Then*

$$f\left(\frac{a+b}{2}\right)(b-a) \le \int_a^b \frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx \le \frac{b-a}{8}\left[f(a)+f(b)+2m\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right) + m^2\left(f\left(\frac{a}{m^2}\right)+f\left(\frac{b}{m^2}\right)\right)\right]. \tag{1.6}$$

Theorem 4. *Let $f : [0,\infty) \to \mathbb{R}$ be an $m-$convex function with $m \in (0,1]$. If $f \in L_1[am, b]$ where $0 \le a < b$, then*

$$\frac{1}{m+1}\left[\int_a^{mb} f(x)dx + \frac{mb-a}{b-ma}\int_{ma}^b f(x)dx\right] \le (mb-a)\frac{f(a)+f(b)}{2}.$$

Remark 1. *A misprint of (1.6) in the orginal paper has been corrected here.*

Definition 2 (see [9, 10, 17])**.** *Let $0 < s \le 1$. A function $f : [0,\infty) \to \mathbb{R}$ is said to be s-convex in the first sense, if for every $x, y \in [0,\infty)$ and $\alpha, \beta \ge 0$ with $\alpha^s + \beta^s = 1$, we have:*

$$f(\alpha x + \beta y) \le \alpha^s f(x) + \beta^s f(y). \tag{1.7}$$

We denote the set of all s-convex functions in the first sense by K_s^1.

Definition 3 (see [9, 10, 17])**.** *Let $0 < s \le 1$. A function $f : [0,\infty) \to \mathbb{R}$ is said to be $s-$convex in the second sense, if for every $x, y \in [0,\infty)$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$ we have the inequality (1.7). The set of all s-convex functions in the second sense is denoted by K_s^2.*

Dragomir and Fitzpatrick [9, 10] proved the following two theorems:

Theorem 5. *Let $f \in K_s^1$ and $a, b \in [0,\infty)$ with $a < b$. Then*

$$(b-a)f\left[2^{-\frac{1}{s}}(a+b)\right] \le \int_a^b f(x)dx \tag{1.8}$$

and

$$f\left(\frac{a+b}{2^{\frac{1}{s}-1}}\right) \le \int_0^1 f\left(\frac{a+b}{2^{\frac{1}{s}}}\left[t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right]\right)dt \le \int_0^1 f\left(at^{\frac{1}{s}} + b(1-t)^{\frac{1}{s}}\right)dt \le \frac{f(a)+f(b)}{2}. \tag{1.9}$$

Theorem 6. *Let $f \in K_s^2$ and $a, b \in [0,\infty)$ with $a < b$. Then*

$$2^{s-1}(b-a)f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)dx \le \frac{(b-a)(f(a)+f(b))}{(s+1)}. \tag{1.10}$$

In this paper, we shall establish some generalizations of Theorems 1 – 6.

2. Main Results

Throughout this section, let $g : [a,b] \to \mathbb{R}$ be nonnegative, integrable and symmetric about $\frac{a+b}{2}$.

Theorem 7. *Let f, m, a and b be defined as in Theorem 1. Then*

$$\int_a^b f(x)g(x)dx \le \min\left\{ \frac{f(a)+mf(\frac{b}{m})}{2}, \frac{f(b)+mf\left(\frac{a}{m}\right)}{2} \right\} \int_a^b g(x)dx. \tag{2.1}$$

Proof. Since f is $m-$convex and g is nonnegative, integrable and symmetric about $\frac{a+b}{2}$, we have

