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# ON SOME NEW SEQUENCE SPACES OF NON-ABSOLUTE TYPE RELATED TO THE SPACES $\ell_p$ AND $\ell_\infty$ I

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**Abstract**

In the present paper, we introduce the sequence space $\ell_p^\lambda$ of non-absolute type and prove that the spaces $\ell_p^\lambda$ and $\ell_p$ are linearly isomorphic for $0 < p \leq \infty$. Further, we show that $\ell_p^\lambda$ is a $p$-normed space and a $BK$-space in the cases of $0 < p < 1$ and $1 \leq p \leq \infty$, respectively. Furthermore, we derive some inclusion relations concerning the space $\ell_p^\lambda$. Finally, we construct the basis for the space $\ell_p^\lambda$, where $1 \leq p < \infty$.

## 1 Introduction

By $w$, we denote the space of all real or complex valued sequences. Any vector subspace of $w$ is called a *sequence space*.

A sequence space $X$ with a linear topology is called a $K$-*space* provided each of the maps $p_n : X \to \mathbb{C}$ defined by $p_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. A $K$-space $X$ is called an $FK$-*space* provided $X$ is a complete linear metric space. An $FK$-space whose topology is normable is called a $BK$-*space* [9, pp.272-273].

We shall write $\ell_\infty$, $c$ and $c_0$ for the sequence spaces of all bounded, convergent and null sequences, respectively, which are $BK$-spaces with the same sup-norm given by

$$\|x\|_{\ell_\infty} = \sup_k |x_k|,$$

where, here and in the sequel, the supremum $\sup_k$ is taken over all $k \in \mathbb{N}$. Also by $\ell_p$ $(0 < p < \infty)$, we denote the sequence space of all sequences associated with $p$-absolutely convergent series. It is known that $\ell_p$ is a complete $p$-normed space and a $BK$-space in the cases of $0 < p < 1$ and $1 \leq p < \infty$ with respect to the usual $p$-norm and $\ell_p$-norm defined by

$$\|x\|_{\ell_p} = \sum_k |x_k|^p; \qquad (0 < p < 1)$$

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and

$$\|x\|_{\ell_p} = \Big(\sum_k |x_k|^p\Big)^{1/p}; \quad (1 \le p < \infty),$$

respectively (see [11, pp.217-218]). For simplicity in notation, here and in what follows, the summation without limits runs from 0 to $\infty$.

Let $X$ and $Y$ be sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers $a_{nk}$, where $n, k \in \mathbb{N}$. Then, we say that $A$ defines a *matrix mapping* from $X$ into $Y$ if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{A_n(x)\}$, the $A$-*transform* of $x$, exists and is in $Y$, where

$$A_n(x) = \sum_k a_{nk} x_k; \quad (n \in \mathbb{N}). \tag{1}$$

By $(X : Y)$, we denote the class of all infinite matrices that map $X$ into $Y$. Thus $A \in (X : Y)$ if and only if the series on the right side of (1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and $Ax \in Y$ for all $x \in X$.

For a sequence space $X$, the *matrix domain* of an infinite matrix $A$ in $X$ is defined by

$$X_A = \big\{x \in w : Ax \in X\big\} \tag{2}$$

which is a sequence space.

We shall write $e^{(k)}$ for the sequence whose only non-zero term is a 1 in the $k^{\text{th}}$ place for each $k \in \mathbb{N}$.

The approach of constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by several authors, e.g., Wang [19], Ng and Lee [18], Malkowsky [12], Başar and Altay [7], Malkowsky and Savaş [13], Aydın and Başar [3, 4, 5, 6], Altay and Başar [1], Altay, Başar and Mursaleen [2, 14] and Mursaleen and Noman [15, 16], respectively. They introduced the sequence spaces $(\ell_\infty)_{N_q}$ and $c_{N_q}$ in [19], $(\ell_\infty)_{C_1} = X_\infty$ and $(\ell_p)_{C_1} = X_p$ in [18], $(\ell_\infty)_{R^t} = r^t_\infty$, $c_{R^t} = r^t_c$ and $(c_0)_{R^t} = r^t_0$ in [12], $(\ell_p)_\Delta = bv_p$ in [7], $\mu_G = Z(u, v; \mu)$ in [13], $(c_0)_{A^r} = a^r_0$ and $c_{A^r} = a^r_c$ in [3], $[c_0(u, p)]_{A^r} = a^r_0(u, p)$ and $[c(u, p)]_{A^r} = a^r_c(u, p)$ in [4], $(a^r_0)_\Delta = a^r_0(\Delta)$ and $(a^r_c)_\Delta = a^r_c(\Delta)$ in [5], $(\ell_p)_{A^r} = a^r_p$ and $(\ell_\infty)_{A^r} = a^r_\infty$ in [6], $(c_0)_{E^r} = e^r_0$ and $c_{E^r} = e^r_c$ in [1], $(\ell_p)_{E^r} = e^r_p$ and $(\ell_\infty)_{E^r} = e^r_\infty$ in [2, 14], $(c_0)_\Lambda = c^\lambda_0$ and $c_\Lambda = c^\lambda$ in [15] and $(c^\lambda_0)_\Delta = c^\lambda_0(\Delta)$ and $(c^\lambda)_\Delta = c^\lambda(\Delta)$ in [16], where $N_q$, $C_1$, $R^t$ and $E^r$ denote the Nörlund, Cesàro, Riesz and Euler means, respectively, $\Delta$ denotes the band matrix defining the difference operator, $G$ and $A^r$ are defined in [13] and [3], respectively, $\Lambda$ is defined in Section 2, below, $\mu \in \{c_0, c, \ell_p\}$ and $1 \le p < \infty$. Also $c_0(u, p)$ and $c(u, p)$ denote the sequence spaces generated from the Maddox's spaces $c_0(p)$ and $c(p)$ by Başarır [8]. The main purpose of the present paper, following [1, 2, 3, 4, 5, 6, 7, 12, 13, 14, 15, 16, 18] and [19], is to introduce the sequence spaces $\ell^\lambda_p$ and $\ell^\lambda_\infty$ of non-absolute type and is to derive some results related to them. Further, we establish some inclusion relations concerning the spaces $\ell^\lambda_p$ and $\ell^\lambda_\infty$, where $0 < p < \infty$. Moreover, we construct the basis for the space $\ell^\lambda_p$, where $1 \le p < \infty$.

## 2 $\lambda$-boundedness and $p$-absolute convergence of type $\lambda$

Throughout this paper, let $\lambda = (\lambda_k)_{k=0}^{\infty}$ be a strictly increasing sequence of positive reals tending to $\infty$, that is

$$0 < \lambda_0 < \lambda_1 < \cdots \quad \text{and} \quad \lambda_k \to \infty \quad \text{as} \quad k \to \infty. \tag{3}$$

We say that a sequence $x = (x_k) \in w$ is $\lambda$-*bounded* if $\sup_n |\Lambda_n(x)| < \infty$, where

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k; \qquad (n \in \mathbb{N}). \tag{4}$$

Also, we say that the associated series $\sum_k x_k$ is *$p$-absolutely convergent of type $\lambda$* if $\sum_n |\Lambda_n(x)|^p < \infty$, where $0 < p < \infty$.

Here and in the sequel, we shall use the convention that any term with a negative subscript is equal to zero, e.g., $\lambda_{-1} = 0$ and $x_{-1} = 0$.

