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ON SOME NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR m – CONVEX FUNCTIONS

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Abstract. Some new inequalities for $m-$convex functions are obtained.

1. Introduction

In [71], G.H. Toader defined the $m-$convexity, an intermediate between the usual convexity and starshaped property.

In the first part of this section we shall present properties of $m-$convex functions in a similar manner to convex functions.

The following concept has been introduced in [71](see also [34]).

Definition 1. The function $f:[0,b]\to\mathbb{R}$ is said to be *m-convex*, where $m\in[0,1]$, if for every $x,y\in[0,b]$ and $t\in[0,1]$ we have:

$$f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y). \tag{1.1}$$

Denote by $K_m(b)$ the set of the $m-$convex functions on $[0,b]$ for which $f(0)\leq 0$.

Remark 1. For $m=1$, we recapture the concept of convex functions defined on $[0,b]$ and for $m=0$ we get the concept of starshaped functions on $[0,b]$. We recall that $f:[0,b]\to\mathbb{R}$ is *starshaped* if

$$f(tx)\leq tf(x)\quad\text{for all } t\in[0,1]\quad\text{and}\quad x\in[0,b]. \tag{1.2}$$

The following lemmas hold [71].

Lemma 1. *If f is in the class $K_m(b)$, then it is starshaped.*

Proof. For any $x\in[0,b]$ and $t\in[0,1]$, we have:

$$f(tx)=f(tx+m(1-t)\cdot 0)\leq tf(x)+m(1-t)f(0)\leq tf(x).$$

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Lemma 2. *If f is $m-$convex and $0 \leq n < m \leq 1$, then f is $n-$convex.*

Proof. If $x, y \in [0, b]$ and $t \in [0, 1]$, then

$$\begin{aligned} f(tx + n(1-t)y) &= f\left(tx + m(1-t)\left(\frac{n}{m}\right)y\right) \\ &\leq tf(x) + m(1-t)f\left(\left(\frac{n}{m}\right)y\right) \\ &\leq tf(x) + m(1-t)\ \frac{n}{m}\ f(y) \\ &= tf(x) + n(1-t)f(y) \end{aligned}$$

and the lemma is proved.

As in paper [48] due to V. G. Miheşan, for a mapping $f \in K_m(b)$ consider the function

$$p_{a,m}(x) := \frac{f(x) - mf(a)}{x - m}$$

defined for $x \in [0, b] \setminus \{ma\}$, for fixed $a \in [0, b]$, and

$$r_m(x_1, x_2, x_3) := \frac{\begin{vmatrix} 1 & 1 & 1 \\ mx_1 & x_2 & x_3 \\ mf(x_1) & f(x_2) & f(x_3) \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ mx_1 & x_2 & x_3 \\ m^2x_1^2 & x_2^2 & x_3^2 \end{vmatrix}},$$

where $x_1, x_2, x_3 \in [0, b]$, $(x_2 - mx_1)(x_3 - mx_1) > 0$, $x_2 \neq x_3$.

The following theorem holds [48].

Theorem 1. *The following assertions are equivalent:*

1°. $f \in K_m(b)$;
2°. $p_{a,m}$ *is increasing on the intervals* $[0, ma)$, $(ma, b]$ *for all* $a \in [0, b]$;
3°. $r_m(x_1, x_2, x_3) \geq 0$.

Proof. $1° \Rightarrow 2°$. Let $x, y \in [0, b]$. If $ma < x < y$, then there exists $t \in (0, 1)$ such that

$$x = ty + m(1-t)a. \tag{1.3}$$

We thus have

$$\begin{aligned} p_{a,m}(x) &= \frac{f(x) - mf(a)}{x - ma} \\ &= \frac{f(ty + m(1-t)a) - mf(a)}{ty + m(1-t)a - ma} \\ &\leq \frac{tf(y) + m(1-t)f(a) - mf(a)}{t(y - ma)} \\ &= \frac{f(y) - mf(a)}{y - ma} \\ &= p_{a,m}(y). \end{aligned}$$

If $y < x < ma$, there also exists $t \in (0,1)$ for which (1.3) holds.

