



Research Article

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On some new quantum midpoint-type inequalities for twice quantum differentiable convex functions

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Abstract: The present paper aims to find some new midpoint-type inequalities for twice quantum differentiable convex functions. The consequences derived in this paper are unification and generalization of the comparable consequences in the literature on midpoint inequalities.

Keywords: Hermite-Hadamard inequality, q -integral, quantum calculus, convex function

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1 Introduction

Quantum calculus, sometimes called calculus without limits, is equivalent to the traditional infinitesimal calculus without the notion of limits. In the field of q -analysis, many studies have recently been carried out. Euler started this field because of the very high demand of mathematics that models quantum computing q -calculus appeared as a connection between physics and mathematics. It has applications in numerous areas of mathematics, such as combinatorics, number theory, basic hypergeometric functions, and orthogonal polynomials, and in fields of other sciences, such as mechanics, theory of relativity, and quantum theory [1–7]. Apparently, Euler was the founder of this branch of mathematics, by using the parameter q in Newton's work on infinite series. Later, the q -calculus was first given by Jackson [8]. In 1908–1909, Jackson defined the general q -integral and q -difference operator [3]. In 1969, Agarwal described the q -fractional derivative for the first time [9]. In 1966–1967, Al-Salam introduced a q -analog of the Riemann-Liouville fractional integral operator and q -fractional integral operator [10]. In 2004, Rajkovic gave a definition of the Riemann-type q -integral which generalized to Jackson q -integral. In 2013, Tariboon introduced ${}_aD_q$ -difference operator [11].

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Many integral inequalities well known in classical analysis, such as Hölder inequality, Simpson's inequality, Newton's inequality, Hermite-Hadamard inequality and Ostrowski inequality, Cauchy-Bunyakovsky-Schwarz, Gruss, Gruss-Cebysev, and other integral inequalities, have been proved and applied in q -calculus using classical convexity. Many mathematicians have carried out studies in q -calculus analysis, and the interested reader can check [11–23].

In this paper, motivated through q -calculus, we found bounds for q -midpoint integral inequalities. We first obtain an identity for twice q -differentiable functions. After that, through the derived identity, we attain some new outcomes for q -midpoint inequalities. With all this, we revealed the results we found in the classical analysis by using $q \rightarrow 1^-$.

2 Preliminaries and definitions of q -calculus

Throughout this paper, let $a < b$ and let $0 < q < 1$ be a constant. The definitions and theorems for q -derivative and q -integral of a function f on $[a, b]$ are as follows.

Definition 1. [4,11] For a function $f : [a, b] \rightarrow \mathbb{R}$, the q_a -derivative of f at $x \in [a, b]$ is characterized by the expression

$${}_a D_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a. \quad (1)$$

If $x = a$, we define ${}_a D_q \phi(a) = \lim_{x \rightarrow a} {}_a D_q \phi(x)$ if it exists and it is finite.

In 2004, Rajkovic *et al.* [24] gave the following definition of the Riemann-type q -integral which was generalized to Jackson q -integral on $[a, b]$:

$$\int_a^x f(s) {}_a d_q s = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), \quad x \in [a, b]. \quad (2)$$

Definition 2. If $a = 0$ in (2), then $\int_0^x f(s) {}_0 d_q s = \int_0^x f(s) d_q s$, where $\int_0^x f(s) d_q s$ is the familiar q -definite integral on $[0, x]$ defined by the expression (see [4])

$$\int_0^x f(s) {}_0 d_q s = \int_0^x f(s) d_q s = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n x). \quad (3)$$

If $c \in (a, x)$, then the q -definite integral on $[c, x]$ is expressed as

$$\int_c^x f(s) {}_a d_q s = \int_a^x f(s) {}_a d_q s - \int_a^c f(s) {}_a d_q s. \quad (4)$$

Alp *et al.* [11] proved the q -Hermite-Hadamard inequality; in [25], the authors proved the same inequality by removing the differentiability assumptions as follows:

Theorem 1. (q -Hermite-Hadamard inequality) Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function on $[a, b]$ and $0 < q < 1$. Then, we have

$$f\left(\frac{qa+b}{1+q}\right) \leq \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \leq \frac{qf(a) + f(b)}{1+q}.$$

On the other hand, in [26], Bermudo et al. gave the following new definitions of quantum integral and derivative. In the same paper, the authors also proved a new variant of the quantum Hermite-Hadamard-type inequality linked to their newly defined quantum integral:

Definition 3. [26] Let $f : [a, b] \rightarrow \mathbb{R}$ be a function. Then, the q^b -definite integral on $[a, b]$ is given by

$$\int_a^b f(x) {}^b d_q x = (1 - q)(b - a) \sum_{n=0}^{\infty} q^n f(q^n a + (1 - q^n)b) = (b - a) \int_0^1 f(sa + (1 - s)b) d_q s.$$

Definition 4. [26] Let $f : [a, b] \rightarrow \mathbb{R}$ be a function. Then, the q^b -derivative of f at $x \in [a, b]$ is given by

$${}^b D_q f(x) = \frac{f(qx + (1 - q)b) - f(x)}{(1 - q)(b - x)}, \quad x \neq b.$$

Theorem 2. [26] If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function on $[a, b]$ and $0 < q < 1$, then, q -Hermite-Hadamard inequalities are given as follows:

$$f\left(\frac{a + qb}{1 + q}\right) \leq \frac{1}{b - a} \int_a^b f(x) {}^b d_q x \leq \frac{f(a) + qf(b)}{1 + q}. \tag{5}$$