Now, let $x = (x_k)$ be a bounded sequence in the ordinary sense of boundedness, i.e., $x \in \ell_\infty$. Then, there is a constant $M > 0$ such that $|x_k| \leq M$ for all $k \in \mathbb{N}$. Thus, we have for every $n \in \mathbb{N}$ that

$$\begin{aligned} |\Lambda_n(x)| &\leq \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) |x_k| \\ &\leq \frac{M}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) = M \end{aligned}$$

which shows that $x$ is $\lambda$-bounded. Therefore, we deduce that the ordinary boundedness implies the $\lambda$-boundedness. This leads us to the following basic result:

**Lemma 2.1.** *Every bounded sequence is $\lambda$-bounded.*

We shall later show that the converse implication need not be true. Further, we shall show that for every $0 < p < \infty$ there is a sequence $\lambda = (\lambda_k)$ satisfying (3) such that the convergence of the series $\sum_n |x_k|^p$ does not imply the convergence of the series $\sum_n |\Lambda_n(x)|^p$, and conversely. Before that, we define the infinite matrix $\Lambda = (\lambda_{nk})_{n,k=0}^{\infty}$ by

$$\lambda_{nk} = \begin{cases} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n}; & (0 \leq k \leq n), \\ 0; & (k > n) \end{cases} \tag{5}$$

for all $n, k \in \mathbb{N}$. Then, for any sequence $x = (x_k) \in w$, the $\Lambda$-transform of $x$ is the sequence $\Lambda(x) = \{\Lambda_n(x)\}$, where $\Lambda_n(x)$ is given by (4) for all $n \in \mathbb{N}$. Therefore, the sequence $x$ is $\lambda$-bounded if and only if $\Lambda(x) \in \ell_\infty$. Also, the notion of $p$-absolute convergence of type $\lambda$ of the sequence $x$ is equivalent to say that $\Lambda(x) \in \ell_p$, where $0 < p < \infty$. Further, it is obvious by (5) that the matrix $\Lambda = (\lambda_{nk})$ is a triangle, i.e., $\lambda_{nn} \neq 0$ and $\lambda_{nk} = 0$ for all $k > n$ ($n \in \mathbb{N}$).

Recently, the sequence spaces $c_0^\lambda$ and $c^\lambda$ have been defined in [15] as the matrix domains of the triangle $\Lambda$ in the spaces $c_0$ and $c$, respectively, that is

$$c_0^\lambda = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \left( \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right) = 0 \right\}$$

and

$$c^\lambda = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \left( \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right) \text{ exists} \right\}.$$

Also, it has been shown that the inclusions $c_0 \subset c_0^\lambda$ and $c \subset c^\lambda$ hold and the inclusion $c_0^\lambda \subset c^\lambda$ strictly holds.

Finally, we define the sequence $y(\lambda) = \{y_k(\lambda)\}$, which will be frequently used, as the $\Lambda$-transform of a sequence $x = (x_k)$, i.e., $y(\lambda) = \Lambda(x)$ and so we have

$$y_k(\lambda) = \sum_{j=0}^{k} \Big( \frac{\lambda_j - \lambda_{j-1}}{\lambda_k} \Big) x_j; \qquad (k \in \mathbb{N}). \tag{6}$$

## 3 The sequence spaces $\ell_p^\lambda$ and $\ell_\infty^\lambda$ of non-absolute type

In the present section, as a natural continuation of Mursaleen and Noman [15], we introduce the sequence spaces $\ell_p^\lambda$ and $\ell_\infty^\lambda$, as the sets of all sequences whose $\Lambda$-transforms are in the spaces $\ell_p$ and $\ell_\infty$, respectively, where $0 < p < \infty$, that is

$$\ell_p^\lambda = \left\{ x = (x_k) \in w : \sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right|^p < \infty \right\}; \qquad (0 < p < \infty)$$

and

$$\ell_\infty^\lambda = \left\{ x = (x_k) \in w : \sup_n \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right| < \infty \right\}.$$

With the notation of (2), we can redefine the spaces $\ell_p^\lambda$ and $\ell_\infty^\lambda$ as follows:

$$\ell_p^\lambda = (\ell_p)_\Lambda \ \ (0 < p < \infty) \ \text{ and } \ \ell_\infty^\lambda = (\ell_\infty)_\Lambda. \tag{7}$$

Then, it is obvious by (7) that $\ell_\infty^\lambda$ and $\ell_p^\lambda$ $(0 < p < \infty)$ are sequence spaces consisting of all sequences which are $\lambda$-bounded and $p$-absolutely convergent of type $\lambda$, respectively. Further, we have the following result which is essential in the text.

**Theorem 3.1.** *We have the following:*

(a) *If $0 < p < 1$, then $\ell_p^\lambda$ is a complete $p$-normed space with the $p$-norm $\|x\|_{\ell_p^\lambda} = \|\Lambda(x)\|_{\ell_p}$, that is*

$$\|x\|_{\ell_p^\lambda} = \sum_n |\Lambda_n(x)|^p; \qquad (0 < p < 1). \tag{8}$$

(b) *If* $1 \leq p \leq \infty$, *then* $\ell_p^\lambda$ *is a BK-space with the norm* $\|x\|_{\ell_p^\lambda} = \|\Lambda(x)\|_{\ell_p}$, *that is*

$$\|x\|_{\ell_p^\lambda} = \Big(\sum_n |\Lambda_n(x)|^p\Big)^{1/p}; \qquad (1 \leq p < \infty) \tag{9}$$

*and*

$$\|x\|_{\ell_\infty^\lambda} = \sup_n |\Lambda_n(x)|. \tag{10}$$

*Proof.* Since the matrix $\Lambda$ is a triangle, this result is immediate by (7) and Theorem 4.3.12 of Wilansky [20, p.63]. □

**Remark 3.2.** One can easily check that the absolute property does not hold on the space $\ell_p^\lambda$, that is $\|x\|_{\ell_p^\lambda} \neq \||x|\|_{\ell_p^\lambda}$ for at least one sequence in the space $\ell_p^\lambda$, and this tells us that $\ell_p^\lambda$ is a sequence space of non-absolute type, where $|x| = (|x_k|)$ and $0 < p \leq \infty$.

**Theorem 3.3.** *The sequence space* $\ell_p^\lambda$ *of non-absolute type is isometrically isomorphic to the space* $\ell_p$, *that is* $\ell_p^\lambda \cong \ell_p$ *for* $0 < p \leq \infty$.

*Proof.* To prove this, we should show the existence of an isometric isomorphism between the spaces $\ell_p^\lambda$ and $\ell_p$, where $0 < p \leq \infty$. For, let $0 < p \leq \infty$ and consider the transformation $T$ defined, with the notation of (6), from $\ell_p^\lambda$ to $\ell_p$ by $x \longmapsto y(\lambda) = Tx$. Then, we have $Tx = y(\lambda) = \Lambda(x) \in \ell_p$ for every $x \in \ell_p^\lambda$. Also, the linearity of $T$ is trivial. Further, it is easy to see that $x = 0$ whenever $Tx = 0$ and hence $T$ is injective.