$$\begin{aligned}
\int_a^b f(x)g(x)dx &= \frac{1}{2}\left[\int_a^b f(x)g(x)dx + \int_a^b f(a+b-x)g(a+b-x)dx\right] \\
&= \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]g(x)dx \\
&= \frac{1}{2}\int_a^b \left[f\left(\frac{b-x}{b-a}a + m\frac{x-a}{b-a}\cdot\frac{b}{m}\right)\right. \\
&\qquad \left. + f\left(\frac{x-a}{b-a}a + m\frac{b-x}{b-a}\cdot\frac{b}{m}\right)\right] g(x)dx \\
&\le \frac{1}{2}\int_a^b \left[\frac{b-x}{b-a}f(a) + m\frac{x-a}{b-a}f\left(\frac{b}{m}\right)\right. \\
&\qquad \left. + \frac{x-a}{b-a}f(a) + m\frac{b-x}{b-a}f\left(\frac{b}{m}\right)\right] g(x)dx \\
&= \frac{f(a)+mf(\frac{b}{m})}{2}\int_a^b g(x)dx. \qquad (2.2)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_a^b f(x)g(x)dx &= \frac{1}{2}\int_a^b \left[f(x)+f(a+b-x)\right] g(x)dx \\
&= \frac{1}{2}\int_a^b \left[f\left(m\frac{b-x}{b-a}\cdot\frac{a}{m} + \frac{x-a}{b-a}b\right)\right. \\
&\qquad \left. +f\left(m\frac{x-a}{b-a}\cdot\frac{a}{m} + \frac{b-x}{b-a}b\right)\right] g(x)dx \\
&\le \frac{1}{2}\int_a^b \left[m\frac{b-x}{b-a}f\left(\frac{a}{m}\right) + \frac{x-a}{b-a}f(b)\right. \\
&\qquad \left. +m\frac{x-a}{b-a}f\left(\frac{a}{m}\right) + \frac{b-x}{b-a}f(b)\right] g(x)dx \\
&= \frac{mf\left(\frac{a}{m}\right)+f(b)}{2}\int_a^b g(x)dx. \qquad (2.3)
\end{aligned}$$

The inequality (2.1) follows immediately from (2.2) and (2.3). ∎

Remark 2. *If we choose $g(x) \equiv 1$, then Theorem 7 reduces to Theorem 1.*

Remark 3. *If $m = 1$, then the inequality (2.1) reduces to the second inequality of (1.2) where $0 \leq a < b < \infty$.*

In order to prove our second theorem, we need the following lemma:

Lemma 1 (see [6] or [13]). *If f is differentiable on $[0, b]$, then $f \in K_m(b)$ if and only if*

$$f(x) - mf(y) \leq f'(x)(x - my) \tag{2.4}$$

for $x, y \in [0, b]$.

Theorem 8. *Let f, m, a and b be defined as in Theorem 2. Then*

$$\begin{aligned} \left[\frac{f(mb)}{m} - \frac{b-a}{2} f'(mb)\right] \int_a^b g(x)dx &\leq \int_a^b f(x)g(x)dx \\ &\leq \int_a^b [(x - ma)f'(x) + mf(a)]g(x)dx. \end{aligned} \tag{2.5}$$

Proof. By Lemma 1, for $x \in [a, b]$, we have

$$f(mb) - mf(x) \leq f'(mb)(mb - mx)$$

and

$$f(mb) - mf(a + b - x) \leq f'(mb)[mb - m(a + b - x)],$$

so that

$$\frac{f(mb)}{m} - (b - x)f'(mb) \leq f(x) \tag{2.6}$$

and

$$\frac{f(mb)}{m} - (x - a)f'(mb) \leq f(a + b - x). \tag{2.7}$$

If we add the inequalities (2.6) and (2.7), then

$$\frac{2f(mb)}{m} - (b - a)f'(mb) \leq f(x) + f(a + b - x) \tag{2.8}$$

for all $x \in [a, b]$. Since g is nonnegative, integrable and symmetric about $\frac{a+b}{2}$, multiplying (2.8) by $\frac{g(x)}{2}$, and integrating the resulting inequalities on $[a, b]$ yields

$$\begin{aligned} &\left[\frac{f(mb)}{m} - \frac{b-a}{2} f'(mb)\right] \int_a^b g(x)dx \\ &\leq \frac{1}{2} \int_a^b [f(x)g(x) + f(a + b - x)g(x)]dx \\ &= \frac{1}{2} \left[\int_a^b f(x)g(x)dx + \int_a^b f(a + b - x)g(a + b - x)dx\right] \\ &= \int_a^b f(x)g(x)dx. \end{aligned}$$