We frequently used the following notations:

$$[n]_q = \begin{cases} \frac{q^n - 1}{q - 1} = \sum_{i=0}^{n-1} q^i, & n \in \mathbb{N}, \\ \frac{q^n - 1}{q - 1}, & n \in \mathbb{C}, \end{cases}$$

and

$$(1 - s)_q^n = (s, q)_n = \prod_{i=0}^{n-1} (1 - q^i s). \tag{6}$$

Lemma 1. [11] For $\alpha \in \mathbb{R} \setminus \{-1\}$, the following formula holds:

$$\int_a^x (s - a)^\alpha {}_a d_q s = \frac{(x - a)^{\alpha+1}}{[\alpha + 1]_q}. \tag{7}$$

Lemma 2. [27] The following equality is valid:

$$\int_{\frac{1}{[2]_q}}^1 (1 - qs)_q^n d_q s = \frac{\left(1 - \frac{1}{[2]_q}\right)_q^{n+1}}{[n + 1]_q}.$$

Lemma 3. [27] The following equality is valid:

$$\int_{\frac{1}{[2]_q}}^1 s(1 - qs)_q^2 d_q s = \frac{2q + 4q^2 + q^3 - q^4 - q^5}{[2]_q^4 [3]_q [4]_q}. \tag{8}$$

Lemma 4. [27] The following equality is valid:

$$\int_{\frac{1}{[2]_q}}^1 (1 - s)(1 - qs)_q^2 d_q s = \frac{-q - q^2 + 2q^3 + 3q^4 + 2q^5 - q^6 - q^7}{[2]_q^4 [3]_q [4]_q}.$$

3 An identity for q^b -integrals

In this section, we will prove an equality which will help to obtain our main results.

Lemma 5. Let $f : [a, b] \rightarrow \mathbb{R}$ be a q -differentiable function on (a, b) and $q \in (0, 1)$. If ${}^bD_q^2 f$ is continuous and integrable on $[a, b]$, then we attain the identity

$$\frac{(b-a)^2}{[2]_q} \left[\int_0^{\frac{1}{[2]_q}} q^3 s^2 {}^bD_q^2 f(sa + (1-s)b) d_q s + \int_{\frac{1}{[2]_q}}^1 (1-qs)^2 {}^bD_q^2 f(sa + (1-s)b) d_q s \right] = \frac{1}{b-a} \int_a^b f(s) {}^b d_q s - f\left(\frac{a+qb}{[2]_q}\right).$$

Proof. From Definition 4, we have the following equality:

$$\begin{aligned} & {}^bD_q^2 f(sa + (1-s)b) \\ &= {}^bD_q ({}^bD_q (f(sa + (1-s)b))) \\ &= {}^bD_q \left(\frac{f(qsa + (1-qs)b) - f(sa + (1-s)b)}{(1-q)(b-a)s} \right) \\ &= \frac{1}{(1-q)(b-a)s} \left[\frac{f(q^2sa + (1-sq^2)b) - f(qsa + (1-qs)b)}{(1-q)q(b-a)s} - \frac{f(qsa + (1-qs)b) - f(sa + (1-s)b)}{(1-q)(b-a)s} \right] \\ &= \frac{f(q^2sa + (1-sq^2)b) - f(qsa + (1-qs)b)}{(1-q)^2q(b-a)^2s^2} - \frac{f(qsa + (1-qs)b) - f(sa + (1-s)b)}{(1-q)^2(b-a)^2s^2} \\ &= \frac{f(q^2sa + (1-sq^2)b) - (1+q)f(qsa + (1-qs)b) + qf(sa + (1-s)b)}{(1-q)^2q(b-a)^2s^2}. \end{aligned} \quad (9)$$

From (9) and the fundamental properties of q -integrals, we have

$$\begin{aligned} & \int_0^{\frac{1}{[2]_q}} q^3 s^2 {}^bD_q^2 f(sa + (1-s)b) d_q s + \int_{\frac{1}{[2]_q}}^1 (1-qs)^2 {}^bD_q^2 f(sa + (1-s)b) d_q s \\ &= \int_0^{\frac{1}{[2]_q}} q^3 s^2 {}^bD_q^2 f(sa + (1-s)b) d_q s + \int_0^1 (1-qs)^2 {}^bD_q^2 f(sa + (1-s)b) d_q s - \int_0^{\frac{1}{[2]_q}} (1-qs)^2 {}^bD_q^2 f(sa + (1-s)b) d_q s \\ &= \int_0^1 (1-qs)^2 {}^bD_q^2 f(sa + (1-s)b) d_q s + \int_0^{\frac{1}{[2]_q}} (q^3 s^2 - (1-qs)^2) {}^bD_q^2 f(sa + (1-s)b) d_q s \\ &= \frac{1}{(1-q)^2(b-a)^2} \int_0^{\frac{1}{[2]_q}} \frac{(1-qs)^2}{s^2} \left(\frac{1}{q} f((q^2sa + (1-q^2s)b)) - \frac{1+q}{q} f(qsa + (1-qs)b) + f(sa + (1-s)b) \right) d_q s \\ &\quad + \frac{1}{(1-q)^2(b-a)^2} \int_0^{\frac{1}{[2]_q}} (q^3 s^2 - (1-qs)^2) \left(\frac{1}{q} f((q^2sa + (1-q^2s)b)) - \frac{1+q}{q} f(qsa + (1-qs)b) + f(sa + (1-s)b) \right) d_q s \\ &= \frac{1}{(1-q)^2(b-a)^2} [I_1 + I_2]. \end{aligned} \quad (10)$$

Now, let us calculate the values of integrals I_1 and I_2 as follows:

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{[2]_q}} \frac{(1-qs)^2}{s^2} \left(\frac{1}{q} f((q^2sa + (1-q^2s)b)) - \frac{1+q}{q} f(qsa + (1-qs)b) + f(sa + (1-s)b) \right) d_q s \\ &= (1-q) \sum_{n=0}^{\infty} q^n \frac{(1-qq^n)^2}{q^{2n}} \left\{ \frac{1}{q} f((q^2q^n a + (1-q^2q^n)b)) - \frac{1+q}{q} f(qq^n a + (1-qq^n)b) + f(q^n a + (1-q^n)b) \right\} \end{aligned}$$

$$\begin{aligned}
 &= (1 - q) \left\{ \frac{1}{q} \sum_{n=0}^{\infty} \frac{(1 - qq^{n+2})_q^2}{q^n} f((q^{n+2}a + (1 - q^{n+2})b)) - \frac{1 + q}{q} \sum_{n=0}^{\infty} \frac{(1 - qq^{n+1})_q^2}{q^n} f(q^{n+1}a + (1 - q^{n+1})b) \right. \\
 &\quad \left. + \sum_{n=0}^{\infty} \frac{(1 - qq^n)_q^2}{q^n} f(q^n a + (1 - q^n)b) \right\} \\
 &= \frac{(1 - q)}{q} \sum_{n=2}^{\infty} \frac{(1 - qq^{n-2})_q^2}{q^{n-2}} f((q^n a + (1 - q^n)b)) - \frac{(1 - q)(1 + q)}{q} \sum_{n=1}^{\infty} \frac{(1 - qq^{n-1})_q^2}{q^{n-1}} f(q^n a + (1 - q^n)b) \\
 &\quad + (1 - q) \sum_{n=0}^{\infty} \frac{(1 - qq^n)_q^2}{q^n} f(q^n a + (1 - q^n)b) \\
 &= \frac{(1 - q)}{q} \sum_{n=0}^{\infty} \frac{(1 - qq^{n-2})_q^2}{q^{n-2}} f((q^n a + (1 - q^n)b)) - \frac{(1 - q)(1 + q)}{q} \sum_{n=0}^{\infty} \frac{(1 - qq^{n-1})_q^2}{q^{n-1}} f(q^n a + (1 - q^n)b) \\
 &\quad + (1 - q) \sum_{n=0}^{\infty} \frac{(1 - qq^n)_q^2}{q^n} f(q^n a + (1 - q^n)b) + \frac{(1 - q)(1 + q)}{q} \frac{(1 - qq^{-1})_q^2}{q^{-1}} f(a) \\
 &\quad - \frac{(1 - q)}{q} \frac{(1 - qq^{-2})_q^2}{q^{-2}} f(a) - \frac{(1 - q)}{q} \frac{(1 - qq^{-1})_q^2}{q^{-1}} f((qa + (1 - q)b)) \\
 &= (1 - q) \sum_{n=0}^{\infty} \left\{ \frac{(1 - qq^{n-2})_q^2}{q^{n-1}} - \frac{(1 + q)(1 - qq^{n-1})_q^2}{q} + \frac{(1 - qq^n)_q^2}{q^n} \right\} f(q^n a + (1 - q^n)b) \\
 &\quad + [(1 - q^2)(1 - qq^{-1})_q^2 - q(1 - q)(1 - qq^{-2})_q^2] f(a) - (1 - q)(1 - qq^{-1})_q^2 f(qa + (1 - q)b) \\
 &= (1 - q) \sum_{n=0}^{\infty} \left\{ \frac{q(1 - qq^{n-2})_q^2 - (1 + q)(1 - qq^{n-1})_q^2 + (1 - qq^n)_q^2}{q^n} \right\} f(q^n a + (1 - q^n)b) \\
 &\quad + [(1 - q^2)(1 - qq^{-1})_q^2 - q(1 - q)(1 - qq^{-2})_q^2] f(a) - (1 - q)(1 - qq^{-1})_q^2 f(qa + (1 - q)b) \\
 &= (1 + q)(1 - q)^2(1 - q) \sum_{n=0}^{\infty} q^n f(q^n a + (1 - q^n)b) \\
 &= \frac{(1 + q)(1 - q)^2}{b - a} (1 - q)(b - a) \sum_{n=0}^{\infty} q^n f(q^n a + (1 - q^n)b) \\
 &= \frac{(1 + q)(1 - q)^2}{b - a} \int_a^b f(s)^b d_q s.
 \end{aligned}$$