Then we have:

$$\begin{aligned} p_{a,m}(x) &= \frac{f(x) - mf(a)}{x - ma} \\ &= \frac{mf(a) - f(ty + m(1-t)a)}{ma - ty - m(1-t)a} \\ &\geq \frac{mf(a) - tf(y) + m(1-t)f(a)}{t(ma - y)} \\ &= \frac{f(y) - mf(a)}{y - ma} \\ &= p_{a,m}(y). \end{aligned}$$

$2^\circ \Rightarrow 3^\circ$. A simple calculation shows that

$$r_m(x_1, x_2, x_3) = \frac{p_{x_1,m}(x_3) - p_{x_1,m}(x_2)}{x_3 - x_2}.$$

Since $p_{x_1,m}$ is increasing on the intervals $[0, mx_1), (mx_1, b]$, one obtains $r_m(x_1, x_2, x_3) \geq 0$.

$3^\circ \Rightarrow 1^\circ$. Let $x_1, x_3 \in [0, b]$ and let $x_2 = tx_3 + m(1-t)x_1,\ t \in (0,1)$. Obviously $mx_1 < x_2 < x_3$ or $x_3 < x_2 < mx_1$, hence

$$r_m(x_1, x_2, x_3) = \frac{tf(x_3) + m(1-t)f(x_1) - f(tx_3 + m(1-t)x_1)}{t(1-t)(x_3 - mx_1)^2}$$

from where we obtain (1.1), i.e., $f \in K_m(b)$.

The following corollary holds for starshaped functions.

Corollary 1. *Let $f : [0, b] \to \mathbb{R}$. The following statements are equivalent*

(*i*) *f is starshaped;*

(*ii*) *The mapping $p(x) := \frac{f(x)}{x}$ is increasing on $(0, b]$.*

The following lemma is also interesting in itself.

Lemma 3. *If f is differentiable on $[0, b]$, then $f \in K_m(b)$ if and only if:*

$$\begin{cases} f'(x) \geq \frac{f(x) - mf(y)}{x - my} & \text{for } x > my,\ y \in (0, b], \\ \\ f'(x) \leq \frac{f(x) - mf(y)}{x - my} & \text{for } 0 \leq x < my,\ y \in (0, b]. \end{cases} \tag{1.4}$$

Proof. The mapping $p_{y,m}$ is increasing on $(my, b]$ iff $p'_{y,m}(x) \geq 0$, which is equivalent with the condition (1.4).

Corollary 2. *If f is differentiable in $[0, b]$, then f is starshaped iff $f'(x) \geq \frac{f(x)}{x}$ for all $x \in (0, b]$.*

The following inequalities of Hermite-Hadamard type for $m-$convex functions hold [34].

Theorem 2. *Let $f:[0,\infty)\to\mathbb{R}$ be a $m-$convex function with $m\in(0,1]$. If $0\le a<b<\infty$ and $f\in L_1[a,b]$, then one has the inequality:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx\le\min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2},\frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}. \tag{1.5}$$

Proof. Since f is $m-$convex, we have

$$f(tx+m(1-t)y)\le tf(x)+m(1-t)f(y),\ \text{for all } x,y\ge 0,$$

which gives:

$$f(ta+(1-t)b)\le tf(a)+m(1-t)f\left(\frac{b}{m}\right)$$

and

$$f(tb+(1-t)a)\le tf(b)+m(1-t)f\left(\frac{a}{m}\right)$$

for all $t\in[0,1]$. Integrating on $[0,1]$ we obtain

$$\int_0^1 f(ta+(1-t)b)\,dt\le\frac{\left[f(a)+mf\left(\frac{b}{m}\right)\right]}{2}$$

and

$$\int_0^1 f(tb+(1-t)a)\,dt\le\frac{\left[f(b)+mf\left(\frac{a}{m}\right)\right]}{2}.$$

However,

$$\int_0^1 f(ta+(1-t)b)\,dt=\int_0^1 f(tb+(1-t)a)\,dt=\frac{1}{b-a}\int_a^b f(x)\,dx$$

and the inequality (1.5) is obtained.