This proves the first inequality in (2.5). Putting in (2.4) $y = a$, we have for $x \geq ma$

$$(x - ma)f'(x) + mf(a) \geq f(x). \tag{2.9}$$

Multiplying (2.9) by $g(x)$ and integrating over x on $[a, b]$, we obtain the second inequality in (2.5). This completes the proof. ∎

Remark 4. *If we choose $g(x) \equiv 1$, then Theorem 8 reduces to Theorem 2.*

Theorem 9. *Let f, m, a and b be defined as in Theorem 3. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx &\leq \int_a^b \frac{f(x)+mf\left(\frac{x}{m}\right)}{2}g(x)dx \\ &\leq \frac{1}{8}\left[f(a)+f(b)+2m\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)\right. \\ &\qquad \left. +m^2\left(f\left(\frac{a}{m^2}\right)+f\left(\frac{b}{m^2}\right)\right)\right]\int_a^b g(x)dx \\ &\leq \frac{m^2\left[f\left(\frac{a}{m^2}\right)+f\left(\frac{b}{m^2}\right)\right]}{2}\int_a^b g(x)dx. \end{aligned} \tag{2.10}$$

Proof. Since f is $m-$convex, $f \in L_1[a,b]$ and g is nonnegtive, integrable and symmetric about $\frac{a+b}{2}$, we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \\ &= \int_a^b f\left[\frac{1}{2}(a+b-x)+\frac{m}{2}\cdot\frac{x}{m}\right]g(x)dx \\ &\leq \int_a^b \left[\frac{1}{2}f(a+b-x)+\frac{m}{2}f\left(\frac{x}{m}\right)\right]g(x)dx \\ &= \int_a^b \frac{1}{2}\left[f(a+b-x)g(a+b-x)+mf\left(\frac{x}{m}\right)g(x)\right]dx \\ &= \frac{1}{2}\left[\int_a^b f(x)g(x)+\int_a^b mf\left(\frac{x}{m}\right)g(x)dx\right] \\ &= \int_a^b \frac{f(x)+mf\left(\frac{x}{m}\right)}{2}g(x)dx, \qquad (2.11) \\ &= \frac{1}{4}\left[\int_a^b f(x)g(x)dx+\int_a^b f(a+b-x)g(a+b-x)dx\right. \\ &\qquad \left.+\int_a^b mf\left(\frac{x}{m}\right)g(x)dx+\int_a^b mf\left(\frac{a+b-x}{m}\right)g(a+b-x)dx\right] \\ &= \frac{1}{8}\left[2\int_a^b f(x)g(x)dx+2\int_a^b f(a+b-x)g(x)dx\right. \\ &\qquad \left.+2\int_a^b mf\left(\frac{x}{m}\right)g(x)dx+2\int_a^b mf\left(\frac{a+b-x}{m}\right)g(x)dx\right] \end{aligned}$$

$$\begin{aligned} = \frac{1}{8}&\left[\int_a^b f\left(\frac{b-x}{b-a}a+\frac{x-a}{b-a}m\frac{b}{m}\right)g(x)dx+\int_a^b f\left(\frac{b-x}{b-a}m\frac{a}{m}+\frac{x-a}{b-a}b\right)g(x)dx\right. \\ &+\int_a^b f\left(\frac{x-a}{b-a}a+\frac{b-x}{b-a}m\frac{b}{m}\right)g(x)dx+\int_a^b f\left(\frac{x-a}{b-a}m\frac{a}{m}+\frac{b-x}{b-a}b\right)g(x)dx \end{aligned}$$