By similar operations, we can establish

$$\begin{aligned}
 I_2 &= \int_0^{\frac{1}{[2]_q}} \frac{(q^3 s^2 - (1 - qs)_q^2)}{s^2} \left(\frac{1}{q} f((q^2 sa + (1 - q^2 s)b)) - \frac{1 + q}{q} f(qsa + (1 - qs)b) + f(sa + (1 - s)b) \right) d_q s \\
 &= \frac{(1 - q)}{[2]_q} \sum_{n=0}^{\infty} q^n \frac{\left(q^3 \left(\frac{q^n}{[2]_q} \right)^2 - \left(1 - q \frac{q^n}{[2]_q} \right)_q^2 \right)}{\left(\frac{q^n}{[2]_q} \right)^2} \\
 &\quad \times \left(\frac{1}{q} f \left(\frac{q^{n+2}}{[2]_q} a + \left(1 - \frac{q^{n+2}}{[2]_q} \right) b \right) - \frac{1 + q}{q} f \left(\frac{q^{n+1}}{[2]_q} a + \left(1 - \frac{q^{n+1}}{[2]_q} \right) b \right) + f \left(\frac{q^n}{[2]_q} a + \left(1 - \frac{q^n}{[2]_q} \right) b \right) \right) \\
 &= \frac{(1 - q)}{[2]_q} \sum_{n=0}^{\infty} \frac{\left(q^3 q^{2n} - [2]_q^2 \left(1 - q \frac{q^n}{[2]_q} \right)_q^2 \right)}{q^n} \\
 &\quad \times \left(\frac{1}{q} f \left(\frac{q^{n+2}}{[2]_q} a + \left(1 - \frac{q^{n+2}}{[2]_q} \right) b \right) - \frac{1 + q}{q} f \left(\frac{q^{n+1}}{[2]_q} a + \left(1 - \frac{q^{n+1}}{[2]_q} \right) b \right) + f \left(\frac{q^n}{[2]_q} a + \left(1 - \frac{q^n}{[2]_q} \right) b \right) \right)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{(1-q)}{[2]_q} \sum_{n=0}^{\infty} \frac{\left(q^3 q^{2n} - [2]_q^2 \left(1 - q \frac{q^n}{[2]_q} \right)_q^2 \right)}{q^{n+1}} f \left(\frac{q^{n+2}}{[2]_q} a + \left(1 - \frac{q^{n+2}}{[2]_q} \right) b \right) \\
&\quad - \frac{(1-q)(1+q)}{[2]_q} \sum_{n=0}^{\infty} \frac{\left(q^3 q^{2n} - [2]_q^2 \left(1 - q \frac{q^n}{[2]_q} \right)_q^2 \right)}{q^{n+1}} f \left(\frac{q^{n+1}}{[2]_q} a + \left(1 - \frac{q^{n+1}}{[2]_q} \right) b \right) \\
&\quad + \frac{(1-q)}{[2]_q} \sum_{n=0}^{\infty} \frac{\left(q^3 q^{2n} - [2]_q^2 \left(1 - q \frac{q^n}{[2]_q} \right)_q^2 \right)}{q^n} f \left(\frac{q^n}{[2]_q} a + \left(1 - \frac{q^n}{[2]_q} \right) b \right) \\
&= \frac{(1-q)}{[2]_q} \sum_{n=2}^{\infty} \frac{\left(q^3 q^{2n-4} - [2]_q^2 \left(1 - q \frac{q^{n-2}}{[2]_q} \right)_q^2 \right)}{q^{n-1}} f \left(\frac{q^n}{[2]_q} a + \left(1 - \frac{q^n}{[2]_q} \right) b \right) \\
&\quad - \frac{(1-q)(1+q)}{[2]_q} \sum_{n=1}^{\infty} \frac{\left(q^3 q^{2n-2} - [2]_q^2 \left(1 - q \frac{q^{n-1}}{[2]_q} \right)_q^2 \right)}{q^n} f \left(\frac{q^n}{[2]_q} a + \left(1 - \frac{q^n}{[2]_q} \right) b \right) \\
&\quad + \frac{(1-q)}{[2]_q} \sum_{n=0}^{\infty} \frac{\left(q^3 q^{2n} - [2]_q^2 \left(1 - q \frac{q^n}{[2]_q} \right)_q^2 \right)}{q^n} f \left(\frac{q^n}{[2]_q} a + \left(1 - \frac{q^n}{[2]_q} \right) b \right) \\
&= \frac{(1-q)}{[2]_q} \sum_{n=0}^{\infty} \frac{\left(q^3 q^{2n-4} - [2]_q^2 \left(1 - q \frac{q^{n-2}}{[2]_q} \right)_q^2 \right)}{q^{n-1}} f \left(\frac{q^n}{[2]_q} a + \left(1 - \frac{q^n}{[2]_q} \right) b \right) \\
&\quad - \frac{(1-q)(1+q)}{[2]_q} \sum_{n=0}^{\infty} \frac{\left(q^3 q^{2n-2} - [2]_q^2 \left(1 - q \frac{q^{n-1}}{[2]_q} \right)_q^2 \right)}{q^n} f \left(\frac{q^n}{[2]_q} a + \left(1 - \frac{q^n}{[2]_q} \right) b \right) \\
&\quad + \frac{(1-q)}{[2]_q} \sum_{n=0}^{\infty} \frac{\left(q^3 q^{2n} - [2]_q^2 \left(1 - q \frac{q^n}{[2]_q} \right)_q^2 \right)}{q^n} f \left(\frac{q^n}{[2]_q} a + \left(1 - \frac{q^n}{[2]_q} \right) b \right) \\
&\quad - \frac{(1-q)}{[2]_q} \frac{\left(q^3 q^{-4} - [2]_q^2 \left(1 - q \frac{q^{-2}}{[2]_q} \right)_q^2 \right)}{q^{-1}} f \left(\frac{a+qb}{[2]_q} \right) - \frac{(1-q)}{[2]_q} \frac{\left(q^3 q^{-2} - [2]_q^2 \left(1 - q \frac{q^{-1}}{[2]_q} \right)_q^2 \right)}{q^0} f \left(\frac{qa+b}{[2]_q} \right) \\
&\quad + \frac{(1-q)(1+q)}{[2]_q} \frac{\left(q^3 q^{-2} - [2]_q^2 \left(1 - q \frac{q^{-1}}{[2]_q} \right)_q^2 \right)}{q^0} f \left(\frac{a+qb}{[2]_q} \right) \\
&= \frac{(1-q)}{[2]_q} \sum_{n=0}^{\infty} \left\{ \frac{\left(\frac{q^3 q^{2n-4} - [2]_q^2 \left(1 - q \frac{q^{n-2}}{[2]_q} \right)_q^2}{q^{n-1}} - \frac{(1+q) \left(q^3 q^{2n-2} - [2]_q^2 \left(1 - q \frac{q^{n-1}}{[2]_q} \right)_q^2 \right)}{q^n} \right)}{\left(\frac{q^3 q^{2n} - [2]_q^2 \left(1 - q \frac{q^n}{[2]_q} \right)_q^2}{q^n} \right)} \right\} f \left(\frac{q^n}{[2]_q} a + \left(1 - \frac{q^n}{[2]_q} \right) b \right)
\end{aligned}$$