Another result of this type which holds for differentiable functions is embodied in the following theorem [34].

Theorem 3. *Let $f:[0,\infty)\to\mathbb{R}$ be a $m-$convex function with $m\in(0,1]$. If $0\le a<b<\infty$ and f is differentiable on $(0,\infty)$, then one has the inequality:*

$$\begin{aligned}\frac{f(mb)}{m}-\frac{b-a}{2}f'(mb)&\le\frac{1}{b-a}\int_a^b f(x)\,dx \\ &\le\frac{(b-ma)f(b)-(a-mb)f(a)}{2(b-a)}.\end{aligned}\tag{1.6}$$

Proof. Using Lemma 3, we have for all $x, y \geq 0$ with $x \geq my$ that

$$(x - my) f'(x) \geq f(x) - mf(y). \tag{1.7}$$

Choosing in the above inequality $x = mb$ and $a \leq y \leq b$, then $x \geq my$ and

$$(mb - my) f'(mb) \geq f(mb) - mf(y).$$

Integrating over y on $[a, b]$, we get

$$m\frac{(b-a)^2}{2} f'(mb) \geq (b-a) f(mb) - m\int_a^b f(y)\, dy,$$

thus proving the first inequality in (1.6).
Putting in (1.7) $y = a$, we have

$$(x - ma) f'(x) \geq f(x) - mf(a), \ x \geq ma.$$

Integrating over x on $[a, b]$, we obtain the second inequality in (1.6).

Remark 2. The second inequality from (1.6) is also valid for $m = 0$. That is, if $f : [0, \infty) \to \mathbb{R}$ is a differentiable starshaped function, then for all $0 \leq a < b < \infty$ one has:

$$\frac{1}{b-a}\int_a^b f(x)\, dx \leq \frac{bf(b) - af(a)}{2(b-a)},$$

which also holds from Corollary 2.

2. The New Results

We will now point out some new results of the Hermite-Hadamard type.

Theorem 4. *Let $f : [0, \infty) \to \mathbb{R}$ be a $m-$convex function with $m \in (0, 1]$ and $0 \leq a < b$. If $f \in L_1[a, b]$, then one has the inequalities*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2} dx \\ &\leq \frac{m+1}{4}\left[\frac{f(a) + f(b)}{2} + m \cdot \frac{f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)}{2}\right]. \end{aligned} \tag{2.1}$$

Proof. By the $m-$convexity of f we have that

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\left[f(x) + mf\left(\frac{y}{m}\right)\right]$$

for all $x, y \in [0, \infty)$.

If we choose $x = ta + (1-t)\,b,\ y = (1-t)\,a + tb$, we deduce

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\left[f\left(ta+(1-t)\,b\right) + mf\left((1-t)\cdot\frac{a}{m} + t\cdot\frac{b}{m}\right)\right]$$

for all $t \in [0,1]$.

Integrating over $t \in [0,1]$ we get

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\left[\int_0^1 f\left(ta+(1-t)\,b\right)dt + m\int_0^1 f\left((1-t)\cdot\frac{a}{m} + t\cdot\frac{b}{m}\right)dt\right]. \tag{2.2}$$

Taking into account that

$$\int_0^1 f\left(ta+(1-t)\,b\right)dt = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

and

$$\int_0^1 f\left(t\cdot\frac{a}{m} + (1-t)\cdot\frac{b}{m}\right)dt = \frac{m}{b-a}\int_{\frac{a}{m}}^{\frac{b}{m}} f(x)\,dx = \frac{1}{b-a}\int_a^b f\left(\frac{x}{m}\right)dx,$$

we deduce from (2.2) the first part of (2.1).