$$
\begin{aligned}
&+\int_a^b mf\left(\frac{b-x}{b-a}\frac{a}{m}+\frac{x-a}{b-a}m\frac{b}{m^2}\right)g(x)dx\\
&+\int_a^b mf\left(\frac{b-x}{b-a}m\frac{a}{m^2}+\frac{x-a}{b-a}\frac{b}{m}\right)g(x)dx\\
&+\int_a^b mf\left(\frac{x-a}{b-a}\cdot\frac{a}{m}+\frac{b-x}{b-a}\cdot m\cdot\frac{b}{m^2}\right)g(x)dx\\
&+\int_a^b mf\left(\frac{x-a}{b-a}m\frac{a}{m^2}+\frac{b-x}{b-a}\cdot\frac{b}{m}\right)g(x)dx\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\le{}& \frac{1}{8}\Bigg\{\int_a^b\left[\frac{b-x}{b-a}f(a)+m\frac{x-a}{b-a}f\left(\frac{b}{m}\right)\right]g(x)\\
&+\int_a^b\left[m\frac{b-x}{b-a}f\left(\frac{a}{m}\right)+\frac{x-a}{b-a}f(b)\right]g(x)dx\\
&+\int_a^b\left[\frac{x-a}{b-a}f(a)+m\frac{b-x}{b-a}f\left(\frac{b}{m}\right)\right]g(x)dx\\
&+\int_a^b\left[m\frac{x-a}{b-a}f\left(\frac{a}{m}\right)+\frac{b-x}{b-a}f(b)\right]g(x)dx\\
&+\int_a^b m\left[\frac{b-x}{b-a}f\left(\frac{a}{m}\right)+m\frac{x-a}{b-a}f\left(\frac{b}{m^2}\right)\right]g(x)dx\\
&+\int_a^b m\left[m\frac{b-x}{b-a}f\left(\frac{a}{m^2}\right)+\frac{x-a}{b-a}f\left(\frac{b}{m}\right)\right]g(x)dx\\
&+\int_a^b m\left[\frac{x-a}{b-a}f\left(\frac{a}{m}\right)+m\frac{b-x}{b-a}f\left(\frac{b}{m^2}\right)\right]g(x)dx\\
&+\int_a^b m\left[m\frac{x-a}{b-a}f\left(\frac{a}{m^2}\right)+\frac{b-x}{b-a}f\left(\frac{b}{m}\right)\right]g(x)dx\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
={}& \frac{1}{8}\left[f(a)+f(b)+2m\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)\right.\\
&\left.+m^2\left(f\left(\frac{a}{m^2}\right)+f\left(\frac{b}{m^2}\right)\right)\right]\int_a^b g(x)dx \tag{2.12}
\end{aligned}
$$

$$
\begin{aligned}
={}& \frac{\int_a^b g(x)dx}{8}\left\{f\left(0\cdot a+m\cdot\frac{a}{m}\right)+f\left(0\cdot b+m\cdot\frac{b}{m}\right)\right.\\
&+2m\left[f\left(0\cdot\frac{a}{m}+m\cdot\frac{a}{m^2}\right)+f\left(0\cdot\frac{b}{m}+m\cdot\frac{b}{m^2}\right)\right]\\
&\left.+m^2\left[f\left(\frac{a}{m^2}\right)+f\left(\frac{b}{m^2}\right)\right]\right\}\\
\le{}& \frac{\int_a^b g(x)dx}{8}\left\{mf\left(\frac{a}{m}\right)+mf\left(\frac{b}{m}\right)+3m^2\left[f\left(\frac{a}{m^2}\right)+f\left(\frac{b}{m^2}\right)\right]\right\}\\
={}& \frac{\int_a^b g(x)dx}{8}\left\{mf\left(0\cdot\frac{a}{m}+m\frac{a}{m^2}\right)+mf\left(0\cdot\frac{b}{m}+m\frac{b}{m^2}\right)\right.\\
&\left.+3m^2\left[f\left(\frac{a}{m^2}\right)+f\left(\frac{b}{m^2}\right)\right]\right\}
\end{aligned}
$$

$$
\le \frac{m^2\left[f\left(\frac{a}{m^2}\right)+f\left(\frac{b}{m^2}\right)\right]}{2}\int_a^b g(x)dx. \tag{2.13}
$$

The inequalities (2.10) follow from (2.11), (2.12) and (2.13). ∎

Remark 5. *If we choose $g(x) \equiv 1$, then Theorem 9 reduces to Theorem 3.*

Remark 6. *If $m = 1$, then the inequalities (2.10) reduce to the inequalities (1.2) when $0 \le a < b < \infty$.*

Theorem 10. *Suppose that $f : [0, \infty) \to \mathbb{R}$ is an $m-$convex function with $f \in L_1[ma, b]$ and $k : [ma, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{ma+b}{2}$ with $\int_{ma}^{b} k(x)dx > 0$, where $m \in [0, 1]$ and $0 \le a < b$.*