$$\begin{aligned}
 & + \frac{(1-q)}{[2]_q} \left\{ (1+q) \left(q^3 q^{-2} - [2]_q^2 \left(1 - q \frac{q^{-1}}{[2]_q} \right)^2 \right) - q \left(q^3 q^{-4} - [2]_q^2 \left(1 - q \frac{q^{-2}}{[2]_q} \right)^2 \right) \right\} f \left(\frac{a+qb}{[2]_q} \right) \\
 & - \frac{(1-q)}{[2]_q} \left(q^3 q^{-2} - [2]_q^2 \left(1 - q \frac{q^{-1}}{[2]_q} \right)^2 \right) f \left(\frac{qa+b}{[2]_q} \right) \\
 = & \frac{(1-q)}{[2]_q} \sum_{n=0}^{\infty} \frac{1}{q^n} \left\{ q \left(q^3 q^{2n-4} - [2]_q^2 \left(1 - q \frac{q^{n-2}}{[2]_q} \right)^2 \right) - (1+q) \left(q^3 q^{2n-2} - [2]_q^2 \left(1 - q \frac{q^{n-1}}{[2]_q} \right)^2 \right) \right\} f \left(\frac{q^n}{[2]_q} a + \left(1 - \frac{q^n}{[2]_q} \right) b \right) \\
 & + \left(q^3 q^{2n} - [2]_q^2 \left(1 - q \frac{q^n}{[2]_q} \right)^2 \right) \\
 & + \frac{(1-q)}{[2]_q} \left\{ (1+q) \left(q^3 q^{-2} - [2]_q^2 \left(1 - q \frac{q^{-1}}{[2]_q} \right)^2 \right) - q \left(q^3 q^{-4} - [2]_q^2 \left(1 - q \frac{q^{-2}}{[2]_q} \right)^2 \right) \right\} f \left(\frac{a+qb}{[2]_q} \right) \\
 & - \frac{(1-q)}{[2]_q} \left(q^3 q^{-2} - [2]_q^2 \left(1 - q \frac{q^{-1}}{[2]_q} \right)^2 \right) f \left(\frac{qa+b}{[2]_q} \right) \\
 = & \frac{(1-q)}{[2]_q} \left\{ (1+q) \left(q - [2]_q^2 \left(1 - \frac{1}{[2]_q} \right)^2 \right) - q \left(\frac{1}{q} - [2]_q^2 \left(1 - \frac{q^{-1}}{[2]_q} \right)^2 \right) \right\} f \left(\frac{a+qb}{[2]_q} \right) \\
 = & \frac{(1-q)}{[2]_q} \{[-1 + q^2 + q^3 - q]\} f \left(\frac{a+qb}{[2]_q} \right) \\
 = & \frac{(1-q)}{1+q} (1+q) \{q^2 - 1\} f \left(\frac{a+qb}{[2]_q} \right) \\
 = & -(1-q)^2 (1+q) f \left(\frac{a+qb}{[2]_q} \right).
 \end{aligned}$$

By substituting the values of integrals I_1 and I_2 in (10), we attain

$$\begin{aligned}
 & \int_0^{\frac{1}{[2]_q}} q^3 s^{2b} D_q^2 f(sa + (1-s)b) d_q s + \int_{\frac{1}{[2]_q}}^1 (1-qs)^2 {}^b D_q^2 f(sa + (1-s)b) d_q s \\
 & = \frac{1}{(1-q)^2 (b-a)^2} \left(\frac{(1+q)(1-q)^2}{b-a} \int_a^b f(s) {}^b d_q s - (1-q)^2 (1+q) f \left(\frac{a+qb}{[2]_q} \right) \right) \\
 & = \frac{[2]_q}{(b-a)^2} \left(\frac{1}{b-a} \int_a^b f(s) {}^b d_q s - f \left(\frac{a+qb}{[2]_q} \right) \right),
 \end{aligned}$$

which completes the proof. □

Remark 1. If we take the limit $q \rightarrow 1^-$ in Lemma 5, then we have the equality

$$\frac{(b-a)^2}{2} \left[\int_0^{\frac{1}{2}} s^2 f''(sa + (1-s)b) ds + \int_{\frac{1}{2}}^1 (1-s)^2 f''(sa + (1-s)b) ds \right] = \frac{1}{b-a} \int_a^b f(s) ds - f \left(\frac{a+b}{2} \right)$$

which was given by Sarikaya et al. in [24].

4 New midpoint-type inequalities for quantum integrals

In this section, we obtain some new midpoint-type inequalities for newly defined quantum integrals utilizing twice q -differentiable convex functions.