By the $m-$convexity of f we also have

$$\begin{aligned}&\frac{1}{2}\left[f\left(ta+(1-t)\,b\right) + mf\left((1-t)\cdot\frac{a}{m} + t\cdot\frac{b}{m}\right)\right] \\ &\le \frac{1}{2}\left[tf(a) + m(1-t)\,f\left(\frac{b}{m}\right) + m(1-t)\,f\left(\frac{a}{m}\right) + m^2 tf\left(\frac{b}{m^2}\right)\right]\end{aligned} \tag{2.3}$$

for all $t \in [0,1]$.

Integrating the inequality (2.3) over t on $[0,1]$, we deduce

$$\frac{1}{b-a}\int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2}dx \le \frac{1}{2}\left[\frac{f(a) + mf\left(\frac{b}{m}\right)}{2} + \frac{mf\left(\frac{a}{m}\right) + m^2 f\left(\frac{b}{m^2}\right)}{2}\right]. \tag{2.4}$$

By a similar argument we can state:

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2}dx \\ &\le \frac{1}{8}\left[f(a) + f(b) + 2m\left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)\right) + m^2\left(f\left(\frac{a}{m^2}\right) + f\left(\frac{b}{m^2}\right)\right)\right]\end{aligned} \tag{2.5}$$

and the proof is completed.

Remark 3. For $m = 1$, we can drop the assumption $f \in L_1[a,b]$ and (2.1) exactly becomes the Hermite-Hadamard inequality.

The following result also holds.

Theorem 5. *Let $f : [0,\infty) \to \mathbb{R}$ be a $m-$convex function with $m \in (0,1]$. If $f \in L_1[am,b]$ where $0 \le a < b$, then one has the inequality:*

$$\frac{1}{m+1}\left[\int_a^{mb} f(x)\,dx + \frac{mb-a}{b-ma}\int_{ma}^{b} f(x)\,dx\right] \le (mb-a)\,\frac{f(a)+f(b)}{2}. \tag{2.6}$$

Proof. By the $m-$convexity of f we can write:

$$\begin{aligned}
f(ta+m(1-t)b) &\le tf(a)+m(1-t)f(b),\\
f((1-t)a+mtb) &\le (1-t)f(a)+mtf(b),\\
f(tb+(1-t)ma) &\le tf(b)+m(1-t)f(a)
\end{aligned}$$

and

$$f((1-t)b+tma) \le (1-t)f(b)+mtf(a)$$

for all $t \in [0,1]$ and a,b as above.

If we add the above inequalities we get

$$\begin{aligned}
& f(ta+m(1-t)b)+f((1-t)a+mtb)\\
& +f(tb+(1-t)ma)+f((1-t)b+tma)\\
& \le f(a)+f(b)+m(f(a)+f(b)) = (m+1)(f(a)+f(b)).
\end{aligned}$$

Integrating over $t \in [0,1]$, we obtain

$$\begin{aligned}
& \int_0^1 f(ta+m(1-t)b)\,dt + \int_0^1 f((1-t)a+mtb)\,dt \\
& + \int_0^1 f(tb+m(1-t)a)\,dt + \int_0^1 f((1-t)b+mta)\,dt \\
& \le (m+1)(f(a)+f(b)).
\end{aligned} \tag{2.7}$$

As it is easy to see that

$$\int_0^1 f(ta+m(1-t)b)\,dt = \int_0^1 f((1-t)a+mtb)\,dt = \frac{1}{mb-a}\int_a^{mb} f(x)\,dx$$

and

$$\int_0^1 f(tb+m(1-t)a)\,dt = \int_0^1 f((1-t)b+mta)\,dt = \frac{1}{b-ma}\int_{ma}^{b} f(x)\,dx,$$

from (2.7) we deduce the desired result, namely, the inequality (2.6).

Remark 4. For an extensive literature on Hermite-Hadamard type inequalities, see the references enclosed.

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