Furthermore, let $y = (y_k) \in \ell_p$ be given and define the sequence $x = \{x_k(\lambda)\}$ by

$$x_k(\lambda) = \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\lambda_j}{\lambda_k - \lambda_{k-1}} y_j; \qquad (k \in \mathbb{N}). \tag{11}$$

Then, by using (4) and (11), we have for every $n \in \mathbb{N}$ that

$$\begin{aligned}
\Lambda_n(x) &= \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k(\lambda) \\
&= \frac{1}{\lambda_n} \sum_{k=0}^{n} \sum_{j=k-1}^{k} (-1)^{k-j} \lambda_j y_j \\
&= \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k y_k - \lambda_{k-1} y_{k-1}) \\
&= y_n.
\end{aligned}$$

This shows that $\Lambda(x) = y$ and since $y \in \ell_p$, we obtain that $\Lambda(x) \in \ell_p$. Thus, we deduce that $x \in \ell_p^\lambda$ and $Tx = y$. Hence $T$ is surjective.

Moreover, for any $x \in \ell_p^\lambda$, we have by (8), (9) and (10) of Theorem 3.1 that

$$\|Tx\|_{\ell_p} = \|y(\lambda)\|_{\ell_p} = \|\Lambda(x)\|_{\ell_p} = \|x\|_{\ell_p^\lambda}$$

which shows that $T$ is $p$-norm and norm preserving in the cases of $0 < p < 1$ and $1 \le p \le \infty$, respectively. Hence $T$ is isometry. Consequently, the spaces $\ell_p^\lambda$ and $\ell_p$ are isometrically isomorphic for $0 < p \le \infty$. This concludes the proof. □

Now, one may expect the similar result for the space $\ell_p^\lambda$ as was observed for the space $\ell_p$, and ask the natural question: Is not the space $\ell_p^\lambda$ a Hilbert space with $p \neq 2$? The answer is positive and is given by the following theorem:

**Theorem 3.4.** *Except the case $p = 2$, the space $\ell_p^\lambda$ is not an inner product space, hence not a Hilbert space for $1 \le p < \infty$.*

*Proof.* We have to prove that the space $\ell_2^\lambda$ is the only Hilbert space among the $\ell_p^\lambda$ spaces for $1 \le p < \infty$. Since the space $\ell_2^\lambda$ is a $BK$-space with the norm $\|x\|_{\ell_2^\lambda} = \|\Lambda(x)\|_{\ell_2}$ by Theorem 3.1 and its norm can be obtained from an inner product, i.e., the equality

$$\|x\|_{\ell_2^\lambda} = \langle x, x\rangle^{1/2} = \langle \Lambda(x), \Lambda(x)\rangle_2^{1/2}$$

holds for every $x \in \ell_2^\lambda$, the space $\ell_2^\lambda$ is a Hilbert space, where $\langle \cdot, \cdot\rangle_2$ denotes the inner product on $\ell_2$.

Let us now consider the sequences

$$u = \{u_k(\lambda)\} = \Big(1, 1, \frac{-\lambda_1}{\lambda_2 - \lambda_1}, 0, 0, \ldots\Big)$$

and

$$v = \{v_k(\lambda)\} = \Big(1, -\frac{\lambda_1 + \lambda_0}{\lambda_1 - \lambda_0}, \frac{\lambda_1}{\lambda_2 - \lambda_1}, 0, 0, \ldots\Big).$$

Then, we have

$$\Lambda(u) = (1, 1, 0, 0, \ldots) \text{ and } \Lambda(v) = (1, -1, 0, 0, \ldots).$$

Thus, it can easily be seen that

$$\|u + v\|_{\ell_p^\lambda}^2 + \|u - v\|_{\ell_p^\lambda}^2 = 8 \neq 4(2^{2/p}) = 2\Big(\|u\|_{\ell_p^\lambda}^2 + \|v\|_{\ell_p^\lambda}^2\Big); \quad (p \neq 2),$$

that is, the norm of the space $\ell_p^\lambda$ with $p \neq 2$ does not satisfy the parallelogram equality which means that this norm cannot be obtained from an inner product. Hence, the space $\ell_p^\lambda$ with $p \neq 2$ is a Banach space which is not a Hilbert space, where $1 \le p < \infty$. This completes the proof. □

**Remark 3.5.** It is obvious that $\ell_\infty^\lambda$ is also a Banach space which is not a Hilbert space.

## 4 Some inclusion relations

In the present section, we establish some inclusion relations concerning the spaces $\ell_p^\lambda$ and $\ell_\infty^\lambda$, where $0 < p < \infty$. We essentially prove that the inclusion $\ell_\infty \subset \ell_\infty^\lambda$ holds and characterize the case in which the inclusion $\ell_p \subset \ell_p^\lambda$ holds for $1 \leq p < \infty$.

We may begin with quoting the following two lemmas (see [15]) which are needed in the proofs of our main results.

**Lemma 4.1.** *For any sequence $x = (x_k) \in w$, the equalities*

$$S_n(x) = x_n - \Lambda_n(x); \quad (n \in \mathbb{N}) \tag{12}$$

*and*

$$S_n(x) = \frac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}}\big[\Lambda_n(x) - \Lambda_{n-1}(x)\big]; \quad (n \in \mathbb{N}) \tag{13}$$

*hold, where $S(x) = \{S_n(x)\}$ is the sequence defined by*

$$S_0(x) = 0 \text{ and } S_n(x) = \frac{1}{\lambda_n}\sum_{k=1}^{n}\lambda_{k-1}(x_k - x_{k-1}); \quad (n \geq 1).$$

**Lemma 4.2.** *For any sequence $\lambda = (\lambda_k)_{k=0}^\infty$ satisfying (3), we have*

(a) $\left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}}\right)_{k=0}^\infty \notin \ell_\infty$ *if and only if* $\liminf_{k\to\infty} \frac{\lambda_{k+1}}{\lambda_k} = 1.$

(b) $\left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}}\right)_{k=0}^\infty \in \ell_\infty$ *if and only if* $\liminf_{k\to\infty} \frac{\lambda_{k+1}}{\lambda_k} > 1.$

It is obvious that Lemma 4.2 still holds if the sequence $\{\lambda_k/(\lambda_k - \lambda_{k-1})\}$ is replaced by $\{\lambda_k/(\lambda_{k+1} - \lambda_k)\}$.

Now, we prove the following:

**Theorem 4.3.** *If $0 < p < q < \infty$, then the inclusion $\ell_p^\lambda \subset \ell_q^\lambda$ strictly holds.*

*Proof.* Let $0 < p < q < \infty$. Then, it follows by the inclusion $\ell_p \subset \ell_q$ that the inclusion $\ell_p^\lambda \subset \ell_q^\lambda$ holds. Further, since the inclusion $\ell_p \subset \ell_q$ is strict, there is a sequence $x = (x_k)$ in $\ell_q$ but not in $\ell_p$, i.e., $x \in \ell_q \setminus \ell_p$. Let us now define the sequence $y = (y_k)$ in terms of the sequence $x$ as follows:

$$y_k = \frac{\lambda_k x_k - \lambda_{k-1}x_{k-1}}{\lambda_k - \lambda_{k-1}}; \quad (k \in \mathbb{N}).$$

Then, we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(y) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k x_k - \lambda_{k-1}x_{k-1}) = x_n$$

which shows that $\Lambda(y) = x$ and hence $\Lambda(y) \in \ell_q \setminus \ell_p$. Thus, the sequence $y$ is in $\ell_q^\lambda$ but not in $\ell_p^\lambda$. Hence, the inclusion $\ell_p^\lambda \subset \ell_q^\lambda$ is strict. This concludes the proof. □

**Theorem 4.4.** *The inclusions $\ell_p^\lambda \subset c_0^\lambda \subset c^\lambda \subset \ell_\infty^\lambda$ strictly hold, where $0 < p < \infty$.*

*Proof.* Since the inclusion $c_0^\lambda \subset c^\lambda$ strictly holds [15, Theorem 4.1], it is enough to show that the inclusions $\ell_p^\lambda \subset c_0^\lambda$ and $c^\lambda \subset \ell_\infty^\lambda$ are strict, where $0 < p < \infty$.