Theorem 3. *Suppose that the assumptions of Lemma 5 hold. If $|{}^bD_q^2f|$ is convex on $[a, b]$, then we have the inequality*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(s) {}^b d_q s - f\left(\frac{a+qb}{[2]_q}\right) \right| \\ & \leq \frac{(b-a)^2}{[2]_q^5 [3]_q [4]_q} [(2q+4q^2+2q^3)|{}^bD_q^2f(a)| + (-q-q^2+2q^3+4q^4+3q^5+q^6)|{}^bD_q^2f(b)|]. \end{aligned} \tag{11}$$

Proof. By taking the modulus in Lemma 5, we have

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(s) {}^b d_q s - f\left(\frac{a+qb}{[2]_q}\right) \right| \\ & \leq \frac{(b-a)^2}{[2]_q} \left[\int_0^{\frac{1}{[2]_q}} q^3 s^2 |{}^bD_q^2f(sa+(1-s)b)| d_q s + \int_{\frac{1}{[2]_q}}^1 (1-qs)_q^2 |{}^bD_q^2f(sa+(1-s)b)| d_q s \right]. \end{aligned}$$

By using the convexity of $|{}^bD_q^2f|$, we can write

$$\begin{aligned} \left| \frac{1}{b-a} \int_a^b f(s) {}^b d_q s - f\left(\frac{a+qb}{[2]_q}\right) \right| & \leq \frac{q^3(b-a)^2}{[2]_q} \int_0^{\frac{1}{[2]_q}} (s^3 |{}^bD_q^2f(a)| + (s^2-s^3)|{}^bD_q^2f(b)|) d_q s \\ & + \frac{(b-a)^2}{[2]_q} \int_{\frac{1}{[2]_q}}^1 (s(1-qs)_q^2 |{}^bD_q^2f(a)| + (1-s)(1-qs)_q^2 |{}^bD_q^2f(b)|) d_q s. \end{aligned}$$

We have the fact that

$$\begin{aligned} \int_0^{\frac{1}{[2]_q}} s^3 d_q s & = \frac{1}{[2]_q^4 [4]_q}, \\ \int_0^{\frac{1}{[2]_q}} (s^2-s^3) d_q s & = \left(\frac{1}{[2]_q^3 [3]_q} - \frac{1}{[2]_q^4 [4]_q} \right), \\ \int_{\frac{1}{[2]_q}}^1 s(1-qs)_q^2 d_q s & = \frac{2q+4q^2+q^3-q^4-q^5}{[2]_q^4 [3]_q [4]_q}, \end{aligned}$$

and

$$\int_{\frac{1}{[2]_q}}^1 (1-s)(1-qs)_q^2 |{}^bD_q^2f(b)| d_q s = \frac{-q-q^2+2q^3+3q^4+2q^5-q^6-q^7}{[2]_q^4 [3]_q [4]_q}.$$

By these equalities, we can write

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(s) {}^b d_q s - f\left(\frac{a+qb}{[2]_q}\right) \right| \\ & \leq \frac{q^3(b-a)^2}{[2]_q} \left(\frac{(1+q+q^2) |{}^b D_q^2 f(a)|}{[2]_q^4 [3]_q [4]_q} + \frac{q+q^2+2q^3+q^4}{[2]_q^4 [3]_q [4]_q} |{}^b D_q^2 f(b)| \right) \\ & \quad + \frac{(b-a)^2}{[2]_q} \left[\frac{2q+4q^2+q^3-q^4-q^5}{[2]_q^4 [3]_q [4]_q} |{}^b D_q^2 f(a)| + \frac{-q-q^2+2q^3+3q^4+2q^5-q^6-q^7}{[2]_q^4 [3]_q [4]_q} |{}^b D_q^2 f(b)| \right] \\ & = \frac{(b-a)^2}{[2]_q^5 [3]_q [4]_q} [(2q+4q^2+2q^3) |{}^b D_q^2 f(a)| + (-q-q^2+2q^3+4q^4+3q^5+q^6) |{}^b D_q^2 f(b)|]. \end{aligned}$$

This completes the proof. □

Remark 2. If we take the limit $q \rightarrow 1^-$ in Theorem 3, then we obtain the inequality

$$\left| \frac{1}{b-a} \int_a^b f(s) ds - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{48} (|f''(a)| + |f''(b)|)$$

which was proved by Sarikaya et al. in [28, Theorem 3].

Theorem 4. Suppose that the assumptions of Lemma 5 hold. If $|{}^b D_q^2 f|^{r_1}$, $r_1 > 1$, is convex function on $[a, b]$, then we have the inequality

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(s) {}^b d_q s - f\left(\frac{a+qb}{[2]_q}\right) \right| \\ & \leq \frac{q^3(b-a)^2}{[2]_q^{\frac{3+r_1}{r_1} + \frac{2r_2+1}{r_2}} [2r_2+1]_q^{\frac{1}{r_2}}} (|{}^b D_q^2 f(a)|^{r_1} + (2q+q^2) |{}^b D_q^2 f(b)|^{r_1})^{\frac{1}{r_1}} \\ & \quad + \frac{q^{\frac{1}{r_1}}(b-a)^2 \left(1 - \frac{1}{[2]_q}\right)_q^{\frac{2r_2+1}{r_2}}}{[2]_q^{\frac{3+r_1}{r_1}} [2r_2+1]_q^{\frac{1}{r_2}}} ((2+q) |{}^b D_q^2 f(a)|^{r_1} + (q+q^2-1) |{}^b D_q^2 f(b)|^{r_1})^{\frac{1}{r_1}}, \end{aligned}$$

where $\frac{1}{r_1} + \frac{1}{r_2} = 1$.