(a) *If $a < mb$ and $h : [a, mb] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+mb}{2}$ with $\int_{a}^{mb} h(x)dx > 0$, then*

$$\frac{1}{m+1}\left(\frac{\int_a^{mb} f(x)h(x)dx}{\int_a^{mb} h(x)dx} + \frac{\int_{ma}^{b} f(x)k(x)dx}{\int_{ma}^{b} k(x)dx}\right) \le \frac{f(a)+f(b)}{2}. \tag{2.14}$$

(b) *If $mb < a$ and $h : [mb, a] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+mb}{2}$ with $\int_{mb}^{a} h(x)dx > 0$, then the inequality (2.14) also holds.*

Proof. (a) Since f is $m-$convex, $f \in L_1[ma, b]$, k is nonnegative, integrable, symmetric about $\frac{ma+b}{2}$ with $\int_{ma}^{b} k(x)dx > 0$, we have

$$\begin{aligned}
\int_{ma}^{b} f(x)k(x)dx &= \frac{\int_{ma}^{b} f(x)k(x)dx + \int_{ma}^{b} f(ma+b-x)k(ma+b-x)dx}{2} \\
&= \frac{\int_{ma}^{b}[f(x)+f(ma+b-x)]k(x)dx}{2} \\
&= \frac{1}{2}\int_{ma}^{b}\left[f\left(\frac{b-x}{b-ma}ma + \frac{x-ma}{b-ma}b\right)\right. \\
&\qquad \left. +f\left(\frac{x-ma}{b-ma}ma + \frac{b-x}{b-ma}b\right)\right]k(x)dx \\
&\le \frac{1}{2}\int_{ma}^{b}\left[m\frac{b-x}{b-ma}f(a) + \frac{x-ma}{b-ma}f(b)\right. \\
&\qquad \left. +m\frac{x-ma}{b-ma}f(a) + \frac{b-x}{b-ma}f(b)\right]k(x)dx \\
&= \frac{mf(a)+f(b)}{2}\int_{ma}^{b} k(x)dx. \qquad (2.15)
\end{aligned}$$

Similarly, we have

$$\int_{a}^{mb} f(x)h(x)dx \le \frac{f(a)+mf(b)}{2}\int_{a}^{mb} h(x)dx \tag{2.16}$$

The inequality (2.14) follows immediately from (2.15) and (2.16).

The proof of part (b) is similar to that of part (a). ∎

Remark 7. *If we choose $h(x) \equiv 1$ and $k(x) \equiv 1$, then Theorem 10 reduces to Theorem 4.*

Remark 8. *If $m = 1$ and $h(x) = k(x) = g(x)$ on $[a, b]$, then the inequality (2.14) reduces to the second inequality of (1.2) when $0 \le a < b < \infty$.*

In order to prove our next theorem, we need the following lemma:

Lemma 2 ([17])**.** *If $0 < s < 1$ and $f \in K_s^1$, then f is nondecreasing on $[0, \infty)$.*

Theorem 11. *Let f, a and b be defined as in Theorem 5. Then*

$$f\left[2^{-\frac{1}{s}}(a+b)\right]\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \tag{2.17}$$

and

$$\begin{aligned} & f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\int_a^b g(x)dx \\ & \le \int_a^b \left\{ f\left[\left(\frac{1}{2}\cdot\frac{b-x}{b-a}\right)^{\frac{1}{s}} + \left(\frac{1}{2}\cdot\frac{x-a}{b-a}\right)^{\frac{1}{s}}\right](a+b)\right\} g(x)dx \\ & \le \int_a^b f\left[a\left(\frac{b-x}{b-a}\right)^{\frac{1}{s}} + b\left(\frac{x-a}{b-a}\right)^{\frac{1}{s}}\right] g(x)dx \\ & \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \end{aligned} \tag{2.18}$$