Firstly, it is trivial that the inclusion $\ell_p^\lambda \subset c_0^\lambda$ holds for $0 < p < \infty$, since $x \in \ell_p^\lambda$ implies $\Lambda(x) \in \ell_p$ and hence $\Lambda(x) \in c_0$ which means that $x \in c_0^\lambda$. Further, to show that this inclusion is strict, let $0 < p < \infty$ and consider the sequence $x = (x_k)$ defined by

$$x_k = \frac{1}{(k+1)^{1/p}}; \qquad (k \in \mathbb{N}). \tag{14}$$

Then $x \in c_0$ and hence $x \in c_0^\lambda$, since the inclusion $c_0 \subset c_0^\lambda$ holds. On the other hand, we have for every $n \in \mathbb{N}$ that

$$\begin{aligned} |\Lambda_n(x)| &= \frac{1}{\lambda_n} \sum_{k=0}^{n} \frac{\lambda_k - \lambda_{k-1}}{(k+1)^{1/p}} \\ &\geq \frac{1}{\lambda_n (n+1)^{1/p}} \sum_{k=0}^{n} (\lambda_k - \lambda_k - 1) \\ &= \frac{1}{(n+1)^{1/p}} \end{aligned}$$

which shows that $\Lambda(x) \notin \ell_p$ and hence $x \notin \ell_p^\lambda$. Thus, the sequence $x$ is in $c_0^\lambda$ but not in $\ell_p^\lambda$. Therefore, the inclusion $\ell_p^\lambda \subset c_0^\lambda$ is strict for $0 < p < \infty$.

Similarly, it is also clear that the inclusion $c^\lambda \subset \ell_\infty^\lambda$ holds. To show that this inclusion is strict, we define the sequence $y = (y_k)$ by

$$y_k = (-1)^k \left( \frac{\lambda_k + \lambda_{k-1}}{\lambda_k - \lambda_{k-1}} \right); \qquad (k \in \mathbb{N}).$$

Then, we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(y) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (-1)^k (\lambda_k + \lambda_{k-1}) = (-1)^n$$

which shows that $\Lambda(y) \in \ell_\infty \setminus c$. Thus, the sequence $y$ is in $\ell_\infty^\lambda$ but not in $c^\lambda$ and hence $c^\lambda \subset \ell_\infty^\lambda$ is a strict inclusion. This completes the proof. □

**Lemma 4.5.** *The inclusion $\ell_p^\lambda \subset \ell_p$ holds if and only if $S(x) \in \ell_p$ for every sequence $x \in \ell_p^\lambda$, where $0 < p \leq \infty$.*

*Proof.* Suppose that the inclusion $\ell_p^\lambda \subset \ell_p$ holds, where $0 < p \leq \infty$, and take any $x = (x_k) \in \ell_p^\lambda$. Then $x \in \ell_p$ by the hypothesis. Thus, we obtain from (12) that

$$\|S(x)\|_{\ell_p} \leq \|x\|_{\ell_p} + \|\Lambda(x)\|_{\ell_p} = \|x\|_{\ell_p} + \|x\|_{\ell_p^\lambda} < \infty$$

which yields that $S(x) \in \ell_p$.

Conversely, let $x \in \ell_p^\lambda$ be given, where $0 < p \leq \infty$. Then, we have by the hypothesis that $S(x) \in \ell_p$. Again, it follows by (12) that

$$\|x\|_{\ell_p} \leq \|S(x)\|_{\ell_p} + \|\Lambda(x)\|_{\ell_p} = \|S(x)\|_{\ell_p} + \|x\|_{\ell_p^\lambda} < \infty$$

which shows that $x \in \ell_p$. Hence, the inclusion $\ell_p^\lambda \subset \ell_p$ holds and this concludes the proof. □

**Theorem 4.6.** *The inclusion $\ell_\infty \subset \ell_\infty^\lambda$ holds. Further, the equality holds if and only if $S(x) \in \ell_\infty$ for every sequence $x \in \ell_\infty^\lambda$.*

*Proof.* The first part of the theorem is immediately obtained from Lemma 2.1, and so we turn to the second part. For, suppose firstly that the equality $\ell_\infty^\lambda = \ell_\infty$ holds. Then, the inclusion $\ell_\infty^\lambda \subset \ell_\infty$ holds which leads us with Lemma 4.5 to the consequence that $S(x) \in \ell_\infty$ for every $x \in \ell_\infty^\lambda$.

Conversely, suppose that $S(x) \in \ell_\infty$ for every $x \in \ell_\infty^\lambda$. Then, we deduce by Lemma 4.5 that the inclusion $\ell_\infty^\lambda \subset \ell_\infty$ holds. Combining this with the inclusion $\ell_\infty \subset \ell_\infty^\lambda$, we get the equality $\ell_\infty^\lambda = \ell_\infty$. This completes the proof. □

Now, the following theorem gives the necessary and sufficient condition for the matrix $\Lambda$ to be stronger than boundedness, i.e., for the inclusion $\ell_\infty \subset \ell_\infty^\lambda$ to be strict.

**Theorem 4.7.** *The inclusion $\ell_\infty \subset \ell_\infty^\lambda$ strictly holds if and only if* $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$.

*Proof.* Suppose that the inclusion $\ell_\infty \subset \ell_\infty^\lambda$ is strict. Then, Theorem 4.6 implies the existence of a sequence $x \in \ell_\infty^\lambda$ such that $S(x) = \{S_n(x)\} \notin \ell_\infty$. Since $x \in \ell_\infty^\lambda$, we have $\Lambda(x) = \{\Lambda_n(x)\} \in \ell_\infty$ and hence $\{\Lambda_n(x) - \Lambda_{n-1}(x)\} \in \ell_\infty$. Combining this with the fact that $\{S_n(x)\} \notin \ell_\infty$, we obtain by (13) that $\{\lambda_{n-1}/(\lambda_n - \lambda_{n-1})\} \notin \ell_\infty$ and hence $\{\lambda_n/(\lambda_n - \lambda_{n-1})\} \notin \ell_\infty$. This leads us with Lemma 4.2 (a) to the consequence that $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$ which shows the necessity of the condition.

Conversely, suppose that $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$. Then, we have by Lemma 4.2 (a) that $\{\lambda_n/(\lambda_n - \lambda_{n-1})\} \notin \ell_\infty$. Let us now consider the sequence $x = (x_k)$ defined by $x_k = (-1)^k \lambda_k/(\lambda_k - \lambda_{k-1})$ for all $k \in \mathbb{N}$. Then, it is obvious that $x \notin \ell_\infty$. On the other hand, we have for every $n \in \mathbb{N}$ that

$$|\Lambda_n(x)| = \frac{1}{\lambda_n} \left| \sum_{k=0}^{n} (-1)^k \lambda_k \right| \leq \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) = 1$$

which shows that $\Lambda(x) \in \ell_\infty$ and hence $x \in \ell_\infty^\lambda$. Thus, the sequence $x$ is in $\ell_\infty^\lambda$ but not in $\ell_\infty$. Therefore, by combining this with the inclusion $\ell_\infty \subset \ell_\infty^\lambda$, we deduce that this inclusion is strict. This concludes the proof. □

Now, as a consequence of Theorem 4.7, the following corollary presents the necessary and sufficient condition for the matrix $\Lambda$ to be equivalent to boundedness.