Proof. From Lemma 5 and using well-known q -Hölder’s integral inequality, we get

$$\begin{aligned} \left| \frac{1}{b-a} \int_a^b f(s) {}^b d_q s - f\left(\frac{a+qb}{[2]_q}\right) \right| & \leq \frac{(b-a)^2}{[2]_q} q^3 \left(\int_0^{\frac{1}{[2]_q}} s^{2r_2} d_q s \right)^{\frac{1}{r_2}} \left(\int_0^{\frac{1}{[2]_q}} |{}^b D_q^2 f(sa + (1-s)b)|^{r_1} d_q s \right)^{\frac{1}{r_1}} \\ & \quad + \frac{(b-a)^2}{[2]_q} \left(\int_{\frac{1}{[2]_q}}^1 (1-qs)^{2r_2} d_q s \right)^{\frac{1}{r_2}} \left(\int_{\frac{1}{[2]_q}}^1 |{}^b D_q^2 f(sa + (1-s)b)|^{r_1} d_q s \right)^{\frac{1}{r_1}}. \end{aligned}$$

Since $|{}^b D_q^2 f|^{r_1}$ is convex on $[a, b]$, we have

$$|{}^b D_q^2 f(sa + (1-s)b)|^{r_1} \leq s |{}^b D_q^2 f(a)|^{r_1} + (1-s) |{}^b D_q^2 f(b)|^{r_1}.$$

By Lemma 2, we get

$$\begin{aligned}
 & \left| \frac{1}{b-a} \int_a^b f(s) {}^b d_q s - f\left(\frac{a+qb}{[2]_q}\right) \right| \\
 & \leq \frac{(b-a)^2}{[2]_q} q^3 \left(\frac{1}{[2]_q^{\frac{2r_2+1}{r_2}} [2r_2+1]_q^{\frac{1}{r_2}}} \left(\int_0^{\frac{1}{[2]_q}} (s |{}^b D_q^2 f(a)|^{r_1} + (1-s) |{}^b D_q^2 f(b)|^{r_1}) d_q s \right)^{\frac{1}{r_1}} \right. \\
 & \quad \left. + \frac{(b-a)^2}{[2]_q} \left(\frac{(1-\frac{1}{[2]_q})^{2r_2+1}}{[2r_2+1]_q} \right)^{\frac{1}{r_2}} \left(\int_{\frac{1}{[2]_q}}^1 (s |{}^b D_q^2 f(a)|^{r_1} + (1-s) |{}^b D_q^2 f(b)|^{r_1}) d_q s \right)^{\frac{1}{r_1}} \right) \\
 & = \left(\frac{q^3(b-a)^2}{[2]_q [2]_q^{\frac{2r_2+1}{r_2}} [2r_2+1]_q^{\frac{1}{r_2}}} \right) \left(\frac{|{}^b D_q^2 f(a)|^{r_1}}{[2]_q^3} + \frac{(2q+q^2) |{}^b D_q^2 f(b)|^{r_1}}{[2]_q^3} \right)^{\frac{1}{r_1}} \\
 & \quad + \frac{(b-a)^2}{[2]_q} \frac{\left(1-\frac{1}{[2]_q}\right)^{\frac{2r_2+1}{r_2}}}{[2r_2+1]_q^{\frac{1}{r_2}}} \left(\frac{(2q+q^2) |{}^b D_q^2 f(a)|^{r_1}}{[2]_q^3} + \frac{(q^2+q^3-q) |{}^b D_q^2 f(b)|^{r_1}}{[2]_q^3} \right)^{\frac{1}{r_1}} \\
 & = \left(\frac{q^3(b-a)^2}{[2]_q^{\frac{3+r_1}{r_1}} [2]_q^{\frac{2r_2+1}{r_2}} [2r_2+1]_q^{\frac{1}{r_2}}} \right) (|{}^b D_q^2 f(a)|^{r_1} + (2q+q^2) |{}^b D_q^2 f(b)|^{r_1})^{\frac{1}{r_1}} \\
 & \quad + \frac{(b-a)^2 \left(1-\frac{1}{[2]_q}\right)^{\frac{2r_2+1}{r_2}}}{[2]_q^{\frac{3+r_1}{r_1}} [2r_2+1]_q^{\frac{1}{r_2}}} ((2q+q^2) |{}^b D_q^2 f(a)|^{r_1} + (q^2+q^3-q) |{}^b D_q^2 f(b)|^{r_1})^{\frac{1}{r_1}} \\
 & = \frac{q^3(b-a)^2}{[2]_q^{\frac{3+r_1}{r_1} + \frac{2r_2+1}{r_2}} [2r_2+1]_q^{\frac{1}{r_2}}} (|{}^b D_q^2 f(a)|^{r_1} + (2q+q^2) |{}^b D_q^2 f(b)|^{r_1})^{\frac{1}{r_1}} \\
 & \quad + \frac{q^{\frac{1}{r_1}}(b-a)^2 \left(1-\frac{1}{[2]_q}\right)^{\frac{2r_2+1}{r_2}}}{[2]_q^{\frac{3+r_1}{r_1}} [2r_2+1]_q^{\frac{1}{r_2}}} ((2+q) |{}^b D_q^2 f(a)|^{r_1} + (q+q^2-1) |{}^b D_q^2 f(b)|^{r_1})^{\frac{1}{r_1}}.
 \end{aligned}$$

Thus, the proof is completed. □

Remark 3. If we take the limit $q \rightarrow 1^-$ in Theorem 4, then we obtain the inequality

$$\left| \frac{1}{b-a} \int_a^b f(s) ds - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{2^{4+\frac{2}{r_1}}(2r_2+1)^{\frac{1}{r_2}}} \left\{ (3|f'''(a)|^{r_1} + |f'''(b)|^{r_1})^{\frac{1}{r_1}} + (|f''(a)|^{r_1} + 3|f''(b)|^{r_1})^{\frac{1}{r_1}} \right\}$$

which was proved by Noor and Awan in [29, Theorem 3 (for $\alpha = s = 1$)].