Proof. Since $f \in K_s^1$ and g is nonnegative, integrable and symmetric about $\frac{a+b}{2}$, we have

$$\begin{aligned} f\left[2^{-\frac{1}{s}}(a+b)\right]\int_a^b g(x)dx &= \int_a^b f\left[2^{-\frac{1}{s}}x + 2^{-\frac{1}{s}}(a+b-x)\right] g(x)dx \\ &\le \int_a^b \left[\frac{1}{2}f(x) + \frac{1}{2}f(a+b-x)\right] g(x)dx \\ &= \frac{1}{2}\left[\int_a^b f(x)g(x)dx + \int_a^b f(a+b-x)g(x)dx\right] \\ &= \frac{1}{2}\left[\int_a^b f(x)g(x)dx + \int_a^b f(a+b-x)g(a+b-x)dx\right] \\ &= \int_a^b f(x)g(x)dx. \end{aligned}$$

This proves (2.17).

Next, if $s = 1$ then (2.18) is (1.2). Let $0 < s < 1$, and $\alpha, \beta \ge 0$, then

$$\left(\frac{\alpha+\beta}{2}\right)^{\frac{1}{s}} \le \frac{1}{2}\left(\alpha^{\frac{1}{s}} + \beta^{\frac{1}{s}}\right).$$

Now, by Lemma 2, f is nondecreasing on $[0, \infty)$. Since g is nonnegative integrable and symmetric about $\frac{a+b}{2}$, we have

$$\begin{aligned} & f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right)\int_a^b g(x)dx \\ & = \int_a^b f\left[\left(\frac{1}{2}\cdot\frac{b-x}{2(b-a)} + \frac{1}{2}\cdot\frac{x-a}{2(b-a)}\right)^{\frac{1}{s}} 2(a+b)\right] g(x)dx \\ & \le \int_a^b f\left[\left(\frac{1}{2}\left(\frac{b-x}{2(b-a)}\right)^{\frac{1}{s}} + \frac{1}{2}\left(\frac{x-a}{2(b-a)}\right)^{\frac{1}{s}}\right) 2(b+a)\right] g(x)dx \end{aligned}$$

$$
\begin{aligned}
(2.19)\qquad &= \int_a^b f\left[\left(\frac{b-x}{2(b-a)}\right)^{\frac{1}{s}} + \left(\frac{x-a}{2(b-a)}\right)^{\frac{1}{s}}\right](a+b)g(x)dx \\
&= \int_a^b f\left\{\left(\frac{1}{2}\right)^{\frac{1}{s}}\left[\left(\frac{b-x}{b-a}\right)^{\frac{1}{s}} a + \left(\frac{x-a}{b-a}\right)^{\frac{1}{s}} b\right]\right. \\
&\qquad \left. + \left(\frac{1}{2}\right)^{\frac{1}{s}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{1}{s}} a + \left(\frac{b-x}{b-a}\right)^{\frac{1}{s}} b\right]\right\} g(x)dx \\
&\leq \int_a^b \left\{\frac{1}{2} f\left[\left(\frac{b-x}{b-a}\right)^{\frac{1}{s}} a + \left(\frac{x-a}{b-a}\right)^{\frac{1}{s}} b\right]\right. \\
&\qquad \left. + \frac{1}{2} f\left[\left(\frac{x-a}{b-a}\right)^{\frac{1}{s}} a + \left(\frac{b-x}{b-a}\right)^{\frac{1}{s}} b\right]\right\} g(x)dx \\
&= \frac{1}{2}\int_a^b f\left[\left(\frac{b-x}{b-a}\right)^{\frac{1}{s}} a + \left(\frac{x-a}{b-a}\right)^{\frac{1}{s}} b\right] g(x)dx \\
&\qquad + \frac{1}{2}\int_a^b f\left[\left(\frac{x-a}{b-a}\right)^{\frac{1}{s}} a + \left(\frac{b-x}{b-a}\right)^{\frac{1}{s}} b\right] g(a+b-x)dx \\
(2.20)\qquad &= \int_a^b f\left[\left(\frac{b-x}{b-a}\right)^{\frac{1}{s}} a + \left(\frac{x-a}{b-a}\right)^{\frac{1}{s}} b\right] g(x)dx.
\end{aligned}
$$