**Corollary 4.8.** *The equality* $\ell_\infty^\lambda = \ell_\infty$ *holds if and only if* $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n > 1$.

*Proof.* The necessity follows immediately from Theorem 4.7. For, if the equality $\ell_\infty^\lambda = \ell_\infty$ holds, then $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n \neq 1$ and hence $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n > 1$.

Conversely, suppose that $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n > 1$. Then, Lemma 4.2 (b) gives us the bounded sequence $\{\lambda_n/(\lambda_n - \lambda_{n-1})\}$ and so $\{\lambda_{n-1}/(\lambda_n - \lambda_{n-1})\} \in \ell_\infty$.

Now, let $x \in \ell_\infty^\lambda$. Then $\Lambda(x) = \{\Lambda_n(x)\} \in \ell_\infty$ and hence $\{\Lambda_n(x) - \Lambda_{n-1}(x)\} \in \ell_\infty$. Thus, we obtain by (13) that $\{S_n(x)\} \in \ell_\infty$. This shows that $S(x) \in \ell_\infty$ for every $x \in \ell_\infty^\lambda$, which leads us with Theorem 4.6 to the equality $\ell_\infty^\lambda = \ell_\infty$. □

Although the inclusions $c_0 \subset c_0^\lambda$, $c \subset c^\lambda$ and $\ell_\infty \subset \ell_\infty^\lambda$ always hold, the inclusion $\ell_p \subset \ell_p^\lambda$ need not be held, where $0 < p < \infty$. In fact, we are going to show, in the following lemma, that if $1/\lambda \notin \ell_p$, then the inclusion $\ell_p \subset \ell_p^\lambda$ fails, where $1/\lambda = (1/\lambda_k)$ and $0 < p < \infty$.

**Lemma 4.9.** *The spaces* $\ell_p$ *and* $\ell_p^\lambda$ *overlap. Further, if* $1/\lambda \notin \ell_p$ *then neither of them includes the other one, where* $0 < p < \infty$.

*Proof.* Obviously, the spaces $\ell_p$ and $\ell_p^\lambda$ overlap, since $(\lambda_1 - \lambda_0, -\lambda_0, 0, 0, \ldots) \in \ell_p \cap \ell_p^\lambda$ for $0 < p < \infty$.

Now, suppose that $1/\lambda \notin \ell_p$, where $0 < p < \infty$, and consider the sequence $x = e^{(0)} = (1, 0, 0, \ldots) \in \ell_p$. Then, we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) e_k^{(0)} = \frac{\lambda_0}{\lambda_n}$$

which shows that $\Lambda(x) \notin \ell_p$ and hence $x \notin \ell_p^\lambda$. Thus, the sequence $x$ is in $\ell_p$ but not in $\ell_p^\lambda$. Hence, the inclusion $\ell_p \subset \ell_p^\lambda$ does not hold when $1/\lambda \notin \ell_p$ $(0 < p < \infty)$.

On the other hand, let $1 \le p < \infty$ and define the sequence $y = (y_k)$ by

$$y_k = \begin{cases} \dfrac{1}{\lambda_k}\,; & (k \text{ is even}), \\[2ex] -\dfrac{1}{\lambda_{k-1}} \Big( \dfrac{\lambda_{k-1} - \lambda_{k-2}}{\lambda_k - \lambda_{k-1}} \Big)\,; & (k \text{ is odd}) \end{cases}$$

for all $k \in \mathbb{N}$. Since $1/\lambda \notin \ell_p$, we have $y \notin \ell_p$. Besides, we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(y) = \begin{cases} \dfrac{1}{\lambda_n} \Big( \dfrac{\lambda_n - \lambda_{n-1}}{\lambda_n} \Big)\,; & (n \text{ is even}), \\[2ex] 0\,; & (n \text{ is odd}) \end{cases}$$

and hence

$$\begin{aligned}\sum_n |\Lambda_n(y)|^p &= \sum_n |\Lambda_{2n}(y)|^p \\ &= \sum_n \frac{1}{\lambda_{2n}^p}\Big(\frac{\lambda_{2n}-\lambda_{2n-1}}{\lambda_{2n}}\Big)^p \\ &\le \frac{1}{\lambda_0^p} + \sum_{n=1}^{\infty} \frac{1}{\lambda_{2n-2}^p}\Big(\frac{\lambda_{2n}-\lambda_{2n-2}}{\lambda_{2n}}\Big)^p \\ &\le \frac{1}{\lambda_0^p} + \sum_{n=1}^{\infty} \frac{1}{\lambda_{2n-2}^p}\Big(\frac{\lambda_{2n}^p-\lambda_{2n-2}^p}{\lambda_{2n}^p}\Big) \\ &= \frac{1}{\lambda_0^p} + \sum_{n=1}^{\infty} \Big(\frac{1}{\lambda_{2n-2}^p} - \frac{1}{\lambda_{2n}^p}\Big) \\ &= \frac{2}{\lambda_0^p} < \infty.\end{aligned}$$

This shows that $\Lambda(y) \in \ell_p$ and so $y \in \ell_p^\lambda$. Thus, the sequence $y$ is in $\ell_p^\lambda$ but not in $\ell_p$, where $1 \le p < \infty$.

Similarly, one can construct a sequence belonging to the set $\ell_p^\lambda \backslash \ell_p$ for $0 < p < 1$. Therefore, the inclusion $\ell_p^\lambda \subset \ell_p$ also fails when $1/\lambda \notin \ell_p$ $(0 < p < \infty)$. Hence, if $1/\lambda \notin \ell_p$ then neither of the spaces $\ell_p$ and $\ell_p^\lambda$ includes the other one, where $0 < p < \infty$. This completes the proof. □

**Lemma 4.10.** *If the inclusion $\ell_p \subset \ell_p^\lambda$ holds, then $1/\lambda \in \ell_p$ for $0 < p < \infty$.*

*Proof.* Suppose that the inclusion $\ell_p \subset \ell_p^\lambda$ holds, where $0 < p < \infty$, and consider the sequence $x = e^{(0)} = (1, 0, 0, \ldots) \in \ell_p$. Then $x \in \ell_p^\lambda$ and hence $\Lambda(x) \in \ell_p$. Thus, we obtain that

$$\lambda_0^p \sum_n \Big(\frac{1}{\lambda_n}\Big)^p = \sum_n |\Lambda_n(x)|^p < \infty$$

which shows that $1/\lambda \in \ell_p$ and this concludes the proof. □

We shall later show that the condition $1/\lambda \in \ell_p$ is not only necessary but also sufficient for the inclusion $\ell_p \subset \ell_p^\lambda$ to be held, where $1 \le p < \infty$. Before that, by taking into account the definition of the sequence $\lambda = (\lambda_k)$ given by (3), we find that

$$0 < \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} < 1; \qquad (0 \le k \le n)$$

for all $n, k \in \mathbb{N}$ with $n + k > 0$. Furthermore, if $1/\lambda \in \ell_1$ then we have the following lemma which is easy to prove.