Theorem 5. Suppose that the assumptions of Lemma 5 hold. If $|{}^b D_q^2 f|^{r_1}$, $r_1 \geq 1$, is convex on $[a, b]$, then we have the inequality

$$\begin{aligned}
 & \left| \frac{1}{b-a} \int_a^b f(s) {}^b d_q s - f\left(\frac{a+qb}{[2]_q}\right) \right| \\
 & \leq \frac{q^3(b-a)^2}{[2]_q^{4-\frac{3}{r_1}} [3]_q^{1-\frac{1}{r_1}}} \left(\frac{|{}^b D_q^2 f(a)|^{r_1}}{[2]_q^4 [4]_q} + \frac{(q+q^2+2q^3+q^4) |{}^b D_q^2 f(b)|^{r_1}}{[2]_q^4 [3]_q [4]_q} \right)^{\frac{1}{r_1}} + \frac{(q+q^2-q^3)^{\frac{1}{r_1}}(b-a)^2}{[2]_q^{4-\frac{3}{r_1}} [3]_q^{1-\frac{1}{r_1}}} \\
 & \quad \times \left(\frac{(2q+4q^2+q^3-q^4-q^5) |{}^b D_q^2 f(a)|^{r_1}}{[2]_q^4 [3]_q [4]_q} + \frac{(-q-q^2+2q^3+3q^4+2q^5-q^6-q^7) |{}^b D_q^2 f(b)|^{r_1}}{[2]_q^4 [3]_q [4]_q} \right)^{\frac{1}{r_1}}.
 \end{aligned}$$

Proof. Using Lemma 5 and power-mean integral inequality, we get

$$\left| \frac{1}{b-a} \int_a^b f(s) {}^b d_q s - f\left(\frac{a+qb}{[2]_q}\right) \right| \leq \frac{(b-a)^2}{[2]_q} \left[q^3 \left(\int_0^{\frac{1}{[2]_q}} s^2 d_q s \right)^{1-\frac{1}{r_1}} \left(\int_0^{\frac{1}{[2]_q}} s^2 |{}^b D_q^2 f(sa + (1-s)b)|^{r_1} d_q s \right)^{\frac{1}{r_1}} \right. \\ \left. + \frac{(b-a)^2}{[2]_q} \left(\int_{\frac{1}{[2]_q}}^1 (1-qs)_q^2 d_q s \right)^{1-\frac{1}{r_1}} \left(\int_{\frac{1}{[2]_q}}^1 (1-qs)_q^2 |{}^b D_q^2 f(sa + (1-s)b)|^{r_1} d_q s \right)^{\frac{1}{r_1}} \right].$$

By using the convexity of $|{}^b D_q^2 f|^{r_1}$ and Lemmas 2–4, we get

$$\left| \frac{1}{b-a} \int_a^b f(s) {}^b d_q s - f\left(\frac{a+qb}{[2]_q}\right) \right| \leq \frac{(b-a)^2}{[2]_q} \frac{q^3}{[2]_q^{3-\frac{3}{r_1}} [3]_q^{1-\frac{1}{r_1}}} \left(|{}^b D_q^2 f(a)|^{r_1} \int_0^{\frac{1}{[2]_q}} s^3 d_q s + |{}^b D_q^2 f(b)|^{r_1} \int_0^{\frac{1}{[2]_q}} (s^2 - s^3) d_q s \right)^{\frac{1}{r_1}} \\ + \frac{(b-a)^2}{[2]_q} \frac{\left[\left(1 - \frac{1}{[2]_q}\right)_q \right]^3}{[3]_q^{1-\frac{1}{r_1}}} \left(|{}^b D_q^2 f(a)|^{r_1} \int_{\frac{1}{[2]_q}}^1 s(1-qs)_q^2 d_q s + |{}^b D_q^2 f(b)|^{r_1} \int_{\frac{1}{[2]_q}}^1 (1-s)(1-qs)_q^2 d_q s \right)^{\frac{1}{r_1}} \\ = \frac{q^3(b-a)^2}{[2]_q^{4-\frac{3}{r_1}} [3]_q^{1-\frac{1}{r_1}}} \left(\frac{|{}^b D_q^2 f(a)|^{r_1}}{[2]_q^4 [4]_q} + \frac{(q+q^2+2q^3+q^4)|{}^b D_q^2 f(b)|^{r_1}}{[2]_q^4 [3]_q [4]_q} \right)^{\frac{1}{r_1}} + \frac{(q+q^2-q^3)^{\frac{1}{r_1}}(b-a)^2}{[2]_q^{4-\frac{3}{r_1}} [3]_q^{1-\frac{1}{r_1}}} \\ \times \left(\frac{(2q+4q^2+q^3-q^4-q^5)|{}^b D_q^2 f(a)|^{r_1}}{[2]_q^4 [3]_q [4]_q} + \frac{(-q-q^2+2q^3+3q^4+2q^5-q^6-q^7)|{}^b D_q^2 f(b)|^{r_1}}{[2]_q^4 [3]_q [4]_q} \right)^{\frac{1}{r_1}}$$

which completes the proof. □

Remark 4. If we take the limit $q \rightarrow 1$ in Theorem 5, then we obtain the inequality

$$\left| \frac{1}{b-a} \int_a^b f(s) ds - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{3\left(2^{4+\frac{3}{r_1}}\right)} \left\{ (5|f''(a)|^{r_1} + 3|f''(b)|^{r_1})^{\frac{1}{r_1}} + (3|f''(a)|^{r_1} + 5|f''(b)|^{r_1})^{\frac{1}{r_1}} \right\}$$

which was proved by Noor and Awan in [29, Theorem 4 (for $\alpha = s = 1$)].

5 Concluding remarks

In this investigation, some new estimates of midpoint-type inequalities for quantum twice differentiable convex functions are attained. It is also proved that the outcomes of the present paper are the strong generalization of the existing comparable consequences in the literature. Upcoming researchers can find the Simpson-like inequalities, Newton-like inequalities, Ostrowski-like inequalities, and the same inequalities by using different kinds of convexities in their future work.

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