On the other hand, using (1.7) we have

$$
\begin{aligned}
&\int_a^b \left\{\frac{1}{2} f\left[\left(\frac{b-x}{b-a}\right)^{\frac{1}{s}} a + \left(\frac{x-a}{b-a}\right)^{\frac{1}{s}} b\right]\right. \\
&\qquad \left. + \frac{1}{2} f\left[\left(\frac{x-a}{b-a}\right)^{\frac{1}{s}} a + \left(\frac{b-x}{b-a}\right)^{\frac{1}{s}} b\right]\right\} g(x)dx \\
&\leq \frac{1}{2}\int_a^b \left[\frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)\right] g(x)dx \\
&\qquad + \frac{1}{2}\int_a^b \left[\frac{x-a}{b-a} f(a) + \frac{b-x}{b-a} f(b)\right] g(x)dx \\
(2.21)\qquad &= \frac{f(a)+f(b)}{2}\int_a^b g(x)dx.
\end{aligned}
$$

The inequalities (2.18) follow from (2.19), (2.20) and (2.21). ∎

Remark 9. *If we choose* $g(x) \equiv 1$, *then Theorem 11 reduces to Theorem 5.*

Remark 10. *If* $s = 1$, *then the inequality (2.17) reduces to the first inequality of (1.2) when* $0 \leq a < b < \infty$.

Theorem 12. *Let f, a and b be defined as in Theorem 6. Then*

$$\begin{aligned} & 2^{s-1} f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx \\ & \leq \int_a^b f(x)g(x)dx \\ & \leq \frac{f(a)+f(b)}{2} \int_a^b \left[\left(\frac{b-x}{b-a}\right)^s + \left(\frac{x-a}{b-a}\right)^s\right] g(x)dx. \end{aligned} \tag{2.22}$$

Proof. Since $f \in K_s^2$, g is nonnegative, integrable and symmetric about $\frac{a+b}{2}$, we have

$$\begin{aligned} & 2^{s-1} f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx \\ & = 2^{s-1} \int_a^b f\left(\frac{x}{2} + \frac{a+b-x}{2}\right) g(x)dx \\ & \leq 2^{s-1} \int_a^b \left[\left(\frac{1}{2}\right)^s f(x) + \left(\frac{1}{2}\right)^s f(a+b-x)\right] g(x)dx \\ & = \frac{1}{2} \int_a^b [f(x) + f(a+b-x)] g(x)dx \\ & = \frac{1}{2} \left[\int_a^b f(x)g(x)dx + \int_a^b f(a+b-x)g(a+b-x)dx\right] \\ & = \int_a^b f(x)g(x)dx. \end{aligned} \tag{2.23}$$

On the other hand, using (1.7) we have

$$\begin{aligned} & \frac{1}{2} \left[\int_a^b [f(x) + f(a+b-x)] g(x)dx\right] \\ & = \frac{1}{2} \int_a^b \left[f\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\right) + f\left(\frac{x-a}{b-a}a + \frac{b-x}{b-a}b\right)\right] g(x)dx \\ & \leq \frac{1}{2} \int_a^b \left[\left(\frac{b-x}{b-a}\right)^s f(a) + \left(\frac{x-a}{b-a}\right)^s f(b) \right. \\ & \qquad \left. + \left(\frac{x-a}{b-a}\right)^s f(a) + \left(\frac{b-x}{b-a}\right)^s f(b)\right] g(x)dx \\ & = \frac{f(a)+f(b)}{2} \int_a^b \left[\left(\frac{b-x}{b-a}\right)^s + \left(\frac{x-a}{b-a}\right)^s\right] g(x)dx. \end{aligned} \tag{2.24}$$

The inequalities (2.22) follow from (2.23) and (2.24). ∎

Remark 11. *If we choose $g(x) \equiv 1$, then Theorem 12 reduces to Theorem 6.*

Remark 12. *If $s = 1$, then the inequalities (2.22) reduces the inequalities (1.2) when $0 \leq a < b < \infty$.*

References

[1] H. Alzer, A note on Hadamard's inequalities, *C.R. Math. Rep. Acad. Sci. Canada*, 11 (1989), 255–258.