**Lemma 4.11.** *If $1/\lambda \in \ell_1$, then*

$$\sup_k \left( (\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \right) < \infty.$$

**Theorem 4.12.** *The inclusion $\ell_1 \subset \ell_1^\lambda$ holds if and only if $1/\lambda \in \ell_1$.*

*Proof.* The necessity is immediate by Lemma 4.10.

Conversely, suppose $1/\lambda \in \ell_1$. Then $M = \sup_k \left[(\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} 1/\lambda_n\right] < \infty$ by Lemma 4.11. Also, let $x = (x_k) \in \ell_1$ be given. Then, we have

$$\begin{aligned}
\|x\|_{\ell_1^\lambda} &= \sum_n |\Lambda_n(x)| \\
&\leq \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})|x_k| \\
&= \sum_{k=0}^{\infty} |x_k|(\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \\
&\leq M \sum_{k=0}^{\infty} |x_k| \\
&= M\|x\|_{\ell_1} < \infty.
\end{aligned}$$

This shows that $x \in \ell_1^\lambda$. Hence, the inclusion $\ell_1 \subset \ell_1^\lambda$ holds. □

**Corollary 4.13.** *If $1/\lambda \in \ell_1$, then the inclusion $\ell_p \subset \ell_p^\lambda$ holds for $1 \leq p < \infty$.*

*Proof.* The inclusion trivially holds for $p = 1$, which is obtained by Theorem 4.12, above. Thus, let $1 < p < \infty$ and take any $x = (x_k) \in \ell_p$. Then, for every $n \in \mathbb{N}$, we obtain by applying the Hölder's inequality that

$$\begin{aligned}
|\Lambda_n(x)|^p &\leq \left[\sum_{k=0}^{n} \Big(\frac{\lambda_k - \lambda_{k-1}}{\lambda_n}\Big)|x_k|\right]^p \\
&\leq \left[\sum_{k=0}^{n} \Big(\frac{\lambda_k - \lambda_{k-1}}{\lambda_n}\Big)|x_k|^p\right]\left[\sum_{k=0}^{n} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n}\right]^{p-1} \\
&= \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})|x_k|^p.
\end{aligned}$$

Therefore, we derive that

$$\begin{aligned}
\sum_n |\Lambda_n(x)|^p &\leq \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})|x_k|^p \\
&= \sum_{k=0}^{\infty} |x_k|^p(\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n}
\end{aligned}$$

and hence

$$\|x\|_{\ell_p^\lambda}^p \leq M \sum_{k=0}^{\infty} |x_k|^p = M\,\|x\|_{\ell_p}^p < \infty,$$

where $M = \sup_k \left[(\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} 1/\lambda_n\right] < \infty$ by Lemma 4.11. This shows that $x \in \ell_p^\lambda$. Hence, we deduce that the inclusion $\ell_p \subset \ell_p^\lambda$ also holds for $1 < p < \infty$. This completes the proof. □

**Corollary 4.14.** *The inclusion $\ell_p \subset \ell_p^\lambda$ holds if and only if $1/\lambda \in \ell_p$, where $1 \leq p < \infty$.*

*Proof.* The necessity is immediate by Lemma 4.10.

Conversely, suppose that $1/\lambda \in \ell_p$, where $1 \leq p < \infty$. Then $1/\lambda^p = (1/\lambda_k^p) \in \ell_1$. Thus, it follows by Lemma 4.11 that

$$\sup_k \left( (\lambda_k - \lambda_{k-1})^p \sum_{n=k}^{\infty} \frac{1}{\lambda_n^p} \right) \leq \sup_k \left( (\lambda_k^p - \lambda_{k-1}^p) \sum_{n=k}^{\infty} \frac{1}{\lambda_n^p} \right) < \infty.$$

Further, we have for every fixed $k \in \mathbb{N}$ that

$$\Lambda_n\big(e^{(k)}\big) = \begin{cases} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n}; & (n \geq k), \\ 0; & (n < k). \end{cases} \qquad (n \in \mathbb{N})$$

Thus, we obtain that

$$\|e^{(k)}\|_{\ell_p^\lambda}^p = (\lambda_k - \lambda_{k-1})^p \sum_{n=k}^{\infty} \frac{1}{\lambda_n^p} < \infty; \qquad (k \in \mathbb{N})$$

which yields that $e^{(k)} \in \ell_p^\lambda$ for every $k \in \mathbb{N}$, i.e., every basis element of the space $\ell_p$ is in $\ell_p^\lambda$. This shows that the space $\ell_p^\lambda$ contains the Schauder basis of the space $\ell_p$ such that $\sup_k \|e^{(k)}\|_{\ell_p^\lambda} < \infty$. Hence, we deduce that the inclusion $\ell_p \subset \ell_p^\lambda$ holds and this concludes the proof. □

Now, in the following example, we give an important special case of the space $\ell_p^\lambda$, where $1 \leq p < \infty$.

**Example 4.15.** Consider the special case of the sequence $\lambda = (\lambda_k)$ given by $\lambda_k = k + 1$ for all $k \in \mathbb{N}$. Then $1/\lambda \notin \ell_1$ while $1/\lambda \in \ell_p$ for $1 < p < \infty$. Hence, the inclusion $\ell_1 \subset \ell_1^\lambda$ does not hold by Lemma 4.9.

On the other hand, by applying the well-known inequality (see [10, p.239])

$$\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{|x_k|}{n+1} \right)^p < \left( \frac{p}{p-1} \right)^p \sum_{n=0}^{\infty} |x_n|^p; \qquad (1 < p < \infty),$$

we immediately deduce that the inequality

$$\|x\|_{\ell_p^\lambda} < \left( \frac{p}{p-1} \right) \|x\|_{\ell_p}$$

holds for every $x \in \ell_p$, where $1 < p < \infty$. This shows that the inclusion $\ell_p \subset \ell_p^\lambda$ holds for $1 < p < \infty$. Further, this inclusion is strict. For example, the sequence $y = \{(-1)^k\}$ is not in $\ell_p$ but in $\ell_p^\lambda$, since

$$\sum_n |\Lambda_n(y)|^p = \sum_n \left| \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \right|^p = \sum_n \frac{1}{(2n+1)^p} < \infty; \qquad (1 < p < \infty).$$

**Remark 4.16.** In the special case of the sequence $\lambda = (\lambda_k)$ given in Example 4.15, i.e., $\lambda_k = k+1$ for all $k \in \mathbb{N}$, we may note that the spaces $\ell_p^\lambda$ and $\ell_\infty^\lambda$ are respectively reduced to the Cesàro sequence spaces $X_p$ and $X_\infty$ of non-absolute type, which are defined as the spaces of all sequences whose $C_1$-transforms are in the spaces $\ell_p$ and $\ell_\infty$, respectively, where $1 \leq p < \infty$ (see [17, 18]).

Now, let $x = (x_k)$ be a null sequence of positive reals, that is

$$x_k > 0 \text{ for all } k \in \mathbb{N} \text{ and } x_k \to 0 \text{ as } k \to \infty.$$

Then, as is easy to see, for every positive integer $m$ there is a subsequence $(x_{k_r})_{r=0}^\infty$ of the sequence $x$ such that

$$x_{k_r} = O\left(\frac{1}{(r+1)^{m+1}}\right)$$

and hence

$$(r+1)x_{k_r} = O\left(\frac{1}{(r+1)^m}\right).$$

Further, this subsequence can be chosen such that $k_{r+1} - k_r \geq 2$ for all $r \in \mathbb{N}$.

In general, if $x = (x_k)$ is a sequence of positive reals such that $\liminf_{k\to\infty} x_k = 0$, then there is a subsequence $x' = (x_{k'_r})_{r=0}^\infty$ of the sequence $x$ such that $\lim_{r\to\infty} x_{k'_r} = 0$. Thus $x'$ is a null sequence of positive reals. Hence, as we have seen above, for every positive integer $m$ there is a subsequence $(x_{k_r})_{r=0}^\infty$ of the sequence $x'$, and hence of the sequence $x$, such that $k_{r+1} - k_r \geq 2$ for all $r \in \mathbb{N}$ and

$$(r+1)x_{k_r} = O\left(\frac{1}{(r+1)^m}\right),$$

where $k_r = k'_{\theta(r)}$ and $\theta : \mathbb{N} \to \mathbb{N}$ is a suitable increasing function.

Now, let $0 < p < \infty$. Then, we can choose a positive integer $m$ such that $mp > 1$. In this situation, the sequence $\{(r+1)x_{k_r}\}_{r=0}^\infty$ is in the space $\ell_p$.

Obviously, we observe that the subsequence $(x_{k_r})_{r=0}^\infty$ depends on the positive integer $m$ which is, in turn, depending on $p$. Thus, our subsequence depends on $p$.

Hence, from the above discussion, we conclude the following result:

**Lemma 4.17.** *Let $x = (x_k)$ be a positive real sequence such that $\liminf_{k\to\infty} x_k = 0$. Then, for every positive number $0 < p < \infty$ there is a subsequence $x^{(p)} = (x_{k_r})_{r=0}^\infty$ of $x$, depending on $p$, such that $k_{r+1} - k_r \geq 2$ for all $r \in \mathbb{N}$ and $\sum_r |(r+1)x_{k_r}|^p < \infty$.*

Now, the following theorem gives the necessary and sufficient conditions for the matrix $\Lambda$ to be stronger than $p$-absolute convergence, i.e., for the inclusion $\ell_p \subset \ell_p^\lambda$ to be strict, where $1 \leq p < \infty$.

**Theorem 4.18.** *The inclusion $\ell_p \subset \ell_p^\lambda$ strictly holds if and only if $1/\lambda \in \ell_p$ and $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$, where $1 \leq p < \infty$.*

*Proof.* Suppose that the inclusion $\ell_p \subset \ell_p^\lambda$ is strict, where $1 \leq p < \infty$. Then, the necessity of the first condition is immediate by Lemma 4.10. Further, since the inclusion $\ell_p^\lambda \subset \ell_p$ cannot be held, Lemma 4.5 implies the existence of a sequence $x \in \ell_p^\lambda$ such that $S(x) = \{S_n(x)\} \notin \ell_p$. Since $x \in \ell_p^\lambda$, we have $\sum_n |\Lambda_n(x)|^p < \infty$. Thus, it follows by applying the Minkowski's inequality that $\sum_n |\Lambda_n(x) - \Lambda_{n-1}(x)|^p < \infty$. This means that $\{\Lambda_n(x) - \Lambda_{n-1}(x)\} \in \ell_p$ and since $\{S_n(x)\} \notin \ell_p$, it follows by the relation (13) that $\{\lambda_{n-1}/(\lambda_n - \lambda_{n-1})\} \notin \ell_\infty$ and hence $\{\lambda_n/(\lambda_n - \lambda_{n-1})\} \notin \ell_\infty$. This leads us with Lemma 4.2 (a) to the necessity of the second condition.

Conversely, since $1/\lambda \in \ell_p$, we have by Corollary 4.14 that the inclusion $\ell_p \subset \ell_p^\lambda$ holds. Further, since $\liminf_{k\to\infty} \lambda_{k+1}/\lambda_k = 1$, we obtain by Lemma 4.2 (a) that

$$\liminf_{k\to\infty} \Big(\frac{\lambda_k - \lambda_{k-1}}{\lambda_k}\Big) = 0.$$

Thus, it follows by Lemma 4.17 that there is a subsequence $\lambda^{(p)} = (\lambda_{k_r})_{r=0}^\infty$ of the sequence $\lambda = (\lambda_k)$, depending on $p$, such that $k_{r+1} - k_r \geq 2$ for all $r \in \mathbb{N}$ and

$$\sum_r \Big|(r+1)\Big(\frac{\lambda_{k_r} - \lambda_{k_r-1}}{\lambda_{k_r}}\Big)\Big|^p < \infty. \tag{15}$$

Let us now define the sequence $y = (y_k)$ for every $k \in \mathbb{N}$ by

$$y_k = \begin{cases} r+1\,; & (k = k_r), \\ -(r+1)\Big(\dfrac{\lambda_{k-1} - \lambda_{k-2}}{\lambda_k - \lambda_{k-1}}\Big)\,; & (k = k_r + 1), \qquad (r \in \mathbb{N}) \\ 0\,; & (\text{otherwise}). \end{cases} \tag{16}$$

Then, it is clear that $y \notin \ell_p$. On the other hand, we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(y) = \begin{cases} (r+1)\Big(\dfrac{\lambda_n - \lambda_{n-1}}{\lambda_n}\Big)\,; & (n = k_r), \\ & \qquad\qquad (r \in \mathbb{N}) \\ 0\,; & (n \neq k_r). \end{cases}$$

This and (15) imply that $\Lambda(y) \in \ell_p$ and hence $y \in \ell_p^\lambda$. Thus, the sequence $y$ is in $\ell_p^\lambda$ but not in $\ell_p$. Therefore, we deduce by combining this with the inclusion $\ell_p \subset \ell_p^\lambda$ that this inclusion is strict, where $1 \leq p < \infty$. This completes the proof. □

Now, as an immediate consequence of Theorem 4.18, the following corollary presents the necessary and sufficient condition for the matrix $\Lambda$ to be equivalent to $p$-absolute convergence, where $1 \leq p < \infty$.

**Corollary 4.19.** *The equality* $\ell_p^\lambda = \ell_p$ *holds if and only if* $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n > 1$, *where* $1 \le p < \infty$.

*Proof.* The necessity follows from Theorem 4.18. For, if the equality holds, then the inclusion $\ell_p \subset \ell_p^\lambda$ holds and hence $1/\lambda \in \ell_p$ by Lemma 4.10. Further, since the inclusion $\ell_p \subset \ell_p^\lambda$ cannot be strict, we have by Theorem 4.18 that $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n \neq 1$ and hence $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n > 1$.

Conversely, suppose that $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n > 1$. Then, there exists a constant $a > 1$ such that $\lambda_{n+1}/\lambda_n \ge a$ and hence $\lambda_n \ge \lambda_0 a^n$ for all $n \in \mathbb{N}$. This shows that $1/\lambda \in \ell_1$ which leads us with Corollary 4.13 to the consequence that the inclusion $\ell_p \subset \ell_p^\lambda$ holds for $1 \le p < \infty$.

On the other hand, we have by Lemma 4.2 (b) that $\{\lambda_n/(\lambda_n - \lambda_{n-1})\} \in \ell_\infty$ and hence $\{\lambda_{n-1}/(\lambda_n - \lambda_{n-1})\} \in \ell_\infty$.