[2] J. L. Brenner and H. Alzer, integral inequalities for concave functions with applications to special functions, *Proc. Roy. Soc. Edinburgh A* 118 (1991), 173–192.
[3] S. S. Dragomir, Two refinements of Hadamard's inequality, *Coll. of Sci. Dep. of the Fac. of Sci. Kragujevaė*, 11 (1990), 23–26.
[4] S. S. Dragomir, Two mappings in connection to Hadamard's inequalities, *J. Math. Anal. Appl.*,167 (1992), 49–56.
[5] S. S. Dragomir, On Hadamard's inequalities for convex functions, *Mat. Balkanica*, 6 (1992), 215–222.
[6] S. S. Dragomir, On Some new inequalities of Hermite-Hadamard type for m-convex functions, *Tamkang J. Math.*, 33 (1) (2002), 55–65.
[7] S. S. Dragomir and C. Buşe, Refinements of Hadamard's inequality for multiple integrals, *Utilitas Mathematica* 47 (1995), 193–198.
[8] S.S. Dragomir and S. Fitzpatrick, S-Orlicz convex functions in linear spaces and Jensen's discrete inequality, *J. Math. Anal. Appl.*, 210(1997), 419–439
[9] S. S. Dragomir and S. Fitzpatrick, The Hadamard's inequality for s-convex function in the first sense, *Demonstratio Math.*, 31 (3) (1998), 633–642.
[10] S. S. Dragomir and S. Fitzpatrick, The Hadamard's inequalitiy for s-convex functions in the second sense, *Demonstratio Math.*, 32(4) (1999), 687–696.
[11] S. S. Dragomir, J. Pečarić and J. E. Sándor, A note on the Jensen-Hadamard inequality, *Anal. Num. Theor. Approx.* 19 (1990), 29–34.
[12] S. S. Dragomir, J. E. Pečarić and L. E. Persson, Some inequalities of Hadamard type, *Soochow J. Math.* 21 (1995), 335–341.
[13] S. S. Dragomir and G. H. Toader, Some inequalities for m-convex functions, Studia Univ. *Babeş-Bolyai, Math.,* 38(1) (1993), 21–28.
[14] L. Fejér, Über die Fourierreihen, II, *Math. Naturwiss, Anz. Ungar. Akad. Wiss.*, 24 (1906), 369–390. (In Hungarian).
[15] A. M. Fink, A best possible Hadamard inequality, *Math. Ineq. & Appl.* 2 (1998), 223–230.
[16] A. M. Fink, Hadamard inequalities for logarithmic concave functions, *Math. Comput. Modelling*, 32 (2000), no. 5-6, 625–629.
[17] H. Hudzik and Maligranda, Some remarks on s-convex functions, *Aequations Math.* 48 (1994), 100–111.
[18] G. H. Toader, Some generalizations of the convexity, *Proc. Colloq. Approx. Optim, Cluj-Napoca (Romania),* 1984, 329-338.
[19] G.-S. Yang and M.-C. Hong, A note on Hadamard's inequality, *Tamkang J. Math.*, 28 (1) (1997), 33–37.
[20] G.-S. Yang and K.-L. Tseng, On certain integral inequalities related to Hermite-Hadamard inequalities *J. Math. Anal. Appl.* 239 (1999), 180–187.
[21] G.-S. Yang and K.-L. Tseng, Inequalities of Hadamard's type for Lipschitizian Mappings, *J. Math. Anal. Appl.*, 260(2001), 230–238.
[22] G.-S. Yang and K.-L. Tseng, On certain multiple integral inequalities related to Hermite-Hadamard inequality, *Utilitas Mathematica* 62 (2002), 131–142.
[23] G.-S. Yang and C.-S. Wang, Some refinements of Hadamard's inequalities, *Tamkang J. Math.*, 28(2) (1997), 87–92.

Department of Mathematics, Aletheia University, Tamsui, Taiwan 25103
E-mail address: kltseng@email.au.edu.tw

China Institute of Technology, Nankang, Taipei, Taiwan11522

School of Computer Science and Mathematics, Victoria University of Technology, PO Box 14428, MCMC, Victoria 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.vu.edu.au/dragomir