Now, let $x \in \ell_p^\lambda$. Then $\Lambda(x) = \{\Lambda_n(x)\} \in \ell_p$ and hence $\{\Lambda_n(x) - \Lambda_{n-1}(x)\} \in \ell_p$. Thus, we obtain by the relation (13) that $\{S_n(x)\} \in \ell_p$, i.e., $S(x) \in \ell_p$ for every $x \in \ell_p^\lambda$. Therefore, we deduce by Lemma 4.5 that the inclusion $\ell_p^\lambda \subset \ell_p$ also holds. Hence, by combining the inclusions $\ell_p \subset \ell_p^\lambda$ and $\ell_p^\lambda \subset \ell_p$, we get the equality $\ell_p^\lambda = \ell_p$, where $1 \le p < \infty$. This concludes the proof. □

**Remark 4.20.** It can easily be shown that Corollary 4.19 still holds for $0 < p < 1$.

Finally, we end this section with the following corollary:

**Corollary 4.21.** *Although the spaces* $\ell_p^\lambda$, $c_0$, $c$ *and* $\ell_\infty$ *overlap, the space* $\ell_p^\lambda$ *does not include any of the other spaces. Furthermore, if* $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$, *then none of the spaces* $c_0$, $c$ *and* $\ell_\infty$ *includes the space* $\ell_p^\lambda$, *where* $0 < p < \infty$.

*Proof.* Let $0 < p < \infty$. Then, it is obvious that the spaces $\ell_p^\lambda$, $c_0$, $c$ and $\ell_\infty$ overlap, since the sequence $(\lambda_1 - \lambda_0, -\lambda_0, 0, 0, \ldots)$ belongs to all these spaces.

Further, the space $\ell_p^\lambda$ does not include the space $c_0$, since the sequence $x$ defined by (14), in the proof of Theorem 4.4, is in $c_0$ but not in $\ell_p^\lambda$. Hence, the space $\ell_p^\lambda$ does not include any of the spaces $c_0$, $c$ and $\ell_\infty$.

Furthermore, if $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$ then the space $\ell_\infty$ does not include the space $\ell_p^\lambda$. To see this, let $0 < p < \infty$. Then, Lemma 4.17 implies that the sequence $y$ defined by (16), in the proof of Theorem 4.18, is in $\ell_p^\lambda$ but not in $\ell_\infty$. Therefore, none of the spaces $c_0$, $c$ and $\ell_\infty$ includes the space $\ell_p^\lambda$ when $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$, where $0 < p < \infty$. This completes the proof. □

## 5 The basis for the space $\ell_p^\lambda$

In this final section, we give a sequence of the points of the space $\ell_p^\lambda$ which forms a basis for this space, where $1 \le p < \infty$.

If a normed space $X$ contains a sequence $(b_n)$ with the property that for every $x \in X$ there is a unique sequence $(\alpha_n)$ of scalars such that

$$\lim_{n\to\infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_n b_n)\| = 0,$$

then $(b_n)$ is called a Schauder basis (or briefly basis) for $X$. The series $\sum_k \alpha_k b_k$ which has the sum $x$ is then called the expansion of $x$ with respect to $(b_n)$, and written as $x = \sum_k \alpha_k b_k$.

Now, because of the transformation $T$ defined from $\ell_p^\lambda$ to $\ell_p$, in the proof of Theorem 3.3, is an isomorphism, the inverse image of the basis $(e^{(k)})_{k=0}^\infty$ of the space $\ell_p$ is the basis for the new space $\ell_p^\lambda$, where $1 \le p < \infty$. Therefore, we have the following:

**Theorem 5.1.** *Let $1 \le p < \infty$ and define the sequence $e_\lambda^{(k)} \in \ell_p^\lambda$ for every fixed $k \in \mathbb{N}$ by*

$$\big(e_\lambda^{(k)}\big)_n = \begin{cases} (-1)^{n-k}\dfrac{\lambda_k}{\lambda_n - \lambda_{n-1}}\,; & (k \le n \le k+1), \\ 0\,; & (\textit{otherwise}). \end{cases} \qquad (n \in \mathbb{N}) \tag{17}$$

*Then, the sequence $\big(e_\lambda^{(k)}\big)_{k=0}^\infty$ is a basis for the space $\ell_p^\lambda$ and every $x \in \ell_p^\lambda$ has a unique representation of the form*

$$x = \sum_k \Lambda_k(x)\, e_\lambda^{(k)}. \tag{18}$$

*Proof.* Let $1 \le p < \infty$. Then, it is obvious by (17) that $\Lambda(e_\lambda^{(k)}) = e^{(k)} \in \ell_p$ $(k \in \mathbb{N})$ and hence $e_\lambda^{(k)} \in \ell_p^\lambda$ for all $k \in \mathbb{N}$.

Further, let $x \in \ell_p^\lambda$ be given. For every non-negative integer $m$, we put

$$x^{(m)} = \sum_{k=0}^m \Lambda_k(x)\, e_\lambda^{(k)}.$$

Then, we have that

$$\Lambda(x^{(m)}) = \sum_{k=0}^m \Lambda_k(x)\, \Lambda(e_\lambda^{(k)}) = \sum_{k=0}^m \Lambda_k(x)\, e^{(k)}$$

and hence

$$\Lambda_n(x - x^{(m)}) = \begin{cases} 0\,; & (0 \le n \le m), \\ \Lambda_n(x)\,; & (n > m). \end{cases} \qquad (n, m \in \mathbb{N})$$

Now, for any given $\epsilon > 0$ there is a non-negative integer $m_0$ such that

$$\sum_{n=m_0+1}^\infty |\Lambda_n(x)|^p \le \Big(\frac{\epsilon}{2}\Big)^p.$$

Therefore, we have for every $m \geq m_0$ that

$$\begin{aligned}\|x - x^{(m)}\|_{\ell_p^\lambda} &= \left(\sum_{n=m+1}^{\infty} |\Lambda_n(x)|^p\right)^{1/p} \\ &\leq \left(\sum_{n=m_0+1}^{\infty} |\Lambda_n(x)|^p\right)^{1/p} \\ &\leq \frac{\epsilon}{2} < \epsilon\end{aligned}$$

which shows that $\lim_{m\to\infty} \|x - x^{(m)}\|_{\ell_p^\lambda} = 0$ and hence $x$ is represented as in (18).

Finally, let us show the uniqueness of the representation (18) of $x \in \ell_p^\lambda$. For this, suppose that $x = \sum_k \alpha_k(x)\, e_\lambda^{(k)}$. Since the linear transformation $T$ defined from $\ell_p^\lambda$ to $\ell_p$, in the proof of Theorem 3.3, is continuous, we have

$$\Lambda_n(x) = \sum_k \alpha_k(x)\, \Lambda_n(e_\lambda^{(k)}) = \sum_k \alpha_k(x)\, \delta_{nk} = \alpha_n(x); \qquad (n \in \mathbb{N}).$$

Hence, the representation (18) of $x \in \ell_p^\lambda$ is unique. This completes the proof. □

Now, it is known by Theorem 3.1 (b) that $\ell_p^\lambda$ $(1 \leq p < \infty)$ is a Banach space with its natural norm. This leads us together with Theorem 5.1 to the following corollary:

**Corollary 5.2.** *The sequence space $\ell_p^\lambda$ of non-absolute type is separable for $1 \leq p < \infty$.*

Finally, we conclude our work by expressing from now on that the aim of the next paper is to determine the $\alpha$-, $\beta$- and $\gamma$-duals of the space $\ell_p^\lambda$ and is to characterize some related matrix classes, where $1 \leq p \leq \infty$.

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